2

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PROPAGATION OF ULTRASHORT LIGHT PULSES
IN A LASER MEDIUM*

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ABSTRACT

The set of coupled Maxwell-Schroedinger equations that govern the propagation of ultrashort pulses through a nonlinear laser medium are solved analytically in the steady-state regime. The periodic energy exchange, between the intense passing electromagnetic field and the stationary aggregate of two level atoms, leads to doubly periodic Jacobian elliptic function solutions. Their modulus k is a function of the characteristic time length of the system τ and the pulse amplitude $\mathcal{E}_0$. The physical properties of these steady-state solutions are considered. It is shown that there are in essence only four periodic pulse train solutions sdu , ndu , cnu and dnu , u is the retarded time $t-z/v$ referred to the characteristic time τ . τ and the length of the pulse T are controlled by the steady population inversion density η_0 established from the outside. Two of these solutions expand with intensified optical pumping while the other two contract under the same conditions.

1. Introduction

We present in this paper results of an analytical investigation into the propagation of ultrashort pulses¹ through a nonlinear laser medium consisting of an aggregate of "atoms" with two energy levels u and l separated by an amount of energy equal to that of the carrier frequency photons. These pulses can be amplified, or attenuated, when passing through this ensemble of two-level atoms when a population inversion is established between their energy levels u and l . The short time duration of these pulses, is realized by a process of mode locking or modulation of the optical carrier wave. The great interest in these pulses is due to their very short time duration, $\sim 10^{-9} - 10^{-12}$ sec., which in turn enables the realization of very high power densities. Proposals, for example, of energy release via fusion of hydrogen isotopes rely on high energy densities in time and space, up to some 10^{17} Joules/cm² sec., of intense coherent laser-produced light beams^{2,3}. With some reasonable approximations the steady state theory of propagation of these pulses through such a resonant nonlinear laser medium can be presented in closed form⁴ with all the solutions given in terms of the well known Jacobian elliptic functions often associated with nonlinear oscillations⁵.

The atomic system is to be treated quantum mechanically while the very intense electromagnetic field can be presented in classical terms. This semiclassical description seems then well justified and sufficient. A coupling term, of the Schroedinger equation of the two-level atoms and the Maxwell equations of the electromagnetic field, representing the light-matter interaction is to be included in the differential equations of the ultrashort pulse's propagating through the resonant laser medium. A density matrix ρ approach is to be used in representing the two level atom to facilitate the necessary statistical summations. The nonlinear

character of the laser medium in which a population inversion is established reduces the coupled Maxwell-Schroedinger equations to a nonlinear set.

Denote by T_1 and T_2 , $T_2 \leq T_1$, the decay constants of the wave amplitudes associated with the atomic energy levels and the atomic dipole moment, respectively. These include contributions of the radiative decay and such damping mechanisms as atomic collisions and averaging over various atomic velocities in gases and phonon interruptions and averaging over various resonance frequencies in solids. The pulse is termed ultrashort when its duration is small compared with any of these relaxation time constants of the two-level atoms.

In the steady state the passage of the ultrashort pulse through the aggregate of resonant two-level atoms, embedded in a nondispersive lossless medium, is associated with a periodic interchange of energy between the reservoirs of the stationary atomic system and the passing electromagnetic field. The steady state solutions of the coupled nonlinear Maxwell-Schroedinger equations must then be periodic waves or pulse trains, not necessarily of zero mean value.

Due to the very high intensity of the propagated electromagnetic field, it is reasonable, by comparison, to neglect the mutual direct interaction of neighboring atoms. Their wave functions are assumed therefore as non-overlapping, and the length of the period of the oscillating wave will be independent of the interatomic distances. It is to be a function of the other physical characteristics of the system such as the wave amplitude and the characteristic time length. We define accordingly a characteristic time length of the system τ . Corresponding characteristic spatial length is $l = \tau v$, where v is the velocity of propagation of the pulse.

A number of researchers investigated systematically and in great detail the propagation of pulses of a wide range of duration⁶⁻¹⁸. The differential equations that govern the pulse propagation phenomenon being nonlinear, most of these investigations treat the subject by numerical methods. McCall and Hahn⁹ were the first to obtain the analytical solution of the sech shaped single pulse-the self-induced transparency effect. Arecchi, DiGiorgio and Smeda¹⁴ reported then the elliptic-function dnu solution, u is the retarded time, and Eberly¹⁵ and Crisp¹⁶ the cnu and dnu solutions. In this paper we shall concern ourselves with the propagation of ultrashort pulses only, for it is these pulses that produce the highest coherent light¹⁹ energy densities. We show that the nonlinear set of the coupled Maxwell-Schroedinger equations can be solved analytically, derive all the solutions of these equations and discuss some of their physical properties including possible ways of controlling the pulse length. These solutions include the elliptic function solutions dnu and cnu mentioned above. The undistorted sech single pulse of infinite support derived by McCall and Hahn follows in the asymptotic limit $k \rightarrow 1$, k is the modulus of the Jacobian elliptic functions, from the dnu¹⁴ and cnu^{15,16} solutions.

