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On the Accuracy and Optimization
of Tracking Systems

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ABSTRACT

A decision theoretic model designed to aid in the creation, development and critical evaluation of automatic tracking systems is presented in this thesis. This model is specifically designed for tracking systems which use two directional Yagi antennas to estimate the unknown location of a radio carrying animal by mathematical triangulation. The approach taken is stochastic in nature and involves the quantification of a tracking system's performance by a probabilistic treatment of radio wave propagation as well as a probabilistic treatment of the propagation of errors during bearing determination and position estimation. Estimation results basic to extending this approach to three tower tracking systems are given. The usefulness and flexibility of the two tower tracking system model is demonstrated by considering possible improvements in the Cedar Creek Tracking System owned and operated by the University of Minnesota. To do this the consequences of proposed system improvements are evaluated by computer simulation. Indications of how this decision theoretic approach can be extended to other design questions in the field of biotelemetry are given.

TABLE OF CONTENTS

ABSTRACT	ii
ACKNOWLEDGEMENTS	v
CHAPTER 1. INTRODUCTION	1
CHAPTER 2. CONCEPTUAL FOUNDATIONS	9
CHAPTER 3. FOUNDATIONS OF TRIANGULATION WITH SPECIAL REFERENCE TO THE CEDAR CREEK SYSTEM	
1. Introduction	18
2. Basic Terminology	19
3. Some Analytic Aspects of Triangulation	21
4. Factors Influencing Acquisition and Accuracy of Observed Bearings	29
CHAPTER 4. CONSIDERATIONS RELATING TO BEARING ERROR DISTRIBUTIONS	
1. Introduction	43
2. Geometrical Interpretation of Errors in Observed Bearings	44
3. Bearing Error Distributions	50
4. Loss Function Construction Using Mean Squared Error	60
5. Asymptotic Uses of Second Moments ..	79
CHAPTER 5. A DECISION THEORETIC MODEL	
1. Introduction	93

2. Probabilistic Loss Functions	94
3. A Stochastic Model for Radio Wave Propagation	99
4. Multivariate Extensions of the Probabilistic Loss Function	106
CHAPTER 6. SOME EXTENSIONS TO 3 TOWER SYSTEMS	
1. Introduction	113
2. Basic Mathematics for Three Tower Systems	113
3. Description of the Set of Triples ..	123
4. Some Estimation Theory and Results .	128
CHAPTER 7. THE TOWER ALLOCATION PROBLEM AND THE CEDAR CREEK SYSTEM	
1. Introduction	142
2. Calculation Procedure	142
3. Performance Characteristics of the Present Cedar Creek System	153
4. The Tower Allocation Problem on the Cedar Creek Natural History Area ...	164
CHAPTER 8. CONCLUDING REMARKS	192
APPENDIX I	196
APPENDIX II	208
APPENDIX III	213
BIBLIOGRAPHY	217

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Biotelemetry is the instrumental technique for gaining and transmitting information from a living organism and its environment to a remote observer. This is a relatively new measurement technique which has received increasing attention from ecologists, field biologists, wildlife managers, electrical engineers and others. Historically, the first ecological application of biotelemetry was published in 1957, when the temperature of an incubating penguin egg was recorded at the South Pole. The technique received its first national attention in 1962 in a conference dealing solely with biotelemetry held at the American Museum of Natural History. The proceedings of this conference were edited and published by Slater (1963). Since the years from 1957 to 1962 to the present, hundreds of research papers have been published in which biotelemetry was applied to various ecological problems. Bibliographies by Adams (1965) and Schladoweiler and Ball (1968) list several hundred such papers, the majority of which were published after 1960. Such an increase in attention has prompted some to refer to biotelemetry as the "new status symbol of the biologist". (See Slater (1965)) Whether this is the case or not, it is clear that the proper use of instrumental techniques in ecological research can provide information not obtainable by any other means. Interesting work which seems to substantiate this point includes

investigations of the response of swimming sharks to olfactory stimulants (Baldwin (1965)), response and movement patterns of unrestrained deer subjected to a drive census (Tester and Heezen (1965)), and grizzly bear tracking in Yellowstone Park (Craighead and Craighead (1965)). Informed individuals in the field are of the opinion that biotelemetry will continue to be used in research particularly in conjunction with other measurement techniques and direct human observation which render the telemetric data more interpretable. (Sanderson (1966)) Technologically, biotelemetry overlaps medical and space telemetry so the present and future developments in these fields can be modified for ecologic use. Slater (1965) feels that as electrical engineering technology develops, resulting in stable equipment of microsize, biotelemetric devices will be designed, in a sense, as part of a living system to allow the collection of more extensive data of both a behavioral and/or physiological nature. He holds out the hope for "closed-loop" biotelemetric systems which will allow information to flow back and forth from the subject to the observer. Presently, most biotelemetric systems are of an "open-loop" nature with unidirectional information flow.

