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THE TRANSIENT BEHAVIOR OF SINGLE-PHASE
NATURAL CIRCULATION WATER LOOP SYSTEMS

by

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Published March, 1956

Operated by The University of Chicago
under
Contract W-31-109-eng-38

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NOMENCLATURE

<u>Symbol</u>	<u>Definition</u>	<u>Units</u>
a	temperature coefficient = $1 - \beta \Delta T - b$	(lb. mass) sec ² /ft ⁵
a _ℓ	loop constant = $\int (\rho/g) dx/A$	
A	cross-sectional area	ft ²
b	mesh ratio = $V \Delta T / \bar{V}_n$	
C	constant	
C _p = c	specific heat of fluid	Btu/(lb. mass)(F)
D	diameter	ft
E	internal energy	Btu/(lb. mass)
ε	total energy = $E + (u^2/2\alpha g_c) + \int (g/g_c) dz$	Btu/(lb. mass)
f	Moody friction factor	
f'	Fanning friction factor	
f _τ	friction factor for transient flow (in this report: f _τ = f)	
F	frictional loss, total	ft(lb. force)/(lb. mass)
$\bar{F}$	frictional loss per unit length	$\frac{\text{ft(lb. force)}}{\text{ft(lb. mass)}}$
g	local acceleration of gravity	ft/sec ²
g _c	conversion factor in Newton's law of motion	$32.2 \left(\frac{\text{lb. mass}}{\text{lb. force}} \right) \left(\frac{\text{ft}}{\text{sec}^2} \right)$
G	mass rate of flow	(lb. mass)/(ft ²)(sec)
gpm	coolant rate of flow	gal/min
h	film heat transfer coefficient	Btu/(sec)(ft ²)(F)
h'	film heat transfer coefficient	Btu/(hr)(ft ²)(F)
H	enthalpy	Btu/(lb. mass)

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<u>Symbol</u>	<u>Definition</u>	<u>Units</u>
H_ℓ	loop hydraulic head = $-\int \rho dz$	(lb. mass)/ft ²
k	thermal conductivity	Btu/(sec)(ft ²)(F/ft)
K	proportionality constant = U/V	Btu/(ft ⁵)(F)
L	length	ft
m	integer, denoting time position	
n	integer, denoting space position	
p	pressure	(lb. force)/ft ²
q	linear rate of heat input to heater element	Btu/(ft)(sec)
q'	rate of heat input to heater element	Btu/sec
q'_f	net heat leaving heater with fluid = $V\rho c(T_0 - T_1)$	Btu/sec
Q_{kw}	rate of heat input to heater element	kilowatts
Q	linear rate of heat input to fluid	Btu/(ft)(sec)
Q'	rate of heat input to fluid	Btu/sec
r	pipe radius	ft
Re	Reynolds' number = $Du\rho/\mu$	
R_ℓ	total loop resistance = $\int \rho \overline{F} dx$	(lb. mass)/ft ²
T	fluid temperature	F
T_c	steady-state cold leg temperature	F
T_h	steady-state hot leg temperature	F
t	external temperature, wall or coolant	F
u	linear velocity	ft/sec
U	over-all heat transfer coefficient for heat exchanger	Btu/(sec)(ft ²)(F)

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<u>Symbol</u>	<u>Definition</u>	<u>Units</u>
v	specific volume = $1/\rho$	$\text{ft}^3/(\text{lb. mass})$
V	volumetric rate of flow	ft^3/sec
$\bar{V}_n$	volume of subdivision = $A_n \Delta x_n$	ft^3
$\bar{V}_c$	total cooler volume	ft^3
$\bar{V}_e$	heater element volume	ft^3
$V \Delta \tau / \bar{V}_n$	mesh ratio	
w	mass rate of flow	$(\text{lb. mass})/\text{sec}$
x	distance along streamline	ft
x_n	length of subdivision	ft
y	distance along streamline in pipe radii, x/r	ft
z	vertical projection	ft
z_n	vertical projection of subdivision	ft
α	coefficient in kinetic energy term = $1/2$ for viscous flow = 1 for turbulent flow	
β	half angle of diverging section	degrees
β_n	$4 U_n / \rho c D_n$	sec^{-1}
γ	angle of inclination of pipe centerline	degrees
δ	$D_0^2 - D_n^2 / 4 D_n \cong$ wall thickness	ft
$\delta_{n,m}$	error in calculated temperature	F
θ	converted time scale = $\tau u / r$	
θ_b	angle subtended by piping bend	degrees
λ	constant = $2h / \rho c u$	
μ	viscosity	$\text{lb}/(\text{ft}) (\text{sec})$

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<u>Symbol</u>	<u>Definition</u>	<u>Units</u>
μ_{cp}	viscosity	centipoise
ρ	density	lb/ft ³
ρc	volumetric heat capacity of fluid = ρc_p	Btu/(ft ³)(F)
$\rho_e c_e$	volumetric heat capacity of heater element	Btu/(ft ³)(F)
$\rho_w c_w$	volumetric heat capacity of walls	Btu/(ft ³)(F)
τ	time	sec
$\tau_{1/2}$	half-life coasting time	sec

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ABSTRACT

Studies were directed toward the development and experimental verification of a practical method for predicting the transient behavior of single-phase, natural circulation water loop systems.

Two experimental loops, along with a complete system of equations used in connection with an illustrative problem, are described in detail. Comparison of experimental results with numerical solutions indicates that the transient behavior of the systems could be satisfactorily predicted for engineering purposes. The reliability of the method is related directly to the accuracy with which the physical parameters of the loop system are known.

I. INTRODUCTION

A number of practical applications arising in boiler and nuclear reactor coolant system designs has demonstrated the desirability of establishing a method for the prediction of temperatures and rates of flow in the transient operation of natural circulation loops. To the authors' knowledge, this paper presents the first published comparisons of calculated and experimental transient behavior of a natural circulation loop involving liquid water as the circulating fluid. The principal objective of this study was to determine the reliability of the calculations based upon numerical solutions of finite-difference energy and flow equations. Solutions were obtained through the use of the Standard Eastern Automatic Computer (SEAC).

The complexities of the physical problems were appreciated at the outset, and thus it was desirable to start with a simplified mathematical model and introduce only those refinements that were indicated as being necessary. For the numerical transient evaluations, it was necessary to calculate and experimentally verify the loop parameters for steady-state flow. The influence of the refinement of the loop parameters on the transient calculations are presented, for example, as revisions for computer problem No. 3.

Of the extensive literature on transient flow, it is of interest to note that Jenny (8)*, among others, presented the derivation of the equation of motion for the unsteady turbulent flow of fluid in a horizontal tube. Accelerated flow may occur in a steady, non-uniform flow, such as at the entrance to a tube, from a reservoir wherein the boundary layer grows along the tube. Among those who have studied this problem is Langhaar (10). Grace (5) considered the oscillatory motion of a fluid in a pipe, while Valensi (21) was concerned with the oscillatory viscous flow in a U-tube. Schonfeld (20) considered isothermal transient viscous flow in pipes for the case where a finite pressure differential is applied to a fluid initially at rest. He found that the flow of fluid approached steady-state conditions asymptotically. He also studied forced oscillation in round tubes and natural oscillation in U-tubes. The mixing length theory of Prandtl and Von Karmen were used to study fully developed turbulent flow in round tubes and wide open canals. Daily and Deemer (3) reported that steady-state friction factors were applicable to the transient flow of fluid. They found that the total pressure drop across a given section could be calculated from the sum of the instantaneous frictional pressure drop and acceleration head. DeJuhasz (4) presented a graphical method for determining the response of a system to unbalances. Velocity and pressure may be determined as functions of time and position for isothermal systems subjected to known disturbances. Binnie (1) treated a somewhat similar problem but obtained an analytical solution. He was able to calculate the pressure surges which occur in long pipelines due to the closure of a valve. Allowance was made for the effects of friction assumed proportional to the velocity. For cases where the friction was not directly proportional to the velocity, the method gives an approximate value. Hall (6) considered pulsating flows and their effect on orifice and flow coefficients. McLain (17) studied the transient non-isothermal flow in a heated horizontal pipe. A numerical method of solution of the three basic equations was used in a qualitative investigation made on the influence of mode of subdivision of the time and space increments on the solutions. The calculated accelerations of fluids in an isothermal U-loop were experimentally verified by Leppert (13).

*Numbers in parentheses pertain to Bibliography references.

II. SUMMARY

Two experimental natural circulation loops were designed, erected, and instrumented to aid in the development of a general theory for the single-phase operation of natural circulation loop systems. The operation of the experimental system was satisfactory and reproducible. The experimental investigations and the comparisons with numerical solutions of the finite-difference equations have led to the following generalizations:

1. Loop Parameters. Unique correlations may be obtained for the steady-state resistance and heat transfer coefficients of natural circulation systems because of the interdependency of the physical parameters. The use of steady-state friction factors in transient flow is valid for natural circulation loop systems since the flow changes are not likely to be subject to large impulsive forces.

2. Mathematical Model. A method for predicting the transient behavior of a single-phase, natural circulation loop was demonstrated. The method employs the iterative solution of the finite-difference energy and momentum balances and has the following points of interest:

(1) The one-dimensional analysis appears to be sufficient for engineering purposes.