The analytical formulation of the propagation phenomenon of ultrashort pulses in a resonant nonlinear laser medium is presented in the next section. It includes the nonlinear set of the Maxwell-Schroedinger equations, (2.3 - 2.6), derived from first principles in previous publications. The following two sections, 3 and 4, introduce the system's characteristic time length τ , the differential equation of the periodic pulse trains and its Jacobian elliptic function solutions. In section 5 we derive the solutions with a pole at d , and in sections 6 and 7 the steady

population inversion density necessary to maintain these pulse trains and also their asymptotic limit. Section 8, 9 and 10 are devoted to the consideration of solutions with a pole at s , c and n , respectively. It is shown that there are in essence only four physically admissible steady state pulse train solutions, two of which expand with intensified optical pumping, while the other two contract under the same conditions. None of the velocities of these pulses exceeds the velocity of light in the host medium. Their associated population inversion densities are within physically acceptable limits $-1 \leq \eta \leq 1$.

2. The Coupled Maxwell-Schroedinger Equations

We consider the propagation of ultrashort pulses through an ensemble of two-level "atoms", with non-overlapping wave functions, embedded in a nondispersive, lossless host medium. At an optical carrier frequency of some 10^{15} c.p.s., a pulse of even 10^{-12} sec duration comprises 10^3 wavelengths. The amplitude of the pulse's envelope can then be viewed as slowly varying over one wavelength and one optical cycle. We assume then the incident field $E(z,t)$ in form of a quasimonochromatic plane wave with a slowly varying amplitude,

$$E(z,t) = p^{-1} \xi(z,t) \cos \phi ; \quad \phi = kz - \omega_0 t \quad (2.1)$$

p -real matrix element of the electric dipole, $p = \langle \psi_u | \hat{p} | \psi_1 \rangle$.

The carrier frequency ω_0 is at the center of the symmetric atomic line shape function $g(\Delta\omega)$; $\Delta\omega = \omega - \omega_0$. The field induced by the dispersion and absorption components of the macroscopic polarization,

$$P(z,t) = n_0 p \left\{ \langle C(\Delta\omega, z, t) \rangle_{\text{av. over } \Delta\omega} \cos \phi + \langle S(\Delta\omega, z, t) \rangle_{\text{av. over } \Delta\omega} \sin \phi \right\} , \quad (2.2)$$

n_0 - number of active atoms per unit volume, must be self-consistent with the incident field (2.1). With the intense incident field treated classically one obtains hence the set of nonlinear coupled Maxwell equations of the propagated field and Schroedinger equation of the atomic system^{20,12,13,18}.

$$d_t \xi = \alpha \langle S \rangle_{\text{av. over } \Delta\omega} ; \quad \alpha = 2\pi\hbar^{-1} n_0 \omega_0 p^2 ; \quad (2.3)$$

$$\partial_t C = -\Delta\omega S ; \quad (2.4)$$

$$\partial_t S = \Delta\omega C + \xi \eta ; \quad d_t = \frac{d}{dt} = \partial_t + c\partial_z ; \quad (2.5)$$

$$\partial_t \eta = -S\xi ; \quad C^2 + S^2 + \eta^2 = 1 , \quad (2.6)$$

$\eta = \frac{n}{n_0}$ - population inversion density per atom ; c- speed of light in the host medium. This set of equations is obtained from a more general one by postulating ultrashort pulse solutions or letting the time constants associated with the atomic system tend to infinity.

3. Differential Equation of Periodic Steady State Pulse Trains.

We define now a characteristic time length τ of the overall system. When computed below, for the various propagated waves, it is shown to depend upon the field amplitude $\mathcal{E}_0$ and the population inversion density η_0 established from the outside.