One ecological application of a telemetric system is that of tracking free roaming animals carrying radio transmitters. Adams (1965) estimates that well over half of

biotelemetric studies in wildlife ecology have been primarily involved with tracking. Such tracking can be done with mobile receivers mounted on vehicles or with hand held units or by more elaborate automated stationary systems. Examples of the latter include a system used by Craighead and Craighead (1965) in Yellowstone Park and the Cedar Creek System (CCS) owned by the University of Minnesota and operated at the Cedar Creek Natural History Area. Both systems have two rotating directional receiving antennas at fixed locations which allow bearings to be measured so that the location of a transmitter can be estimated by triangulation.

The CCS came into existence approximately ten years ago as one of the first automatic radio-tracking systems for monitoring animal movements. Over the past decade the system has been used in investigations on animals such as rabbits, deer, ruffed grouse and raccoons. Improvements are currently being made on the system as a computer is being introduced to carry out preliminary data reduction and analysis. The question of relocating the present two towers or of introducing a third has been discussed. Improvements in the transmitters and other hardware are always considered along with the possibility of collecting physiological data on various animals in addition to their position. Thus, questions concerning the optimum design and operation of the CCS are of importance to those who work with the system and it is their questions and concerns

which have prompted the research presented in this paper.

When developing a model to aid with decision making in a field that is growing rapidly one must be careful how the model is designed so that it is not tied too closely to a given system design which may be soon outdated. Consequently, the primary emphasis of this thesis will be with questions relating to the location of the rotating directional antennas, and these questions constitute what will be referred to as the tower allocation problem. It is hoped this approach to tracking system optimization and the resultant model will be sufficiently general and flexible so as to be adaptable to future developments in biotelemetry; that the model will provide a framework for the rational assessment of the present and future uses of a facility such as the CCS; and that it will serve by focusing attention on prior considerations and experiments which need to be resolved and conducted before a system is constructed.

The basic viewpoint held in proposing this paper's decision theoretic model is that before one can compare different systems (i.e. - tower arrangements plus certain engineering parameters such as transmitter range, observed bearing accuracy, etc.) one needs to decide what general questions are to be answered by the data and what quality of data is required. In view of the fact that some questions might necessitate collecting data of different qualities and in different locations on some study area, research

priorities may have to be established to provide a consistent basis of comparison. Once a level of performance is decided upon and expressed in terms of the desired quality of data to be collected, the performance of the measurement system can be evaluated as to the quality of data it collects. To do this with a model of a tracking system, the model must involve parameters descriptive of the measurement system in such a way that the parameters can be varied to reflect changes in system performance. Then one can seek those parameters which will yield the desired quality of data; or alternatively seek that system which will come closest to fulfilling the stated experimental needs subject to constraints placed on the parameters. Both searches can be conducted via the model developed herein.

An example of the need for the latter type of search can be found at the CCS. Given two fixed towers and some information regarding the accuracy of bearing determination, where should a third tower with a rotating directional antenna be placed to enhance most the current research goals of the scientists using the system? Also, for a fixed or ceiling cost where should the tower be located for maximum benefit? Or, where should the present set of towers be located for maximum benefit? These questions will be considered in later sections, but they serve to substantiate the need for a model of the nature described above.

Reviews of the bibliographies mentioned earlier indicate that most of the previous work (up to 1968) involving ecological uses of electronic instrumentation seems to have been directed toward (1) the development of suitable and reliable electronic hardware and (2) general exploration of the ramifications of such instrumental techniques and their research potential. As far as automatic tracking systems are concerned, no indication of work similar in nature to that conducted in this paper is cited. Since the military and NASA are concerned with tracking planes, satellites and other objects, a review of the literature was conducted at the Space Sciences Center and discussions were engaged in with different people at the Center and in aeronautical engineering. Two papers were found, one by Broste (1970) and another by RCA (1968) on a defense project. Discussions with individuals attending the International Institute on Statistical Ecology held at Pennsylvania State University during 1972 resulted in reference being made to Choong Yun Cho and his work on micro-navigation with the U. S. Army.