(2) The assumption of uniform average volumetric flow around the loop at a given instant is sufficiently valid for most conditions to be expected.

(3) The finite-difference equations may be simplified considerably by the use of an average fluid density in the flow equation except for the hydraulic head term.

(4) The degree of reliability of the mathematical model will depend upon the accuracy with which the physical parameters are known or can be estimated. For example, two main factors are loop resistance and heat transferred to the fluid.

(5) Relatively large subdivisions may be employed in the finite-difference equations if the proper mesh ratio is chosen.

3. System Response The flow equation indicates that the response of a circulation loop system may be increased by increasing the diameter at any point in the loop, thereby decreasing the frictional resistance of the loop, and also decreasing the inertia lag. However, a decrease in the diameter of the hot leg will more rapidly propagate the influence of changes in the heater section, and to that extent may increase the response of the system.

III. EXPERIMENTAL EQUIPMENT AND PROCEDURE

A. Apparatus

Two experimental loops were designed, erected, and instrumented. The first convection loop is shown schematically in Fig. 1. To obtain faster accelerations, the top horizontal section of the first loop was replaced by a much shorter, larger diameter, section (Fig. 2), and this arrangement with its new method of measuring flow constituted the second circulation loop.

1. Flow Measurements

The flow meter for the first experimental loop was a differential pressure transmitter and recorder which measured the pressure drop across the horizontal pressure drop test section. This test section consisted of 3/4-inch brass tubing (0.681 in. average ID) with pressure taps placed at 10-foot intervals. Less than a 5-second transmission lag time was noted for full-scale deflection at the recorder, and the uncertainty of the time measurements for the recorder charts was 2 seconds. Though this method of evaluating the transient flow from the pressure drop readings was cumbersome (see Appendix A for details of calculations), it was found to be satisfactory for the relatively slow transient in the first circulation loop.

An electromagnetic flow meter (Magnaflow Type #HB.HC-60, manufactured by the Mittleman Electronics Corporation of Chicago) was used to measure the flow in the second loop. The complete Magnaflow unit consisted of the sensing head, amplifier, and recorder. The design of this instrument is based on Faraday's law of electromagnetic conduction which states that a voltage will be induced in any conducting medium when moving through a magnetic field. Some of the advantages of such a flow meter are that it presents no additional head losses in metering the flow, is not affected by the viscosity, density and temperature of the fluid flowing, and gives instantaneous volumetric rates of flow since the measuring system possesses no inertia. Because the induced voltage is dependent only on the velocity distribution in the meter section, the instrument gives true volumetric rates of flow for symmetrical velocity profiles. The sensing head was placed in the bottom horizontal return section.

2. Heater Section

The heater section consisted of four 7.5-kw Chromalox immersion heaters (Type M-375), one inserted in each end of two 35-1/2-inch lengths of standard 2-1/2-inch galvanized pipe having reducing tees on each end. The two lengths were coupled in the form of a V by a 45-degree, 1-1/4-inch elbow close-coupled to the tees (Fig. 3). The

FIG. 1
MINNESOTA NATURAL CIRCULATION LOOP

FIG. 2
TOP SECTION OF SECOND CIRCULATION LOOP

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Fig. 3
HEATER SECTION

immersion heaters were connected through switches to a single-phase, 215-volt power supply, which resulted in a maximum output of slightly more than 6 kw per heater or a total heat load of about 84,000 Btu/hr. The top three heaters were connected to four-way switches so that power input to each of these heaters could be varied independently in increments of one-third of full power. The total power to these three heaters was recorded on an Esterline Angus recording wattmeter (Model AW) connected through transformers to indicate 1/20 kw consumption. The bottom heater was connected to a powerstat so that the input to the heater could be varied continuously from 0 to 120% of the rated input. A second recording wattmeter was used to record the heater load.

3. Temperature Measurement

Bare platinum/platinum-rhodium thermocouples (24 gage) were directly exposed to the fluid to obtain maximum response. The fluid temperatures T_1 to T_4 were determined at locations noted in Fig. 1 and were recorded on a Leeds & Northrup Speedomax recorder.

4. Heat Exchangers (Coolers)

Concentric tube heat exchangers were used to cool the flow. The outer shell was 1-1/2-inch brass tubing and the internal tube was 3/4-inch brass tubing for the first loop and 1-inch tubing for the second loop. Tap water was employed as the coolant and most runs were made with a constant setting for the coolant flow.

5. Vertical Riser and Downcomer, and Bottom Horizontal Return Section

The vertical riser and downcomer sections were made of 1-inch (OD) hard brass tubing (0.872 in. average ID), giving an over-all height of the loop of about 24 feet. As shown in Fig. 1, the system of valving in the bottom horizontal return section permitted runs also to be

made with forced circulation using the pump and rotameters. Compressed air was introduced to a small surge tank (4 in. dia. brass pipe, 11-1/2 in. long) to regulate the pressures over a range of about 40 to 80 psi. A minimum liquid level was maintained in the surge tank to reduce pressure fluctuations during the transient operations.

B. Loop Parameters

Steady-state flow runs were made to determine the physical parameters of the experimental loops. These parameters include the friction factors, over-all heat transfer coefficients for the coolers, fluid film heat transfer coefficients, total loop resistance, and heater lag. It was possible to demonstrate that simplified relations could be used for some of the parameters. These simplifications, arising from the interdependency of the factors affecting loop performance, were more readily adaptable to the numerical analyses than were the relations based upon conventional heat transfer and fluid flow correlations.

1. Friction Factors

The friction factor data permitted the pressure-drop test section to be used as a flow meter for the transient runs in the first circulation loop. A plot of Moody Friction Factor versus Reynolds' Number is given in Fig. 4.

2. Over-all Heat Transfer Coefficient for Heat Exchangers

The over-all heat transfer coefficient for the cooler (Fig. 5) was effectively correlated by the simple equation

$$U = 26.0 (\text{gpm})^{0.13} V$$

where

U = over-all heat transfer coefficient, Btu/(sec)(ft²)(F)

gpm = coolant rate of flow in gpm

V = volumetric rate of flow in loop, ft³/sec.

Fluid temperatures are not explicitly given in this equation but will have a definite effect on the value of U , since they are implicitly related to the volumetric rate of flow. The volumetric rate of flow term therefore accounts for the influence of both fluid temperature and velocity on the fluid film coefficient and its relation to the over-all coefficient. The coolant flow term accounts for the influence of the coolant velocity on the external film coefficient and its effect on the over-all coefficient. The form of the equation is particularly desirable for the finite-difference equation used to solve the transient flow problems. A more rigorous analytical approach did not lead to a better correlation and, as noted in the next section,

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FIG. 4
FRICTION FACTORS

FIG. 5
OVER-ALL HEAT TRANSFER COEFFICIENT
PARAMETERS FOR HEAT EXCHANGER

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the individual fluid film coefficient was determined from the over-all correlation. It was later recognized that the coefficient should be corrected for the number of subdivisions of the cooler. This correction is given in Appendix B.

A value of $U = 16.1V$ was found satisfactory for the second loop for a constant coolant flow of 10 gpm.

3. Fluid Film Heat Transfer Coefficient

The interchange of heat between the fluid and the confining piping needs to be taken into account if the heat capacity of the metal walls is relatively large or if there are significant heat losses from the loop system. Sufficient insulation was provided for the experimental loop; therefore only the raising and lowering of the pipe wall temperature needed to be taken into account. For water over a range of 40 to 220F, McAdams (16) presents an empirical correlation for the film heat transfer coefficient; e.g., for a 3/4-in. tubing, the coefficient is given by

$$h = 6.41 \left(1 + \frac{0.011 T}{V^{0.2}} \right) V,$$

where T is in degrees F and V in ft^3/sec . This equation would be too inconvenient to handle in the numerical solution, so an average value of $(1 + 0.011 T/V^{0.2})$ was estimated from experimental runs to give $h = 71.1V$. However, from tests made on the cooler in the first circulation loop, a value for $h = 43V$ was indicated and this correlation was used in the initial calculation. In the later calculations it was found desirable to increase the coefficient to an intermediate value.

4. Total Loop Resistance

Steady-state determinations of total loop resistance are given in Fig. 6(a) for the first circulation loop. For these runs the total loop resistances may be expressed as

$$R_\ell = 10^6 V^{1.65} \text{ lb/ft}^2.$$

For a coolant flow of 10 gpm, the experimental data may be correlated very accurately by

$$R_\ell = 690,000 V^{1.58}$$

and this empirical correlation was used in the later calculations.

The evaluation of the loop resistances by calculation of the individual resistances for each section of the loop compared very favorably with the empirical determinations; however, the form of the empirical correlation was much more convenient for the finite-difference calculations.

As in the case of the over-all heat transfer coefficient, temperature effects are accounted for in the volumetric rate of flow. The coolant flow affects the total loop resistance because of changes in the operating temperature level. We may generalize to say that the lower the coolant flow, the lower will be the total pressure drop for a given circulating flow.