The steady state solutions of the above set of nonlinear equations are functions of the retarded time, $u = \tau^{-1} (t-z/v)$, measured in units of the characteristic time length τ . Equations (2.3, 2.4) point to the same u dependence for $\mathcal{E}$ and C.S and η follow then from eqs. (2.4) and (2.6), respectively

$$\mathcal{E} = \mathcal{E}_0 e(u) \quad (3.1)$$

$$C = \tau \Delta \omega A(\Delta \omega) e(u) \quad (3.2)$$

$$S = -A(\Delta \omega) e(u) \quad (3.3)$$

$$\eta - \eta_0 = \frac{1}{2} \tau \mathcal{E}_0 A(\Delta \omega) e^2(u) \quad (3.4)$$

No use was made hitherto of eq. (2.5). Expressions (3.2, 3.3 and 3.4) for C,S and η must, of course, satisfy this equation. It is shown below that with the solutions C,S and η inserted in eq. (2.5), this latter equation defines the characteristic time length τ of the two-level atoms-field structure in terms of the amplitude $\mathcal{E}_0$ and the constant established population inversion density η_0 .

The $\Delta \omega$ dependent part $A(\Delta \omega)$ of C,S and η is defined by means of the normalization condition, $C^2 + S^2 + \eta^2 = 1$. We do thereby consider also eq. (2.5) indirectly for this normalization condition results also from eqs. (2.4, 2.5 and 2.6).

4. Steady State Solutions

The steady state solutions of the coupled Maxwell-Schroedinger equations (2.3-2.6) form the set of periodic waves the two-level atomic structure is capable of sustaining. These periodic amplitude modulated optical carrier waves are viewed as a train of pulses with each period representing a single pulse.

The set of solutions of the nonlinear differential equations (2.3-2.6) of the periodic steady state pulse trains consists of the well known Jacobian elliptic functions defined as follows: Denote the corners of the rectangle $o, K, K+iK', iK'$ in the Argand plane by the letters s, c, d and n . K is the complete elliptic integral of the first kind, $K(k) = \int_0^{\pi/2} d\theta (1-k^2 \sin^2 \theta)^{-1/2}$, with the modulus k . $K'(k') = K(k)$ where $k^2 + k'^2 = 1$ and k' is the complementary modulus. The Jacobian elliptic function pqu , p and q are any of the points s, c, d and n , has a simple zero at p and single pole at q ^{21,22}. With four possible positions of the simple pole and three associated positions of the zero there are altogether twelve Jacobian elliptic functions, all solutions of the set of the nonlinear Maxwell-Schroedinger equations (2.3-2.6).

These are doubly periodic functions. Both of their periods, proportional to K and K' , appear in the wave solutions of the nonlinear set of equations (2.3-2.6). All the solutions of the Maxwell-Schroedinger equations (2.3-2.6) are the set of the twelve Jacobian elliptic functions. They do not, of course, form a complete set in the sense of a linear space.

The twelve Jacobian elliptic functions can be divided in three groups of four functions each. The elliptic function solution of one group can be obtained from another function within the same group by means of

displacement transformations that involve shifting the retarded time argument u of the elliptic function along the real or imaginary axis of the Argand plane by the amounts $K(k)$ and $iK'(k)$, and multiples thereof, see reference 22, section 16.8. However, these transformations involve also the modulus k of the Jacobian elliptic functions, which is amplitude $\mathcal{E}_0$ and characteristic retarded time τ dependent. This may lead to new physical results, not just different formal presentations of previous solutions. It is necessary then when performing these transformations to interpret in physical terms the processes involved and the newly obtained wave amplitudes, field-polarization-population inversion density relations, and the various functional dependences. We prefer, therefore, to solve the nonlinear set of eqs. (2.3-2.6) directly.

5. The Copolar Solutions sdu, ndu and cdu

We consider first the three copolar elliptic function solutions, with a pole at d , of the coupled Maxwell-Schroedinger eqs. (2.3-2.6)

$$\xi = \xi_0 \text{ sdu}; \quad \xi_0 \text{ ndu}; \quad \xi_0 \text{ cdu}. \quad (5.1)$$

The corresponding, in phase and $\frac{\pi}{2}$ out of phase, components of the polarization follow directly from eqs. (3.2 and 3.3),

$$\begin{aligned} C &= \tau \Delta \omega A(\Delta \omega) \text{ sdu}; & \tau \Delta \omega A(\Delta \omega) \text{ ndu}; & \tau \Delta \omega A(\Delta \omega) \text{ cdu}; \\ S &= -A(\Delta \omega) \text{ cdu ndu}; & -A(\Delta \omega) k'^2 \text{ sdu cdu}; & A(\Delta \omega) k'^2 \text{ sdu ndu}, \end{aligned} \quad (5.2)$$

and the related population inversions from eq. (2.5)

$$\begin{aligned} \eta &= -A(\Delta \omega) (\tau \xi_0)^{-1} [2k^2 c d^2 u - 1 + (\tau \Delta \omega)^2]; \\ \eta &= -A(\Delta \omega) (\tau \xi_0)^{-1} [2k^2 c d^2 u - k^2 + (\tau \Delta \omega)^2]; \\ \eta &= A(\Delta \omega) (\tau \xi_0)^{-1} [2k^2 k'^2 s d^2 u + k'^2 - (\tau \Delta \omega)^2]. \end{aligned} \quad (5.3)$$