Broste (1970) was concerned with errors in range estimation by triangulation. His mathematical discussion limited itself to long distance "boresight" range estimation and although informative is not really helpful from the viewpoint of animal tracking and optimal system design. The

RCA (1968) report is concerned with the feasibility of using satellites to aid in navigation and control traffic on the North Atlantic. The report is not particularly helpful to the biologist or electrical engineer concerned with animal tracking or physiological monitoring. However, the report is a good example of the procedure involved in creating and optimizing a complicated system. It begins with a definition of the necessary level of performance needed from the system and builds from that point, which is similar to the approach taken here. Dr. Cho (personal communication) felt that his work would not be helpful to this worker and recommended contacting the Bureau of Sport Fisheries and Wildlife, Department of Interior. Thus, it seems the questions of optimum design as formulated here are unique to radio-telemetry triangulation systems.

In conclusion, as the potential of the technique of biotelemetry is realized and more extensive and complicated data is collected, interesting problems of data reduction and analysis will likely appear. Some workers argue the need for new and more appropriate analysis techniques in view of their current needs. Sanderson (1968) feels the need is pressing now and that it will increase in the future. This paper provides a model which hopefully will be useful in certain situations. Many more quantitative questions remain to be answered in the area of ecological applications of electronic telemetry.

Chapter 2 discusses biotelemetry systems and equipment allocation problems in general and examines the tower allocation problem further by considering the types of information that will be used in this thesis to solve this problem. Chapter 3 introduces the basic terminology and mathematics for two tower tracking systems and analyzes field problems involved in the acquisition of data from such systems. Chapter 4 studies the mathematical consequences of certain data acquisition problems and relates these consequences to two tower tracking systems. On this basis the fundamental approach of this thesis to the quantification of a tracking system's performance is introduced in Chapter 5 and is developed into a general decision theoretic model for use in selecting optimal tower allocations. Problems of position estimation are considered in Chapter 6 and possibilities for future research are indicated. Chapter 7 contains an evaluation of the performance of the current Cedar Creek monitoring system and a discussion of the tower allocation problem on the Cedar Creek Natural History Area. Some concluding remarks are made in Chapter 8.

Two different approaches to the optimization of biotelemetric systems are recognized in this chapter and various aspects of the tower allocation problem are more fully delineated. The CCS is used to provide a concrete basis for various claims in developing the conceptual foundation for the decision theoretic model of this thesis as certain problems of quantification are discussed.

Since biotelemetric measurement systems have many facets, one might suspect that there is more than one way to improve upon the functioning of such systems and the quality of data collected. Investigations over the past decade and at present have been concerned with optimizing the electronic design of such systems by increasing the durability, reliability, functioning life and miniaturization of various components. As indicated in the introduction such optimization, and the range of the electronic possibilities, will not be considered directly in this thesis. The particular optimization problem discussed herein is that of equipment allocation and geographic tower allocation for automatic electronic triangulation systems. However, it seems that both types of optimization (i.e. - equipment design and use or allocation) go hand in hand and should be studied jointly in the evaluation of biotelemetric systems since a shrewd allocation of equipment could enhance the quality of data collected for various specified objectives.

Figure 2.1 - Schematic representation of a "Closed-loop" biotelemetry system.

Figure 2.2 - Schematic representation of the logical features of the CCS illustrating "Open-Loop" nature.

This point is further substantiated by Figure 2.1 page 10. Following Haahn (1965) and Slater (1965), Figure 2.1 gives a schematic representation of a general "closed-loop" biotelemetric system. One can electronically optimize the equipment placed on the subject and also the observer's receiver and transmitter but for free roaming animals the question still remains as where to locate the observer's instrumentation for the best results in view of some specified research purpose. Clearly this question of equipment allocation grows in importance as the observer's instrumentation increases in size, complexity and cost and subsequently tends to be less mobile.

An example of a stationary "open-loop" biotelemetry system is the CCS. This system has a replacement cost of approximately \$80,000 and automatically measures bearings subsequently used to estimate a subject's position. It serves as a prototypic example of a system where tower allocation is of key importance for optimal position estimation. Biotelemetric systems not concerned with position estimation and tracking but rather with physiological monitoring would also be concerned with equipment allocation since one would want to locate reception instrumentation at enough points over a study area to insure that high quality data could be collected on a continuous basis. Both issues are raised here since they are simultaneously important at the CCS as physiological monitoring is planned for future studies.

Descriptions of the CCS can be found in Cochran, Warner, Tester and Kuechle (1965) and Cochran, Warner and Tester (1964). Indications of the nature of data collected and its analysis are contained in Siniff and Tester (1965) and Sargeant, Forbes and Warner (1965). Figure 2.2 page 11 summarizes the relevant "logical" features of the system for this discussion. Current planned improvements in the system include the introduction of a computer on line to speed up bearing determination, data reduction and preliminary analysis. This will eliminate the optical display and photography but have only indirect influence on the TA problem. Therefore these improvements will not be considered further in this thesis.