The equations presented are valid for $V \geq 0.00072 \text{ ft}^3/\text{sec}$. There is a sharp drop in the loop resistance for lower rates of flow in the first circulation loop corresponding to a change from turbulent flow to viscous flow in the 3/4-inch section. The determination of a simple empirical correlation of the above form for viscous flow throughout the loop was not possible; however, the total loop resistance as calculated from the dimensions of the system may be simplified for viscous flow to give:

$$R_\ell = 7410 \mu_{av} V + 580,000 V^2,$$

which is applicable for all coolant flows and for $V < 0.00072 \text{ ft}^3/\text{sec}$.

Figure 6(b) shows the correlation of R_ℓ versus V for the second circulation loop. The relation used in the finite-difference equations was

$$R_\ell = 332,000 V^{1.64}.$$

The head losses as calculated from the direct evaluation of all of the individual resistances proved to be quite accurate for the runs checked on the second loop. For viscous flow, the temperature and density differences were so small that the uncertainties in these differences ruled out the experimental determination of the hydraulic head. Consequently, the total loop resistance was calculated using the sizes and equivalent lengths of the piping. The simplified equation used for the calculated viscous loop resistance in the second circulation loop was

$$R_\ell = 2500 \mu V,$$

where μ represents the average viscosity (in centipoises) of the fluid in the loop. The transition from viscous to turbulent flow was then dependent on the temperature level of the loop. The critical value of $Re = 2,000$ was used to determine the volumetric flow at this transition point. The transition from viscous to turbulent flow occurred at a much higher rate of flow in the second loop than in the first loop because of the increased diameter of the top section.

5. Heater Lag

All the transient runs reported involved a sudden increase in the heat input to the loop. It was recognized that a step change in heater switch settings would not introduce instantaneously the corresponding power change to the fluid. Consequently, it was deemed desirable to introduce

FIG. 6
LOOP RESISTANCE

heater lag equations of various degrees of complexity into the finite-difference method calculations. The initial calculations made for all runs on the first circulation loop involved no corrections for the heater lag and, as illustrated in Table VII, page 40, heater lag provisions along with other refinements in loop parameters were introduced into the succeeding revisions.

The determination of the heater lag time was approached from both a theoretical and experimental point of view. If we consider an isolated heater section, neglect the heat capacity of the walls, and use the simplifying assumption that the average temperature of the heater element and the average temperature of the fluid change at equal rates, we obtain the following expression for the heat flow to fluid:

$$Q' = q' (1 - 0.189 e^{-18.3 V \tau}),$$

where

Q' = rate of heat input to fluid, Btu/sec

q' = rate of heat input to heater element, Btu/sec

V = volumetric flow rate of fluid through heater section,
ft³/sec

τ = time in seconds from step change in heat input to
heater element.

The constants were evaluated from the volumetric heat capacities of heater element and fluid and the heater element volume ($\bar{V}$). This simple equation was checked experimentally for a single heater and, as shown in Fig. 7, the agreement is good except for minor departures near zero time. The

FIG. 7
HEATER LAG FOR TOP HEATER

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experimental runs were made under conditions of forced circulation with flow rate and inlet temperature of the fluid kept constant, and the outlet temperature was recorded versus time following a step change in power input to the heater element. The analytical solution was discontinued at values of $\tau = \bar{V}/V$, which are beyond the range of application of the simplified heater lag equation.

The preceding oversimplified analysis is not satisfactory at zero time and is limited to the use of only one subdivision per heater element. Since, however, the forced convection runs induced considerable mixing, and perhaps secondary natural circulation, the use of one subdivision per heater is to be preferred. Further, it was found that a rigorous approach of integrating the heat balance over a differential element for the entire heater was not as satisfactory as the heat balance in which it is assumed that the outlet fluid temperature was equal to the average value in the heater section. An improved heater lag equation developed for one subdivision per heater element taking into account the heat transfer coefficient for the fluid is

$$Q' = q' (1 - 0.173 e^{-18.3 V \tau} - 0.827 e^{-197 V \tau}).$$

The evaluation of the constants, which depend upon the physical parameters of the heater sections, heat transfer coefficients and properties of the fluid, was simplified through the use of the previous heater lag equation.

Though the improved heater lag equation was satisfactory for use with the top heater, the application to all four heaters required additional refinements. It was determined through comparison of experiments and calculations that adequate corrections could be achieved simply by taking into account the heat transfer from the fluid to the walls of the heater sections. It is to be noted that all refinements introduced for the heater lag were made independently of the transient operation of the natural circulation loops and were based only on the performance of steady-state flow runs.

IV. TRANSIENT ANALYSES

A. Unsteady-State Equations

This analysis is restricted to the one-dimensional, non-steady, rectilinear flow of real fluids. The assumption of one-dimensional flow is strictly applicable to cases where the velocity of the fluid is uniform over the entire cross section of flow. The analysis is therefore valid only for an ideal case, but in practice an average velocity may be used for a variety of velocity profiles and the problem considered ideal. We shall further assume that the fluid is continuous so that no sources or sinks exist and that no heat is generated or lost within the body, such as by chemical reaction. The effects due to the conduction of heat along the longitudinal axis are assumed to be negligible.

1. Continuity Equation

The continuity equation may be written as

$$\frac{\partial \rho}{\partial \tau} + \frac{1}{A} \frac{\partial (\rho u A)}{\partial x} = 0 \quad , \quad (1)$$

where the fluid density is designated by ρ , the fluid velocity by u , cross-sectional area for flow by A , time by τ , and distance along the flow path by x .

2. Energy Equation

Consider a system consisting of the volume element of fluid as illustrated. Assume that there is no shaft work. The total rate of energy input to the system is

$$w(\epsilon + pv)|_x + Q\Delta x \quad ,$$

where ϵ is defined by the relation

$$\epsilon = E + \frac{u^2}{2\alpha g_c} + \frac{g}{g_c} z$$

(Consistent units are to be used throughout so the mechanical equivalent of heat factor is omitted.) Note that the heat input, Q , is used on a linear basis, i.e., heat used per foot per second. The total rate of energy output is

$$w(\epsilon + pv)|_{x+\Delta x}$$

and the rate of energy accumulation is

$$A_{av} \Delta x \cdot \frac{\partial(\rho \epsilon)}{\partial \tau} .$$

The limit of the energy balance as Δx approaches zero is

$$- \frac{\partial [w(\epsilon + pv)]}{\partial x} + Q = A \frac{\partial(\rho \epsilon)}{\partial \tau} . \quad (2)$$

Introduction of the continuity equation leads to

$$u \frac{\partial}{\partial x} \left(E + \frac{u^2}{2\alpha g_c} + \frac{g}{g_c} z \right) + \frac{\partial}{\partial \tau} \left(E + \frac{u^2}{2\alpha g_c} + \frac{g}{g_c} z \right) + \frac{1}{\rho A} \frac{\partial}{\partial x} (puA) = \frac{Q}{\rho A} , \quad (3)$$

which may be written as

$$\frac{D\epsilon}{D\tau} + \frac{1}{\rho A} \frac{\partial(puA)}{\partial x} = \frac{Q}{\rho A} . \quad (4)$$

An alternate form is

$$\frac{D}{D\tau} \left(H + \frac{u^2}{2\alpha g_c} + \frac{g}{g_c} z \right) - \frac{1}{\rho} \frac{\partial p}{\partial \tau} = \frac{Q}{\rho A} . \quad (5)$$

3. Equation of Motion

Consider the volume element of fluid as illustrated. Since considerations are restricted to a continuous medium, any internal forces must be counterbalanced by equal and opposite forces so that in the isolation of the volume element only external forces will remain. Pipe friction will therefore exert its force on the element and has been represented simply by $\bar{F}$. The units of this term are pound force per pound mass and may be visualized as shearing stress times area for unit mass of fluid flowing. At this point no assumptions need be made concerning the functional relationship between $\bar{F}$ and any other variable. The forces acting on

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the isolated volume element of the fluid at a given instant are also illustrated. According to Newton's second law of motion, mass times acceleration equals net forces in direction of motion. The following terms are evaluated:

$$\text{mass} \times \text{acceleration} = \frac{\rho_{av} A_{av} \Delta x}{g_c} \frac{Du}{D\tau}$$

$$\text{net end forces} = (pA)_x - (pA)_{x+\Delta x}$$

$$\text{component of potential force} = -\rho_{av} A_{av} \Delta x \frac{g}{g_c} \sin \gamma = -\rho_{av} A_{av} \Delta x \frac{g}{g_c} \frac{\Delta z}{\Delta x}$$

$$\text{component of wall pressure forces} = p_{av} \pi D_{av} (\sin \beta) \Delta x = p_{av} \frac{\Delta A}{\Delta x} \Delta x \cos \beta$$

$$\text{component of frictional force} = -\bar{F}_{av} \rho_{av} A_{av} \cos \beta$$

The direct substitution of these equations in Newton's second law of motion, after division through by $\rho_{av} A_{av} \Delta x$, with the limit taken as $\Delta x \rightarrow 0$, yields

$$\frac{1}{g_c} \frac{Du}{D\tau} = -\frac{1}{\rho A} \frac{\partial(pA)}{\partial x} + \frac{p}{\rho A} \frac{\partial A}{\partial x} \cos \beta - \frac{g}{g_c} \frac{\partial z}{\partial x} - \bar{F} \cos \beta \pi \quad (6)$$