The last equation (2.6) expresses the characteristic length of the retarded time τ in terms of the modulus k , with the amplitude ξ_0 as a parameter,

$$\tau \xi_0 = 2kk'; \quad 2k'; \quad -2ik. \quad (5.4)$$

A change in modulus, $k \rightarrow ik$, in the cdu solution is now needed for the pulse train amplitude ξ_0 and the characteristic length of the retarded time τ to be real. This in turn leads to the transformation²³

$$\xi = \xi_0 \text{ cd}(u|ik) = \xi_0 \text{ cn}(uk_1'^{-1}|k_1) \quad (5.5)$$

or

$$\xi = \xi_0 \text{ cn}(u|k_1); \quad u = \tau_1^{-1}(t-z/v). \quad (5.6)$$

The characteristic length of the retarded time is now

$$\tau_1 = \tau k_1' , \quad (5.7)$$

where

$$k_1^2 = k^2(1+k^2)^{-1} ; \quad k_1'^2 = 1-k_1^2 = (1+k^2)^{-1}. \quad (5.8)$$

The modulus k_1 is positive and less than one when $0 < k < \infty$ and

$$\tau_1 \mathcal{E}_0 = 2k_1 , \quad (5.9)$$

see eqs. (5.4), (5.7), and (5.8). To render the notation uniform we shall omit in the following the subscript 1 in τ_1 and k_1 .

The corresponding changes in C , S and η , given by eqs. (5.2) and (5.3), are

$$\begin{aligned} C &= \tau \Delta \omega A(\Delta \omega) \text{cnu} ; \\ S &= A(\Delta \omega) \text{snu} \text{dnu} ; \\ \eta &= A(\Delta \omega) (2k)^{-1} [2\text{dn}^2 u - 1 - (\tau \Delta)^2] . \end{aligned} \quad (5.10)$$

The function $A(\Delta \omega)$, of the amount of the off-resonance of the irradiated two level atoms and the incident carrier frequency, that appears in the above expressions of C , S and η follows the normalization condition $C^2 + S^2 + \eta^2 = 1$, see eq. (2.6),

$$\begin{aligned} A(\Delta \omega) &= 2kk' \{ 4(k\tau\Delta\omega)^2 + [1 - (\tau\Delta\omega)^2]^2 \}^{-1/2} ; \\ A(\Delta \omega) &= 2k' \{ 4(\tau\Delta\omega)^2 + [k^2 - (\tau\Delta\omega)^2]^2 \}^{-1/2} ; \\ A(\Delta \omega) &= 2k \{ [1 + (\tau\Delta\omega)^2]^2 - 4(k'\tau\Delta\omega)^2 \}^{-1/2} . \end{aligned} \quad (5.11)$$

The macroscopic values of the population inversion density per atom η and the polarization density components C , S are given by expressions

(5.2) (5.3) and (5.11) averaged over all values of $\Delta\omega$.

It is easily shown that the population inversion densities η of the solutions sdu , ndu and cdu reduced to cnu given by eqs. (5.3) and (5.11) never exceed their asymptotic values ± 1 , $-1 \leq \eta \leq 1$.

In the Jacobian elliptic function pqu , where p and q are any of the rectangle corners s , c , d , n in the Argand plane, see Fig. 1, the step $p \rightarrow q$ amounts to a half-period of pqu . Any of the sides of the rectangle K , iK' and its diagonal $K + iK'$ that differ from the $p \rightarrow q$ step are quarter periods of pqu .^{21,22} The length of the periods of the train pulses sdu , ndu and cnu are then

$$T = 4\tau K(k) \quad 2\tau K(k) \quad 4\tau K(k) \quad (5.12)$$

The functional relationship $T = T(\tau, \xi_0)$ is given by eqs. (5.4), (5.9) and (5.12). It is independent of the particular line shape $g(\Delta\omega)$ of the atomic system. It is a function of the system's characteristic time length and the amplitude ξ_0 .