For flow in a uniform pipe, we have

$$\frac{1}{g_c} \left(\frac{\partial u}{\partial \tau} + u \frac{\partial u}{\partial x} \right) + \frac{1}{\rho} \frac{\partial p}{\partial x} + \bar{F} + \frac{g}{g_c} \frac{\partial z}{\partial x} = 0 \quad (7)$$

$\bar{F}$ has not yet been given an explicit definition. The general equation of motion must, however, reduce to the simple mechanical energy balance for steady-state conditions. We shall therefore assume that

$$\bar{F} = \frac{f_T u^2}{2g_c D} \quad (8)$$

in order that the equation of motion will reduce to a steady-state counterpart. The resultant equation of motion is

$$\frac{1}{g_c} \left(\frac{\partial u}{\partial \tau} + u \frac{\partial u}{\partial x} \right) + \frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{g}{g_c} \frac{\partial z}{\partial x} + \frac{f_T u^2}{2g_c D} = 0 \quad (9)$$

It is not necessarily true that f_T will correspond with the Moody friction factor, since we have only assumed the form of the frictional losses, which may in itself be in error. Recent experiments (3), however, have shown

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the validity of Eq. (9), at least for isothermal flow. These results indicate that f_τ corresponds to the Moody friction factor of steady state using transient velocities from 15 to 75 fps and accelerations from 0 to 35 ft²/sec. It was reported that the total observed pressure drops could be calculated from steady-state velocity losses and acceleration heads except during periods of initial impulse.

B. Isothermal Transients in Loop Systems - Coasting

The analysis of isothermal transients in a convective loop for coasting will illustrate the important role that reduced sections display in determining the transient response of the entire system. A convenient yardstick for comparing coasting response is the half-life defined as the time required for a system to decelerate from some initial volumetric rate of flow $V_0 = V_i$ to one-half this rate of flow, $V = V_i/2$. Consider the isothermal flow of an incompressible fluid in the horizontal compound loop illustrated below. The flow may be developed by a pump in the by-pass line. The by-pass line is then opened while the valves in the pump circuit are closed. We may examine the flow at any arbitrary time after the initial pump disturbances have been damped out. Kinetic energy of the fluid will be dissipated by the frictional resistance. The equation of motion for this case may be written as

$$-\frac{1}{\rho} \frac{\partial p}{\partial x} = \frac{1}{g_c} \frac{\partial u}{\partial \tau} + \frac{u}{g_c} \frac{\partial u}{\partial x} + \bar{F} \quad (10)$$

where $\bar{F}$ is defined by the form of the equation and includes pipe friction, bend losses, contraction losses, etc. The integration of Eq. (10) around the loop simplifies to

$$0 = \frac{1}{g_c} \left[L_2 + L_1 \left(\frac{D_2}{D_1} \right)^2 + L_3 \left(\frac{D_2}{D_1} \right) + L_4 \left(\frac{D_2}{D_1} \right) \right] \frac{du_2}{d\tau} + \int \bar{F} dx \quad (11)$$

if we make use of the fact that the total pressure drop around the loop is zero and the sum of all terms involving $\frac{\partial u}{\partial x}$ can be shown to be equal to 0. ($\frac{\partial u}{\partial x}$ is 0 for the constant area sections, and the converging and diverging sections balance each other if the sections are conical and symmetrical about the axis.) The limit of this expression as L_3 and L_4 approach zero will give the rate of deceleration for a compound loop having abrupt contractions and expansions. For convenience we shall assume that the

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lengths of mixing piping are long enough so that the contraction and expansion losses may be neglected. This will allow us to write Eq. (11) in a more symmetric form so that the responses may be determined without specifying the physical dimensions of the system. With these assumptions and using steady-state friction loss correlations, we obtain the following expression for the half-life:

$$\tau_{1/2} = \left[\frac{1 + \frac{L_1}{L_2} \left(\frac{D_2}{D_1} \right)^2}{1 + \frac{f_1}{f_2} \frac{L_1}{L_2} \left(\frac{D_2}{D_1} \right)^5} \right] 2 D_2 A_2 \int_{\frac{V_0}{2}}^{V_0} \frac{dV}{f_2 V^2} \quad (12)$$

1. Turbulent Coasting

Table I lists several cases of coasting in the turbulent range where the friction factor has been taken as

$$f = 0.18/(\text{Re})^{0.2} = 0.18 \left(\frac{\pi D \mu}{4 V \rho} \right)^{0.2}$$

The values show for turbulent flow that higher initial velocities cause more rapid decelerations, whereas enlarged sections increase the half-life.

Table I

HALF-LIVES FOR ISOTHERMAL TURBULENT COASTING

Case	Condition	Half-Life, $\tau_{1/2}$
1	Uniform loop of dia. D_0 with an initial volumetric rate of flow of $V_0 = V_i$. ($L_1 = 0$, $D_2 = 0$)	C_1
2	Uniform loop of dia. D_0 with an initial volumetric flow rate of $V_0 = V_i/2$.	$1.74 C_1$
3	Uniform loop of dia. $2D_0$ with an initial volumetric rate of flow $V_0 = 2V_i$. (Same Re No. as in Case 1, but with only half the linear velocity. Linear velocity same as in Case 2.)	$4 C_1$
4	Compound loop with $L_1 = L_2$, $D_1 = 2D_0$, $D_2 = D_0$, and initial volumetric flow rate of $V_0 = V_i$.	$1.21 C_1$
5	Compound loop with $L_1 = 8L_2$, $D_1 = 2D_0$, $D_2 = D_0$, and initial volumetric flow rate of $V_0 = V_i$.	$2.33 C_1$

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2. Viscous Coasting

For isothermal viscous coasting using

$$f = 64/\text{Re} = 16 \frac{\pi D \mu}{V_0}$$

we obtain the following expression:

$$\tau_{1/2} = \left[\frac{1 + \frac{L_1}{L_2} \left(\frac{D_2}{D_1} \right)^2}{1 + \frac{L_1}{L_2} \left(\frac{D_2}{D_1} \right)^4} \right] \frac{\rho D_2^2}{32 \mu} \ln 2 \quad (13)$$

Several cases are listed in Table II. It will be noted that reduced sections display the same marked tendency to control the coasting times. The main difference between the viscous half-life values and those of turbulent flow is that in the viscous regime the half-life is constant for a given system and is independent of velocity.

Table II

HALF-LIVES FOR ISOTHERMAL VISCOUS COASTING

<u>Case</u>	<u>Condition</u>	<u>Half-Life, $\tau_{1/2}$</u>
1	Uniform loop of dia. D_0 with an initial volumetric rate of flow of $V_0 = V_i$. ($L_1 = 0$, $D_2 = D_0$)	C_2
2	Uniform loop of dia. $2D_0$ with an initial volumetric rate of flow $V_0 = V_i/2$.	C_2
3	Uniform loop of dia. $2D_0$ with an initial rate of flow of $V_0 = 2 V_i$.	$4 C_2$
4	Compound loop with $L_1 = L_2$, $D_1 = 2 D_0$, $D_2 = D_0$, an initial volumetric rate of flow of $V_0 = V_i$.	$1.18 C_2$
5	Compound loop with $L_1 = 8 L_2$, $D_1 = 2 D_0$, $D_2 = D_0$, and an initial volumetric rate of flow of $V_0 = V_i$.	$2 C_2$

The kinetic response of a loop system will be directly related to its half-life coasting time. A high half-life value will correspond to a low resistance to flow, and such systems will possess the ability to accelerate rapidly and to counteract the effects of any applied unbalance.

C. Non-Isothermal Transients in Loop Systems

The method of approach in applying the unsteady-state partial differential equations to loop systems is to reduce the equations of motion and the energy balance to simplified finite-difference equations.

1. Equation of Motion

The equation of motion, Eq. (9), is applied to the schematic loop illustrated below. For this application, $\bar{F}$ represents the frictional losses which are given by the ordinary steady-state correlations. For the transient case the thermal loads of the two heat exchangers will undoubtedly not be equal and will fluctuate with time. The non-isothermal operation of a convective loop would require the addition of a surge tank to the system as shown (dotted lines). This could be located at any convenient point in the loop. The liquid level of the surge tank will vary with time in the transient operations due to the expansion and contraction of the fluid in the loop. Thus the total mass of the entire system will be invariant but the mass in the loop will change. A surge tank may be readily designed with a sufficient void volume to insure against severe pressure fluctuations. We shall therefore assume that the rate of change of standard pressure in the surge tank is negligibly small, equivalent to saying that the static pressure in the surge tank is essentially constant. The systems considered are restricted to all-liquid systems. We shall further presuppose that the force required to accelerate the small flow to the surge tank is insignificant when compared with the force required to accelerate a large mass flowing in the main loop.