The pulse trains sdu , ndu and cnu are shown in Figs. 2-4 for various values of k . Fig. 5 shows the functional dependence of the length of the period T , divided by ξ_0 , on the modulus k for the ndu solution. The sdu and the cdu , reduced to dnu , curves oscillate around their zero average values while the area under ndu , taken over the period $2\tau K(k)$, equals 2π corresponding to a whole turn of the C, S, η unit length vector,

$$\xi_0 \int_0^{2\tau K(k)} ndu \, dt = 2\pi \quad (5.13)$$

we used here the formula $\int ndu = k'^{-1} \arccos(cdu)$ ²⁴.

The atomic lineshape $g(\Delta\omega)$, which appeared in none of the above computations of the area under the pulse, population inversion density, and induced polarization of the resonant medium, strongly affects the

velocity v of the periodic pulse train. The ratio of the speed of light in the medium c to the velocity v of the steady state pulse train is obtained by inserting the values of S and $\mathcal{E}$ from eqs. (5.1) and (5.2), or (5.10), into the reduced Maxwell equation (2.3). For the sdu, ndu and cnu shaped pulse trains we obtain the identical expression

$$\frac{c}{v} = 1 + \alpha\tau \mathcal{E}_0^{-1} \int_{-\infty}^{\infty} \Lambda(\Delta\omega) g(\Delta\omega) d(\Delta\omega) \quad (5.14)$$

with the explicit expression for $\Lambda(\Delta\omega)$ given by the respective eqs. (5.11).

6. The Constant η_0 and the Characteristic Time Length τ .

The integration constant $\eta_0 = \eta_0(\Delta\omega)$ in equation (3.4) is the value of the population inversion density η , in the absence of any incident electric field. Its macroscopic value is obtained by averaging $\eta_0(\Delta\omega)$ over all values $\Delta\omega$ of the irradiated atoms,

$$\langle \eta_0 \rangle = \int_{-\infty}^{+\infty} \eta_0(\Delta\omega) g(\Delta\omega) d(\Delta\omega) . \quad (6.1)$$

This quantity is induced from the outside by some optical pumping process. η_0 is a function of the characteristic time length of the system τ , see equations (5.4, 5.9 and 6.3 below) and provides thereby the means of controlling the length of the passing pulse,

For the sdu, ndu and cnu solutions of the previous section we obtain from eqs. (3.1), (3.4) and (5.1)

$$\begin{aligned} \eta - \eta_0 &= \frac{1}{2}\tau \mathcal{E}_0 A(\Delta\omega) s d^2 u ; \\ \frac{1}{2}\tau \mathcal{E}_0 A(\Delta\omega) n d^2 u &; \\ \frac{1}{2}\tau \mathcal{E}_0 A(\Delta\omega) c n^2 u &. \end{aligned} \quad (6.2)$$

Subtracting these from the expressions for the overall population inversion densities η given by eqs. (5.3) and (5.10) we obtain hence,

$$\begin{aligned} \frac{\eta_0}{A(\Delta\omega)} &= (2kk')^{-1} [1 - 2k^2 - (\tau\Delta\omega)^2] ; \\ (2k')^{-1} [k^2 - 2 - (\tau\Delta\omega)^2] &; \\ (2k)^{-1} [2k'^2 - 1 - (\tau\Delta)^2] &, \end{aligned} \quad (6.3)$$

with $A(\Delta\omega)$ given by the respective eqs. (5.11).

When the modulus $k \rightarrow 1$, all the above values of $\eta_0(\Delta\omega)$, and their averaged values taken over all $\Delta\omega$, tend to -1

$$\lim_{k \rightarrow 1} \eta_0(\Delta\omega) = \lim_{k \rightarrow 1} \langle \eta_0(\Delta\omega) \rangle = -1 . \quad (6.4)$$

All the atoms are then in the pure state associated with their lower energy level.

7. The Asymptotic Limit $k \rightarrow 1$ of the sdu, ndu and cnu Pulses.

Figs. 2, 3 and 4 show that, as k increases the sdu and ndu pulses contract while their nondimensional amplitudes ξ/ξ_0 , equal to k'^{-1} , increase. The amplitude of the cnu pulse is constant, $\xi/\xi_0 = 1$, while its length increases with k .

It is then of interest to investigate the asymptotic limit, $k \rightarrow 1$. We note that the characteristic time lengths associated with the sdu and ndu solutions vanish then, see eq. (5.4). The complete elliptic integral of the first kind tends to infinity, for $k \rightarrow 1$, as,

$$\lim_{k \rightarrow 1} [K(k) - \ln \frac{4}{k'}] = 0 . \quad (7.1)$$

The asymptotic values of the lengths of the pulses of the sdu, ndu and cnu solutions follow then from their general expressions given by eq. (5.12),

$$\begin{aligned} T &= \lim_{k \rightarrow 1} (-8kk' \xi_0^{-1} \ln k') = 0 ; \\ \lim_{k \rightarrow 1} (-4k' \xi_0^{-1} \ln k') &= 0 ; \\ \lim_{k \rightarrow 1} (-8 \xi_0^{-1} \ln k') &\rightarrow \infty . \end{aligned} \quad (7.2)$$

The sdu pulse oscillates around its zero mean value, the area under the single ndu pulse is conserved at 2π , see eq. (5.13). The area under

the cnu pulse, extending from $-\tau K(k)$ to $\tau K(k)$, is

$$2 \xi_0 \int_0^{\tau K(k)} \text{cnu} \, dt = \text{arc cos } k' \rightarrow 2\pi \quad \text{as } k \rightarrow 1 \quad (7.3)$$

As $k \rightarrow 1$, the Jacobian elliptic functions sdu , nud and cnu reduce to^{21,22}

$$\begin{array}{lll} \text{sdu} \xrightarrow{k \rightarrow 1} \sinh u & ; & \text{ndu} \xrightarrow{k \rightarrow 1} \cosh u \\ & & \text{cnu} \xrightarrow{k \rightarrow 1} \text{sech } u \end{array} \quad (7.4)$$

all the active atoms are brought to the state of their lower energy level, the sdu solution contracts to a zero- π pulse in form of a generalized function of infinite height and zero support, the ndu solution shrinks to a 2π pulse of infinite height and vanishing support, while the cnu solution expands into a 2π sech shaped pulse of infinite support

It is of importance to note here that the approximations used in deriving the basic set of differential equations (2.3 through 2.6) are no longer valid in the asymptotic limit. The coupled Maxwell-Schroedinger equations assumed small pulse lengths, compared with any of the relaxation time constants of the two-level atomic medium, and slowly varying pulse amplitudes. In the limit, $k \rightarrow 1$, when the pulse takes on the shape of a generalized function on infinite height and vanishing support the amplitude cannot be considered slowly varying and the validity of the solutions is to be limited to some appropriately smaller values of k .

8. Solutions with a Pole at z .

It is easily seen that neither of the three copolar functions csu , dsu and nsu , which are solutions of coupled Maxwell-Schroedinger eqs. (2.3-2.6) represents a physically realizable pulse train. They all tend to infinity at the origin, $u = 0$, and at, $u = 2K$, and integer multiples

thereof. The derivative, with respect to u , of either of these functions equals the product of the two remaining function taken with a negative sign. Following eqs. (3.1), (3.2), and (3.3) we can write therefore,

$$\begin{aligned} \xi &= \xi_0 \text{csu} ; & \xi_0 \text{dsu} ; & \xi_0 \text{nsu} ; \\ C &= \tau \Delta \omega A(\Delta \omega) \text{csu} ; & \tau \Delta \omega A(\Delta \omega) \text{dsu} ; & \tau \Delta \omega A(\Delta \omega) \text{nsu} ; \\ S &= A(\Delta \omega) \text{dsu nsu} ; & A(\Delta \omega) \text{nsu csu} ; & A(\Delta \omega) \text{csu dsu} , \end{aligned} \quad (8.1)$$

compute the population inversion density η with the aid of eq. (2.5), and insert the expressions for η , S and ξ into eq. (2.6). We obtain then an identical, modulus k free, condition for the characteristic time length τ and amplitude ξ_0 , common to all these three functions

$$(\tau \xi_0)^2 = -4 \quad (8.2)$$

A real amplitude ξ_0 involves then an imaginary τ and in turn an imaginary u . Hence, with $u \rightarrow iu$, we obtain, by means of the Jacobi imaginary transformation,^{21,22} a pure imaginary amplitude $-i \xi_0$,

$$\xi_0 \text{cs}(u|k) \rightarrow \xi_0 \text{cs}(iu|k) = i \xi_0 \text{ns}(u|k') \quad (8.3)$$

Similar expressions result for the other two functions dsu and nsu .

It can also be shown that the population inversion density η for all these solutions is outside its physically feasible range, $-1 \leq \eta \leq 1$, for any value of the modulus k .

9. Periodic Pulse Train Solutions with a Pole at c .

The functions scu , dcu and ncu , all solutions of the Maxwell Schrodinger equations (2.3-2.6) display infinities at K , $3K$, due to the pole at these points of the $\text{Re}u$ axis. They all lead to pure

imaginary moduli k which reduces the scu and ncu solutions to the previously encountered pulse trains sdu and ndu, respectively. The dcu solution transforms into dnu¹⁴ to be considered now.