The final assumption will be that the volumetric rate of flow is uniform around the loop at any instant (independent of position). It is recognized that the latter assumption, in general, will not be true, since even for steady-state conditions the volumetric rate of flow must be greater in the hot riser leg than in the cold return leg. (For steady-state conditions the weight rate of flow will be constant around the entire loop.) The order of magnitude of this error or approximation can be calculated for specific examples. Thus, if the hot leg temperature is 300F and the cold leg temperature 100F, the density of water at these

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temperatures is, respectively, 57.31 and 62.00 lb/ft². The weight rate of flow (w lb/sec) is constant so the volumetric rate of flow for the hot side is 0.01745 w, for the cold side 0.01613 w, and an average value would be 0.01679 w. The difference (based on the average) between either volumetric rate and the average rate would be less than 4%. The 200-degree difference in temperature levels would be quite severe for a practical case. From this we may see that an average volumetric rate of flow is quite uniform around the loop. However, although the volumetric rate of flow may actually vary around the loop, the time rate of change will vary but little around the circuit corresponding to a small accumulation or depletion of liquid in the surge tank.

Another basis for the flow equation would be a constant weight rate of flow around the loop which definitely satisfies steady-state conditions. The use of weight rate of flow along with the specification of a uniform density around the loop at any instant is equivalent to the use of a uniform volumetric rate of flow. If no simplification is introduced for the density, then this term would appear as a coefficient of the temperatures in the heat balance. This would be undesirable, since this coefficient determines the stability of the finite-difference equations. In either case, experimental justification would be necessary to validate all assumptions made.

With the assumptions that $\frac{\partial V}{\partial x} = 0$ and $\frac{\partial V}{\partial \tau}$ is independent of x, the equation of motion can be simplified to

$$\frac{dV}{d\tau} = \frac{-\int \rho (g/g_c) dz - \int \rho \bar{F} dx}{\int \frac{\rho}{g_c} \frac{dx}{A}} \quad (14)$$

Equation (14) may be symbolized as follows

$$\frac{dV}{d\tau} = \frac{H_\ell - R_\ell}{a_\ell} \quad (15)$$

This equation expresses the volumetric acceleration in terms of the net driving force, which is the difference between the loop hydraulic head and total loop resistance and the loop geometry. The total loop resistance to flow, R_ℓ , is identical with the steady-state values for similar temperatures and rates of flow. H_ℓ corresponds to the loop hydraulic head, and a_ℓ corresponds to the loop constant. In order to solve this equation simultaneously with the energy equation, it will be necessary to replace the derivative by its finite-difference approximation. This gives

$$V' = V + \Delta\tau \left[\frac{H'_\ell - R_\ell}{a_\ell} \right] \quad , \quad (16)$$

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where the prime denotes a forward extrapolation in time. The time interval should be small enough so that H_ℓ , R_ℓ , and a_ℓ could be evaluated at either end of the time interval without appreciably affecting the solution. The energy equation, as noted below, permits the evaluation of H'_ℓ .

2. Energy Equation

It can be readily ascertained that a suitable form of the energy equation is simply a heat balance:

$$\frac{DH}{D\tau} = C_p \left(\frac{\partial T}{\partial \tau} + u \frac{\partial T}{\partial x} \right) = \frac{Q}{\rho A} \quad (17)$$

Heat conduction along the axis of flow and the mechanical energy terms can be neglected. Replacing the partial derivatives by their approximate finite differences we obtain

$$\frac{T'_{x+\Delta x} - T_{x+\Delta x}}{\Delta \tau} + \frac{V}{A \Delta x} (T_{x+\Delta x} - T_x) = \frac{Q}{\rho C_p A} \quad (18)$$

or

$$T'_{x+\Delta x} = \left(1 - \frac{V \Delta \tau}{A \Delta x} \right) T_{x+\Delta x} + \frac{V \Delta \tau}{A \Delta x} T_x + \frac{Q \Delta \tau}{\rho C_p A} \quad (19)$$

This equation determines the change of temperature with time over a volume element of fluid subject to give conditions of rate of flow and rate of feed input.

3. Loop Equations

The analysis would begin by subdividing the loop into a number, say b , of finite elements. Equation (19) would then be applied to each subdivision with the appropriate values of Q , ρ , A , and Δx to give b equations of the form

$$T'_n = \left(1 - \frac{V \Delta \tau}{\bar{V}_n} \right) T_n + \frac{V \Delta \tau}{\bar{V}_n} T_{n-1} + \frac{Q_n \Delta \tau}{\rho_n C_p A_n} \quad (20)$$

where $\bar{V}_n = A_n \Delta x_n$, the volume of the subdivision. For a particular value of the time increment, the right side of this equation is known from initial conditions and method of subdivision. The temperature distribution and therefore the hydraulic head, H_ℓ , may then be evaluated at the new instant. With the aid of Eq. (16) the rate of flow may also be determined at the new instant. Repeated applications of these two equations will produce a record of temperature and rates of flow.

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The heat added from the surroundings, Q , will in general be composed of both direct heat addition, such as by an immersion heater, and heat gained through the walls.

For example:

$$Q = q + U \pi D(t-T) \quad ,$$

where q = direct input (immersion heater, etc.); U = local over-all or film heat transfer coefficient; and t = local external temperature (wall temperature). Equation (20) may be generalized as

$$T_{n,m+1} = \left(1 - \beta_n \Delta\tau - \frac{V\Delta\tau}{V_n}\right) T_{n,m} + \frac{V\Delta\tau}{V_n} T_{n-1,m} + \beta_n \Delta\tau t_{n,m} + \frac{q_n \Delta\tau}{\rho_n C_p A_n} \quad , \quad (21)$$

where

$$\beta_n = \frac{U_n \pi D_n}{\rho_n C_p A_n} \quad \text{and} \quad \tau = m \Delta\tau \quad \text{for } m \text{ an integer} \quad .$$

A simple heat balance (no longitudinal conduction, no external heat losses, and no radial temperature distributions) for the confining walls of the subdivision yields

$$\frac{dt_n}{d\tau} = \frac{h_n}{\frac{(D_0^2 - D_n^2)}{4 D_n} \rho_w c_w} (t_n - T_n) \quad . \quad (22)$$

In finite difference form, Eq. (22) is written as

$$t_n^1 = \left(1 - \frac{h_n \Delta\tau}{\delta \rho_w c_w}\right) t_n + \frac{h_n \Delta\tau}{\delta \rho_w c_w} T_n \quad , \quad (23)$$

where

$$\delta = \frac{D_0^2 - D_n^2}{4 D_n} \quad \text{approximates the wall thickness, in feet.}$$

Automatic digital computing machines are particularly suited for the solutions of such problems, and an illustration of the computer set-up is given in a later section for the second circulation loop. Table III lists the number of subdivisions for the computer problems on the first loop.

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Table III

NUMBER OF SUBDIVISIONS FOR COMPUTER PROBLEMS ON FIRST LOOP

Loop Component		Computer Problem							
		1	2	3	1st revision of No. 3	2nd revision of No. 3	3rd revision of No. 3	1st revision of No. 2	4th revision of No. 3
1-in. Riser Leg	{fluid	15	15	15	15	15	15	15	8
	{wall	15	15	15	15	15	15	15	8
Top Horizontal Section	{fluid	13	13	13	13	13	13	13	8
	{wall	13	13	13	13	13	13	13	7
Heat Exchanger	fluid	6	6	6	6	6	6	6	4
1-in. Return Leg	{fluid	21	21	21	21	21	21	21	11
	{wall	21	21	21	21	21	21	21	11
Heater	{fluid	8	8	8	8	4	4	4	4
	{wall	0	0	0	0	0	4	4	4

4. Stability and Convergence of Finite Difference Equations

In the cases where it is impossible to obtain an exact solution of a partial differential equation, an approximate solution may be obtained from a corresponding finite-difference equation. The merit of such a numerical solution will then depend upon the convergence of the finite-difference solution to the exact solution. The magnitude of the increments used not only affects the convergence of the numerical solution, but also affects the stability of the solution. Errors may be compounded in the computation and result in instability. The general problem of stability and convergence of numerical solutions has been discussed in detail in the literature (2, 7, 11, 12, 14, 15, 18, 19, among others), primarily for the wave equation and heat conduction equation. Though the flow equation (Eq. 16) would be difficult to analyze formally for convergence and stability, a relatively simple consideration for suppressing the propagation forward of an error in time and distance for the heat balance leads to the following criteria for stability:

$$\text{For } b = \frac{V\Delta T}{V_n} = \text{mesh ratio}$$

$$a = 1 - \beta_n \Delta T - b$$

$$|a| \leq 1, |b| \leq 1, |a + b| \leq 1$$

The convergence of the solution of the finite-difference heat balance for a related problem on non-isothermal, steady flow through a heat exchanger is considered in Appendix C. For the transient loop systems, we have assumed, in general, that for a stable mesh ratio, the smaller the finite increment the better the convergence of the numerical solution to the exact solution. For a practical problem the proper size of space subdivisions and time increments will depend on the approach of the physical system to the ideal one-dimensional system represented by the mathematical model. If the flow in the convective loop resembles ideal slug flow, then the mathematical model should predict the transients and small subdivisions would be desirable. If, on the other hand, the flow is considerably different from the ideal case, due possibly to secondary natural circulation or peculiar velocity profiles in the non-isothermal flow, then it may not be possible to predetermine how well the mathematical model will simulate the physical loop. Since these non-idealities represent deviations from the abstract mathematical model, it is not inconceivable that proper increments of space and time might be found so that the numerical solution would be in agreement with experimental results. It will be noted that a choice of the mesh ratio $\frac{V\Delta T}{V_n} = 1$ permits a step input of temperature to be transmitted as a step. Further work is required to develop criteria to ascertain the appropriate magnitude of the finite increments for a practical loop system.