With the incident electric field given by

$$\mathcal{E} = \mathcal{E}_0 \text{scu} ; \quad \text{dcu} ; \quad \text{ncu} , \quad (9.1)$$

the in phase and 90° out of phase components of the induced polarization are

$$\begin{aligned} C &= \tau \Delta \omega A(\Delta \omega) \text{scu} ; & \tau \Delta \omega A(\Delta \omega) \text{dcu} ; & \tau \Delta \omega A(\Delta \omega) \text{ncu} ; \\ S &= -A(\Delta \omega) \text{dcu} \text{ncu} ; & -A(\Delta \omega) k^2 \text{scu} \text{ncu} ; & -A(\Delta \omega) \text{scu} \text{dcu} . \end{aligned} \quad (9.2)$$

The population inversion density follows from eq. (2.5),

$$\begin{aligned} \eta &= -(\tau \mathcal{E}_0)^{-1} A(\Delta \omega) [2 \text{dc}^2 \text{u} - k^2 + (\tau \Delta \omega)^2] ; \\ &\quad -(\tau \mathcal{E}_0)^{-1} A(\Delta \omega) [2 k'^2 \text{nc}^2 \text{u} - k'^2 + (\tau \Delta \omega)^2] ; \\ &\quad -(\tau \mathcal{E}_0)^{-1} A(\Delta \omega) [2 k'^2 \text{sc}^2 \text{u} + 1 + (\tau \Delta \omega)^2] , \end{aligned} \quad (9.3)$$

and the system's characteristic time length from eq. (2.6),

$$(\tau \mathcal{E}_0)^2 = -4 k'^2 ; \quad -4 ; \quad -4 k'^2 , \quad (9.4)$$

In solutions scu and ncu we choose a pure imaginary k' , for τ and $\mathcal{E}_0$ to be real, $k' \rightarrow ik'$, which leads to a modulus k that is greater than one, $k^2 = 1 - k'^2 \rightarrow k^2 = 1 + k'^2 \geq 1$. Jacobi's real transformation,^{21,22} leads then to previously discussed sdu and ndu solutions shown in figures 2 and 3, respectively.

$$\begin{aligned}
\mathcal{E}_0^{sc}(u|k) &= \mathcal{E}_0^{sd}(u|k_1) \quad ; \quad \mathcal{E}_0^{ol} = \mathcal{E}_0^{k_1} \quad ; \\
\mathcal{E}_0^{nc}(u|k) &= \mathcal{E}_0^{nd}(u|k_1) \quad ; \quad k_1 = k^{-1} < 1 \quad .
\end{aligned}
\tag{9.5}$$

The characteristic time length associated with the dcu solution appears to independent of the modulus k . We choose then a pure imaginary τ or $\tau \rightarrow i\tau$, and hence a pure imaginary $u \rightarrow iu$. Following Jacobi's imaginary transformation $dc(iu|k)$ evolves into the dnu solution, with a pole at n .

$$\mathcal{E}_0^{dc}(iu|k) = \mathcal{E}_0 \frac{dn(iu|k)}{cn(iu|k)} = \mathcal{E}_0 \frac{dc(u|k')}{nc(u|k')} = \mathcal{E}_0^{dn}(u|k') \quad . \tag{9.6}$$

The symmetry of the elliptic functions with respect to the modulus k and the complementary modulus k' permits us to consider in the following the $\mathcal{E}_0^{dn}(u|k) \equiv \mathcal{E}_0^{dnu}$ solution. The induced polarization components C, S and the population inversion η follow the appropriate transformation of their values given in eq. (9.2) and (9.3), with $u \rightarrow iu$,

$$\begin{aligned}
C &= \tau \Delta \omega A(\omega) dnu \quad ; \quad S = +k^2 A(\Delta \omega) snu \, cnu \quad ; \quad \eta = 1/2A(\Delta \omega) [2k^2 cn^2 u - k'^2 - (\tau \Delta \omega)^2] \quad ; \\
A(\Delta \omega) &= 2\{[k'^2 - (\tau \Delta \omega)^2]^2 + 4(\tau \Delta \omega)^2\}^{-1/2} \quad ; \quad \tau \mathcal{E}_0 = 2 \quad ; \quad T = 2\tau K(k) \quad .
\end{aligned}
\tag{9.7}$$

The periodic dnu curve is shown in fig. 6. The area bounded by the curve and the time axis, taken over one period is

$$\mathcal{E}_0 \int_0^{2\tau K(k)} dnu \, dt = 2\pi \quad . \tag{9.8}$$

The minimum ordinate equals k' and the length of the period $4\mathcal{E}_0^{-1}K(k)$ increases with k . The population inversion density η_0 induced by outside pumping is, in accordance with eq. (3.4) and (9.7),

$$\eta_0 = - \frac{k'^2 + 1 + (\tau \Delta \omega)^2}{\sqrt{[k^2 - (\tau \Delta \omega)^2]^2 + 4(\tau \Delta \omega)^2}} \quad (9.9)$$

As $k \rightarrow 1$, $\eta_0 \rightarrow -1$, all active atoms are in the pure state of their lower energy level and the dnu curves evolves in a 2π sech shaped pulse of infinite support.

The velocity of propagation of the periodic dnu wave, derived by insertion of the expression (9.7) for S into the reduced Maxwell eq. (2.3), is given again by eq. (5.14) with $\Lambda(\Delta \omega)$ defined by eq. (9.7).

10. Solutions With a Pole at n at $iK'(k)$.

Of the three copolar solutions of the Maxwell-Schroedinger equations (2.3 - 2.6), with a pole at n , cnu, dnu and snu, the first two were discussed in sections 5 and 9, respectively. By the same method one derives easily the characteristic time length τ - modulus k relationship for the snu solution,

$$(\tau \mathcal{E}_0)^2 = -4k^2 \quad (10.1)$$

The modulus k is now to be chosen pure imaginary, $k \rightarrow ik$, and the ensuing transformation^{21,22},

$$\mathcal{E}_0 \text{sn}(u|ik) = \mathcal{E}_1 \text{sd}(u_1|k_1) \quad ; \quad \mathcal{E}_1 = \mathcal{E}_0 k_1' \quad ; \quad u_1 = uk_1'^{-1} \quad ; \quad k_1 = k^2(1+k^2)^{-1}; \quad (10.2)$$

$$0 \leq k_1' \leq 1.$$

reduces snu to the previously discussed periodic solution sdu.

Summary

Propagation of ultrashort pulses of an optical carrier frequency through a resonant nonlinear laser medium involves a periodic exchange of energy between the stationary aggregate of two level atoms and the passing electromagnetic field. It is governed by a nonlinear set of Maxwell-Schroedinger equations. All their steady-state solutions are expressed in closed form in terms of the well known Jacobian elliptic functions. A closer investigation of these functions shows that there are essentially only four periodic pulse trains sdu, ndu, cnu and dnu that can be sustained, in the steady state regime, by the two energy level atoms structure. The characteristic time length of the system can be controlled by outside optical pumping. Intensified pumping lengthens the pulses associated with the first two solutions sdu, ndu while shortening the ones represented by the two latter ones, cnu, dnu.

FIGURE CAPTIONS

- Figure 1 The s,c,d,n rectangle used to define the Jacobian elliptic functions.
- Figure 2 The $\mathcal{E} = \mathcal{E}_0$ sdu pulse train for $k = .71, .98$ and $.995$.
- Figure 3 The $\mathcal{E} = \mathcal{E}_0$ ndu pulse train for $k = .71, .98$ and $.995$.
- Figure 4 The $\mathcal{E} = \mathcal{E}_0$ cnu pulse train for $k = .71, .98$ and $.995$.
- Figure 5 The period $T/\mathcal{E}_0$ of the ndu pulse train as a function of the modulus 0k .
- Figure 6 The $\mathcal{E} = \mathcal{E}_0$ dnu curve for $k = .71, .98$ and $.995$.

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23. See, for example, reference 18, section 16.10.
24. See reference 18, section 16.24.

FIGURE 1

FIGURE 2

FIGURE 3

FIGURE 4

FIGURE 5

FIGURE 6

Distribution

Internal:

E. Bliss
S. Bodner
R. Carman
G. Dahlbacka
J. Emmett
J. Fleck
R. Fisher
A. Glass
R. Harney
R. Harrach
H. Haus
C. Hausmann
P. Hoff
J. Holtzrichter
R. Jacobs
G. Kachen
R. Kidder
W. Kruer
W. Krupke
G. Leppelmeir
K. Manes
E. Mishkin (10)
J. Nuckolls
C. Rhodes
J. Shearer
E. Teller
J. Trenholme
L. Wood

TID (5)

External:

F. Hopf
Optical Sciences Center
University of Arizona
Tucson, Arizona

J. Marburger
University of Southern California
Los Angeles, California

W. Condell
Office of Naval Research
Arlington, Virginia

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New York, N. Y.

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