V. ILLUSTRATION OF FINITE-DIFFERENCE EQUATIONS FOR COMPUTER EVALUATIONS

The method of loop subdivision and the evaluations of the constants in the finite-difference equations are illustrated for computer problem No. 4. This problem concerns the response of the second experimental loop to an abrupt input of heat starting with the fluid at uniform temperature and at rest. The operating conditions are summarized in Table IV. The method of subdivision is illustrated in Fig. 8. In the subdivision of the top section of the second loop, the cooler was considered as one unit 154.5 in. long with 9 subdivisions, and the rest of the top section was allotted to the hot and cold legs. Equivalent lengths of 1-in. tubing were used for the small irregular sections in the lower portion of the loop, including the 1-in. pipe fittings and the 3/4-in. sensing element for the Magnaflo. Table V summarizes the loop parameters used in the calculations.

Temperatures of the fluid and wall for each subdivision are evaluated through the use of Eqs. (21) and (23), respectively. A summary of the actual equations employed is given in Table VI, including the numerical evaluation of the coefficients based upon the values given in Fig. 8 and Table V. Equation (16) was used to determine the flow. The approximation used for the computation of the hydraulic head, H_h , was

$$H_h = \int \rho' dx = \sum_{n=1}^N \rho' Z_n ,$$

where Z_n represents the vertical projection of the subdivision. A negative value was used if the flow was upward in the subdivision and a positive value was used for downward flow. The total number of subdivisions was N . The densities were computed from the fluid temperature and the relation

$$\rho_n = 62.404 + 0.4556 \left(\frac{T_n}{100} \right) - 0.9605 \left(\frac{T_n}{100} \right)^2 + 0.08164 \left(\frac{T_n}{100} \right)^3 . \quad (24)$$

Equation (24) was obtained by applying the method of least squares to the tabulated densities of water given in McAdams (16) and in the International Critical Tables. The equation is valid for temperatures up to 300F and is accurate to 0.01% for low pressures. The evaluation of the loop constant, a_h , is given in Table V.

Table IV

FLOW CONDITIONS FOR COMPUTER PROBLEMS

Computer Problem No.	Initial Volumetric Rate of Flow, ft ³ /sec	Initial Temperature, F	Nature of Step Change	Coolant Temp., F	Comments
1*	0	Uniform around loop- 80.5	Abrupt Heat Input 4 Heaters, 25.08 kw	40	First Loop
2	0	Uniform around loop- 122.0	Abrupt Heat Input 4 Heaters, 24.75 kw	37.5	First Loop
3	0.00137	144.0 in riser 77.5 in downcomer	Bottom Heater, 6.30 kw Abrupt Heat Input 3 Heaters, 18.58 kw	36.5	First Loop
4	0	Uniform around loop- 109.0	Abrupt Heat Input Top Three Heaters, 19.17 kw	40	Second Loop

*Not reported.

(Revisions to computer problems pertain to changes in loop parameters and number of subdivisions. See Table VII.)

UNIT	TOTAL EQUIVALENT LENGTH inches	VOLUME ft ³	NO. OF SUBDIVISIONS n	VOLUME OF SUBDIVISION $\frac{V}{n}$	$\frac{1}{V_n}$	LENGTH OF SUBDIVISION ft.
HOT LEG	257.6	0.0890	15	0.005933	168.5	1.431
COOLER	154.5	0.0534	9	0.005933	168.5	1.431
COLD LEG	352.5	0.1218	20	0.006091	164.2	1.469
HEATER	-	0.1844	4	0.0461	21.7	-

FIG. 8
SUBDIVISION OF SECOND EXPERIMENTAL LOOP

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Table V

LOOP PARAMETERS FOR SECOND EXPERIMENTAL LOOPHeat Transfer CoefficientsOver-all Coefficient for Heat
Exchanger (Cooler)

$$U = 16.1 V$$

Film Coefficient

$$3/4 \text{ in. Tubing} \quad h_{3/4} = 56.7 V$$

$$1 \text{ in. Tubing} \quad h_1 = 31.7 V$$

$$\text{Heater Section} \quad h_h = 16.0 V$$

Loop Resistances

$$\text{Turbulent Flow} \quad R_\ell = 332,000 V^{1.64}$$

$$\text{Viscous Flow} \quad R_\ell = 2500 \mu V$$

Dimensions and Other Values

$$D_0 = 1.000 \text{ in.}; \quad D_n = 0.872 \text{ in.}; \quad \delta = 0.00573 \text{ ft.};$$

$$\text{and} \quad \rho_w c_w = 53.0 \text{ Btu}/(\text{ft}^3)(F) \text{ for } 1 \text{ in. tubing}$$

$$a_\ell = \frac{\rho}{g} \int \frac{dx}{A} = (\rho/g) \sum_{n=1}^N \frac{x_n}{A_n} = 16,990 \rho/g = 32,200 \text{ lb. sec}^2/\text{ft}^5$$

$$1 \text{ in. Tubing} \quad \frac{L}{A} = 14,530 \text{ ft}^{-1}$$

$$3/4 \text{ in. Tubing} \quad \frac{L}{A} = 1,880$$

$$1 \text{ in. pipe} \quad \frac{L}{A} = 280$$

$$\text{Heater} \quad \frac{L}{A} = 290$$

Average
density used.

$$\rho c = 60.9 \text{ Btu}/(\text{ft}^3)(F)$$

Table VI

FINITE-DIFFERENCE EQUATIONS FOR COMPUTER PROBLEM NO. 4

Subdivision	Equation			
Hot Leg 1-15	$T_n' = (1 - 197.1 V \Delta \tau) T_n + 168.5 V \Delta \tau T_{n-1} + 28.6 V \Delta \tau t_n$ $t_n' = (1 - 105 V \Delta \tau) t_n + 105 V \Delta \tau T_n$			
Cooler 16-24	$T_n' = (1 - 183.1 V \Delta \tau) T_n + 168.5 V \Delta \tau T_{n-1} + 14.6 V \Delta \tau t_c$			
Cold Leg 25-44	$T_n' = (1 - 192.8 V \Delta \tau) T_n + 164.2 V \Delta \tau T_{n-1} + 28.6 V \Delta \tau t_n$ $t_n' = (1 - 105 V \Delta \tau) t_n + 105 V \Delta \tau T_n$			
Heater 45-48	$T_n' = (1 - 27.7 V \Delta \tau) T_n + 21.7 V \Delta \tau T_{n-1} + 6.0 V \Delta \tau t_n$ $+ 0.338 (Q_{kw})_n \Delta \tau (1 - 0.173 e^{-18.3 V \tau} - 0.827 e^{-0.1 \tau})$ $t_n' = (1 - 16.5 V \Delta \tau) t_n + 16.5 V \Delta \tau T_n$			
Flow Equations	$V' = V + \Delta \tau (0.0000311 H_\ell' - 0.0481 V) \text{ for } V < 0.00085$ $V' = V + \Delta \tau (0.0000311 H_\ell' - 10.32 V^{1.64}) \text{ for } V > 0.00085$ $H_\ell' = \sum_{n=1}^{48} \rho_n Z_n$			
Conditions of Restraint	$Z_1 = -1.3475$ $Z_{2-13} = -1.4308$ $Z_{14} = -1.1075$	$Z_{15-24} = 0$ $Z_{25} = 0.3640$ $Z_{26-41} = 1.4687$	$Z_{42} = 0.1156$ $Z_{43} = 0$ $Z_{44} = -0.8542$ $Z_{45-48} = -0.8750$	
	$\rho_n = 62.404 + 0.4556 \left(\frac{T_n}{100} \right) - 0.9605 \left(\frac{T_n}{100} \right)^2 + 0.08164 \left(\frac{T_n}{100} \right)^3$			
	$n_{49} = n_1$			
Operating Conditions	$V_{\tau=0}; \quad t_c = 40.0; \quad (Q_{kw})_{45} = 0; \quad (Q_{kw})_{46-49} = 6.39;$ $(T_{1-48})_{\tau=0} = 109.0; \quad (T_n)_{\tau=0} = (t_n)_{\tau=0}$			

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VI. COMPARISON OF EXPERIMENTAL AND CALCULATED TRANSIENT LOOP RUNS

A. First Experimental Loop

A total of 64 transient natural circulation runs were made on the first loop. These runs served to demonstrate the reproducibility of the test results and to characterize the response to abrupt additions and release of heat to the fluid under various initial conditions. Attention is focused on only two runs for purposes of comparing the experimental results with the numerical solution of the finite-difference equations. The conditions of the two runs for computer problems No. 2 and 3 are listed in Table IV.

The progressive refinement of the relations for the loop parameters including a reduction in the number of subdivisions and the effects on the numerical evaluations of computer problem No. 3 are summarized in Table VII and are illustrated in Fig. 9. The comparisons of calculated and experimental values of flow and temperatures given in Figs. 9(a) and 9(b) indicate that the transient operation of the loop can be predicted with a fair degree of assurance. Several points of departure may be summarized as follows:

- (1) Difference between the initial maximum values of T_1 and T_3 indicate that the film heat transfer coefficient should be increased.
- (2) Lag between experimental and calculated temperatures indicates the importance of the heater lag.
- (3) Difference between initial flow rates indicates more precision required in specifying loop resistance.
- (4) Difference between steady-state temperatures indicates the need for an improved cooler equation.

The successive improvement of the loop parameters for revisions 2 and 3 and the approach of the calculated values to the experimental values are revealed in Figs. 9(c) and (d), respectively.

The fourth revision [Fig. 9(d)] shows little or no departure from the third revision even though the number of subdivisions in the loop were nearly halved.

Computer problem No. 2 represents a more exacting test of the analytical equations for the volumetric flow starts at zero. The same finite-difference equations which were so successful for the third revision of computer problem No. 3 were applied in the second revision of computer

Table VII

NATURE OF REVISIONS FOR COMPUTER PROBLEM NO. 3

	Comments	Heat Transfer Coefficient		Loop Resistance	Heater Lag	Heat Transfer to Heater Walls	Heater Volume, ft ³
		Cooler (Over-all)	Vertical Riser (Film)				
Initial Problem		U = 35.1V	h = 43V	$R_f = 10^6 \text{ V}^{1.65}$	Not taken into account	Not taken into account	0.176 (calc'd.)
First Revision	Error in heat capacity of tubing walls corrected. Simple heater lag taken into account.	Same	Same	Same	$Q' = q'(1 - 0.189e^{-18.3V})$	Same	Same
Second Revision	Film heat transfer coefficient revised. Cooler coefficient revised to take into account cooler subdivisions. Loop resistance evaluated on basis of 10 gpm for coolant flow. Improved heater lag equation used. Heater volume measured.	U = 41.5V	h = 58.7V	$R_f = 690,000 \text{ V}^{1.58}$	$Q' = q'(1 - 0.173e^{-18.3V} - 0.827e^{-197V})$	Same	0.184 (measured)
Third Revision	Heat transfer to heater walls taken into account.	Same	Same	Same	Same	Taken into account	Same
Fourth Revision	Size of subdivisions nearly doubled.	Equation corrected to take into account subdivisions.	Same	Same	Same	Same	Same

Notes: "Same" designates condition of previous revision not changed.

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problem No. 2. As an approximation for the heater lag equation, a viscous flow of $0.00076 \text{ ft}^3/\text{sec}$ was used in the term involving the larger exponential:

$$Q' = q' (1 - 0.173 e^{-18.3V\tau} - 0.827 e^{0.15})$$

The calculated and experimental results are shown in Fig. 10, and, in general, the agreement is satisfying. Further improvement should be achieved by increasing the heat transfer coefficient and refining the viscous flow loop resistances.

B. Second Experimental Loop

The second loop, with its faster response, provided a relatively severe test of the analytical approach. Computer problem No. 4 was calculated for operating conditions corresponding to transient run No. 65 which was made on the second loop. The finite volume subdivisions used were of the same order of magnitude as those employed in the third revision of computer problem No. 3 and the details of the equations have been previously noted. The comparisons of the calculated and experimental results are shown in Fig. 11. The agreement is generally very good. Had the heat transfer coefficient for the cooler been corrected for the number of subdivisions, better agreement of steady-state values would have been attained.

FIG. 10
 COMPARISON OF EXPERIMENTAL WITH
 CALCULATED FLOWS AND TEMPERATURES
 (COMPUTER PROBLEM NO. 2)

FIG. 11
COMPARISON OF EXPERIMENTAL WITH CALCULATED
FLOWS AND TEMPERATURES
(COMPUTER PROBLEM NO. 4)

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APPENDIX A

METHOD FOR CALCULATING TRANSIENT FLOW
FROM PRESSURE DROP MEASUREMENTS

The transient flow may be determined from the pressure drop recorder reading only after the average temperature in the test section has been determined, the so-called steady-state flow calculated, and the steady-state flow corrected to account for acceleration effects. The method is complex but satisfactory for the uniform, relatively slow transients encountered in the first circulation loop.

Average Temperature in Pressure Drop Test Section

Since the temperature at the hot leg thermocouples are known as a function of time, temperatures may be estimated all along the test section. This is accomplished by dividing the total volume between any point in the section and thermocouple No. 3 by an average volumetric rate of flow to obtain the time required for the temperature leaving the thermocouple to reach the desired point. Since the flows and the temperature change in such a regular manner, extrapolation is not difficult. If necessary, the assumed average flow may be corrected to redetermine the temperatures. There will be a considerable period of time before the initial temperature wave reaches this section, so the temperature and flow will be known for approximately the first 100 seconds. The method assumes no heat loss by the fluid to the walls during the given time interval. The test section was broken up into some two to six subdivisions and the arithmetic average of the separate temperatures was used as the average temperature.

Steady-State Flows

So-called steady-state flow were calculated from the instantaneous pressure drop recorder readings using the steady-state pressure drop correlations and the average viscosity as determined by the average temperature. It is to be noted with the interpretation used, the steady-state flows will vary with time if Δp varies with time.

Transient Flows

The equation of motion for the horizontal pressure drop test section expressed in terms of the volumetric rate of flow reduces to

$$V^2 = \frac{\pi^2 g D^5 \Delta p}{8fL} - \frac{\pi}{2} \frac{D^3}{f} \frac{dV}{d\tau}$$

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The first term on the right hand sides corresponds to the steady-state flow, V_s^2 , calculated on the basis of the instantaneous pressure drop, Δp . In terms of the parameters of the pressure drop test section,

$$V = V_s \sqrt{1 - \frac{71,500}{R} \frac{dV}{d\tau}} \quad ,$$

where R is the recorder reading for the pressure drop. In most instances $dV/d\tau$ will be very nearly equal to $dV_s/d\tau$ and this substitution is made as a first approximation.

Comparisons of the transient flow as determined from the pressure drop measurement and by the Magnaflo are shown for both isothermal and non-isothermal flow in Fig. 12. The general agreement is good.

FIG. 12
COMPARISON OF TRANSIENT FLOW
DETERMINED BY TWO TECHNIQUES

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APPENDIX B

OVER-ALL COEFFICIENT FOR HEAT EXCHANGER

(Correction for Number of Subdivisions)

A steady-state heat balance over the entire cooler is given by:

$$V \rho c_p (T_h - T_c) = UA \left(\frac{T_h + T_c}{2} - t_{av} \right) ,$$

where T_h is the steady-state hot leg temperature; T_c is the steady-state cold leg temperature; and t_{av} is the average temperature of the coolant. The empirical correlation for U is:

$$U = 26.0 (\text{gpm})^{0.13} V ;$$

for a 10-gpm coolant flow,

$$U = 35.1 V .$$

The following equation obtained for T_c correlates well with the experimental steady-state data:

$$T_c = 0.357 T_h + 0.643 t_{av} .$$

If we now consider the heat exchanger divided into "n" subdivisions, the finite-difference equations over each subdivision reduce to the following form for steady state:

$$T_n = a T_{n-1} + (1-a) t_{av} .$$

By repeated application over the "n" subdivisions, we obtain

$$T_c = a^n T_h + (1-a^n) t_{av} ,$$

where

$$a = \frac{1}{1 + \frac{1.275 K \bar{V}_c}{n}} ,$$

with $K = U/V$ and $\bar{V}_c$ = total cooler volume. In order that the transient cooler equation should give the correct steady-state results, it is therefore sufficient to have

$$0.357 = \left(\frac{1}{1 + \frac{1.275 K \bar{V}_c}{n}} \right)^n .$$

Thus for any number of cooler subdivisions, "n," a fictitious value of K can be found and the cooler will give satisfactory steady-state performance.

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APPENDIX C

EXAMPLES OF CONVERGENCE OF A SOLUTION FOR A FINITE-DIFFERENCE HEAT BALANCE

A problem, considered very similar to conditions existing in the experimental heat exchanger for constant rate of flow, was studied for convergence. The results of this study indicated that for the experimental loop the volume of subdivision, in the main, determined the convergence and that a maximum stable time increment should be used.

Consider the steady-state flow of a fluid through a uniform pipe. An inlet temperature wave is impressed on the fluid and heat is transferred from the fluid to a constant wall (or coolant, or ambient) temperature. If T represents the fluid temperature above the constant value which is taken as zero, and if we neglect the axial heat conduction, we obtain for the one-dimensional heat balance equation:

$$\frac{\partial T}{\partial \tau} + u \frac{\partial T}{\partial x} + \frac{2hT}{\rho C_p r} = 0 \quad . \quad (C-1)$$

Letting $x = yr$, $\theta = \frac{\tau u}{r}$ and $\lambda = \frac{2h}{\rho C_p u}$,

Eq. (C-1) reduces to

$$\frac{\partial T}{\partial y} + \frac{\partial T}{\partial \theta} + \lambda T = 0 \quad \text{for } y > 0, \theta > 0 \quad . \quad (C-2)$$

For a constant fluid velocity, u , and constant heat transfer coefficient, h , the quantity λ will be constant. In the experimental loop, h was linear in u so that λ was constant. The boundary conditions are taken as

$$\begin{aligned} T &= 0 & \theta &= 0 \\ T &= A \sin(\beta\theta + \gamma) + B = F(\theta), & y &= 0 \end{aligned} \quad .$$

The choice of the sinusoidal temperature input was based upon the behavior of the experimental loop.

The complete solution to Eq. (C-2) is obtained through the use of Laplace transforms:

$$T_{(y,\theta)} = \begin{cases} 0 & \text{for } \theta < y \\ e^{-y} (A \sin[\beta(\theta - y) + \gamma] + B) & \text{for } \theta > y. \end{cases} \quad (C-3)$$

Equation (C-3) represents the analytical solution of the partial differential, heat balance equation and gives the values of the temperature at any distance y pipe radii from the inlet and for any value of θ (θ representing the number of pipe radii the initial inlet fluid slug has travelled in time τ).

Equation (C-2) in finite-difference form is

$$\frac{T(y, \theta) - T(y - \Delta y, \theta)}{\Delta y} + \frac{T(y, \theta + \Delta \theta) - T(y, \theta)}{\Delta \theta} + \lambda T(y, \theta) = 0 \quad (C-4)$$

Letting $y = n\Delta y$; $\theta = m\Delta \theta$; $b = \Delta \theta / \Delta y = V\Delta \tau / \bar{V}_n$ (mesh ratio);
and $a = 1 - (\Delta \theta / \Delta y) - \lambda \Delta \theta$,

Equation (C-4) becomes:

$$T_{n\Delta y, (m+1)\Delta \theta} = aT_{n\Delta y, m\Delta \theta} + bT_{(n-1)\Delta y, m\Delta \theta} \quad (C-5)$$

The temperature at any given point, say the cooler outlet, may be obtained through the use of the generating functions described by Jordan (9). This eliminates the need for determining the entire previous history. The theory of generating functions leads to

$$T_{n\Delta y, m\Delta \theta} = b^n \sum_{j=0}^{m-n-1} A_{m-n-j} B_j \quad (C-6)$$

where

$$A_m = A \sin(\beta_m \Delta \theta + \gamma) + B$$

$$B_j = \left(\frac{n+j-1}{j} \right) a^j$$

If "a" is sufficiently small, the series converges rapidly and the temperature at any point in space and time (n, m) may be evaluated with as few as three terms.

Six problems were calculated using the given analytical solutions. For both sets a finite length of heat exchanger was used corresponding to the 122-inch long experimental unit. The temperatures were all evaluated for outlet conditions ($x = 122/12$, or $y = 395.8$ pipe radii). A constant rate of flow of $0.00149 \text{ ft}^3/\text{sec}$ was used together with a value of $60.9 \text{ Btu}/(\text{ft}^3)(\text{sec})$ for ρc and a value of $h = 0.0875 \text{ Btu}/(\text{sec})(\text{ft}^2)(\text{F})$. The first set of problems used the following equation for the sinusoidal input temperature wave

$$T(0, \theta) = 100 \sin(0.0005655 \theta - \pi/2) + 100 \quad (C-7)$$

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This gave a period of 400 sec and a variation in inlet temperatures of 200F. These conditions were therefore very similar to actual experimental results. The true solution of the original partial-differential equations is illustrated in Fig. 13. Also shown are six solutions obtained with the finite-difference equations subject to the above conditions. The method of subdivision, time increments, and mesh ratios used are given in Table VIII.

Table VIII

FINITE-DIFFERENCE SUBDIVISIONS FOR COOLER PROBLEMS

Problem	No. of Subdivisions, n	$\Delta\theta = \frac{\Delta\tau u}{r}$	$\Delta\tau$, seconds	Mesh Ratio $b = \frac{\Delta\theta}{\Delta y} = \frac{V\Delta\tau}{V_n}$
1	3	55.552	2.00	0.422
2	6	55.552	2.00	0.844
3	7	55.552	2.00	1.000
4	6	27.776	1.00	0.422
5	12	27.776	1.00	0.844
6	24	13.888	0.50	0.844

A study of Fig. 13 reveals the following points of interest:

(1) The time increments and mesh ratios are not important factors in determining the convergence of finite-difference solutions to the analytical solution of the partial differential equation for the hypothetical problem considered. The mesh ratios vary from 0.422 to 1.000 and the time interval varies twofold.

(2) The magnitudes of the distance increments appear to be the major reason for the divergence of the finite-difference solutions.

A second set of problems was calculated using a sinusoidal inlet temperature wave having a period of only 50 sec. For this case

$$T(0, \theta) = 100 \sin (0.004524 \theta - \pi/2) + 100 \quad . \quad (C-8)$$

FIG. 13
CONVERGENCE OF FINITE - DIFFERENCE SOLUTIONS FOR COOLER PROBLEMS

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The solution of the partial differential equation remains the same as before, except the period of oscillation is now only one-eighth as large. Thus if the origin is shifted to $\bar{V}_c/V$ sec (the time required for the initial slug to travel the length of the cooler) and if the period of oscillation is used instead of the time, the two solutions would be coincident. The solution of the finite-difference equation was obtained for the same mode of subdivision as was used in Problem No. 2: $n = 3$, $\Delta\tau = 2$ sec, $b = 0.844$. This solution was also coincidental with the solution of Problem No. 2 if the period of oscillation is used as the abscissa. The problem illustrates the rather obvious point that the convergence of the finite-difference solution to the solutions of the partial differential equations is insensitive to the period of the impressed input as long as this period is sufficiently greater than the magnitude of the time increments employed in the finite-difference equations.

BIBLIOGRAPHY

1. Binnie, A. M., "The Effect of Friction on Surges in Long Pipe-Lines," *Quarterly Journal of Mech. and Applied Math.*, 4, 330 (1951).
2. Crank, J., and Nicolson, P., "A Practical Method for Numerical Evaluation of Solutions of Partial Differential Equations of the Heat Conduction Type," *Proc. Cambridge Philosophical Soc.*, 43, 50 (1947).
3. Daily, J. W., and Deemer, K. C., "The Unsteady-Flow Water Tunnel at M.I.T.," *Trans. A.S.M.E.*, 76, 87 (1954).
4. DeJuhasz, K. J., "Graphical Analysis of Transient Phenomena in Linear Flow," *Journal of Franklin Inst.*, 223, 463 (1937).
5. Grace, S. F., "Oscillatory Motion of a Viscous Liquid in a Long Straight Tube," *Philosophical Magazine*, 5, 933 (1928).
6. Hall, N. A., "Orifice and Flow Coefficients in Pulsating Flows," *Trans. A.S.M.E.*, 74, 925 (1952).
7. Hildebrand, F. B., "On the Convergence of Numerical Solutions of the Heat Flow Equation," *Journal of Math. and Physics*, 31, 35 (1952).
8. Jenny, E., "Berechnungen und Modelversuche uber Druckwellen Grossen Amplituden in Auspuff Leitungen," Zurich, E.T.H. Promotionarbeit Basel (1949).
9. Jordan, Karoly, *Calculus of Finite Differences*, 2nd Ed., New York, Chelsea Pub. Co., 1947.
10. Langhaar, H. L., "Steady Flow in the Transition Length of a Straight Tube," *Journal of Applied Mechanics*, 9, No. 2 (1942).
11. Leppert, George, "The Numerical Solution of Unsteady State Heat Conduction Problems by the Method of Crank and Nicolson," *Journal of Am. Soc. of Naval Engineers, Inc.*, 64, No. 3, August 1952.
12. Leppert, George, "A Stable Numerical Solution for Transient Heat Flow," A.S.M.E. paper No. 53-F-4 (advance copy).
13. Leppert, George, "Derivation and Experimental Verification of a Transient Flow Equation," Argonne National Laboratory Memo., May 29, 1952.

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14. Leutert, Werner, "On the Convergence of Unstable Approximate Solutions of the Heat Equation to the Exact Solution," Journal of Math. and Physics, 30, 245 (1951).
15. Leutert, Werner, and O'Brien, George G., "On the Convergence of Approximate Solutions of the Wave Equation to the Exact Solution," Journal of Math. and Physics, 30, 252 (1951).
16. McAdams, Heat Transmission, John Wiley and Sons, Inc. (1950).
17. McLain, H. A., "Transient Analysis of Some Fluid Flow Problems," M.S. Thesis, University of Minnesota (1953).
18. O'Brien, George G., Hyman, Morton A., and Kaplan, Sidney, "A Study of the Numerical Solutions of Partial Differential Equations," Journal of Math. and Physics, 29, 223 (1950).
19. Richter, H., Forschungsarbeiten, 338 (1930).
20. Schonfeld, J. C., Applied Scientific Research, A1, 169 (1948).
21. Valensi, J., "Oscillations d'un liquids pesant et visqueaux dans un tube en U de faible diameter," Comptes Rendus de l'Academie des Sciences, 224, 446 (1947).