The goal of optimal tower allocation in this thesis is to locate the towers in such a way on a study area so as to reduce in some sense the overall errors in position estimation. At least three types of information will be used in arriving at a selection of optimum tower sites. These types are:

- (1) a specification of the field performance characteristics (the "range") of the radio transmitters used on the various animals and the overall accuracy of bearing determination,
- and (2) a listing of the animals to be studied by the system and some specification of the relative importance of different subjects to the overall research mission of the monitoring system, and (3) a specification of the relative importance of the different geographical regions to be monitored by the

measurement system.

It seems clear why information of the first type is needed. Without knowledge of transmitter field performance it is difficult to quantify the variation in signal strength introduced by the distance and terrain between transmitter and receiver. Also, since one is seeking to locate the towers in such a way as to enhance position estimation relative to given research pursuits, one needs to know how well bearings are measured to investigate the estimation properties of the system. Type (2) information is needed since no measurement system such as the CCS will yield data of uniformly high quality on every study animal because of their different locations relative to the towers and because of the different transmitter characteristics from species to species; larger animals generally carrying larger stronger transmitters. Any tower placement will yield a measurement system producing data; but, to determine which one will do the best job of producing the highest overall quality of data relative to the research pursuits of those scientists using the CCS, one must evaluate the losses and gains in data quality across species in search of an optimum compromise. Such an evaluation requires a specification of the relative importance of the different subjects in view of the relative importance of the different animals to the research mission of the CCS. Informations of type (3) is needed to provide fixed reference regions about and within which various potential tower sites

can be evaluated, so that an allocation can be selected which will monitor the study area and surrounding regions to yield the highest overall data quality. Without information of types (2) and (3), one is forced to treat the study subjects equally and the habitat as homogeneous with respect to the locations of the differing subjects.

The chief objection to specifying the information of types (1), (2), and (3) would seem to be that it is not available or is unobtainable. This is not the case. Information of type (3) can be obtained from the biological literature, personal observational knowledge or data which has been collected in pilot studies with hand held receivers on animals with relatively unknown habits.

Since elaborate monitoring systems such as the CCS are expensive to build and operate, it seems reasonable to expect that considerable thought will have been given to the research policy and mission of the system. This policy would naturally be based on such factors as grants for research, the range of subjects suitable for tracking due to their size and catchability, nature of study area, etc. The quantification of the importance of different species to the overall research effort of a system may seem difficult but methods for quantifying preference patterns are available having been studied by psychometricians and statisticians (Guilford (1954), David (1963)). It should be mentioned at this juncture that one possible use of the model developed in this thesis could

be the evaluation of a tracking system with regard to its usefulness in different research projects so that as the mission of the system changes with time, an evaluation can be made of the adequacy of the tracking system and plans for the future can be formulated.

One problem arises regarding the availability of type (1) information. It would not be difficult to field test the transmitters to ascertain their performance characteristics due to their low cost and easy maneuverability. However, the specification of how well the triangulation system can measure bearings before the system is operational is a difficult problem. Since the CCS has been in use for a decade, changes to it can be based on past and present performance. Unfortunately, someone building a new system with different electrical components than the CCS will encounter new problems. There can be considerable uncertainty associated with prior plans in this area. More than anything else this one issue necessitates that the tower allocation problem be considered from the viewpoint of mathematical decision theory so that attempts can be made to handle the uncertainty involved.

The three types of requested information can be augmented by a fourth type to further constrain the optimal tower allocation being sought. Namely, one can introduce cost factors directly into the problem by only seeking an optimum tower allocation within regions of the study area where

such factors as accessibility, construction costs, etc. are desirable. Such an additional constraint can in principle result in a different system being selected as the optimum system from the system which is optimum without the constraint.

In conclusion, it is the goal of this thesis to present a general quantitative technique and model that will allow the rational selection of tower sites which will optimize position estimation in some overall manner across subjects at a fixed ceiling cost. Due to the uncertainty involved, the resulting model will of necessity be developed from a decision theoretic viewpoint. The next few chapters will develop a suitable loss function which is central to such an approach.

Section 3.1 - Introduction

The Cedar Creek System (CCS) is a two tower triangulation system employed to locate animals carrying small radio transmitters. Two independent towers at known locations with rotating directional Yagi antennas detect incoming radio signals emanating from a distant transmitter so that the azimuths of these signals from the towers can be measured and thus the transmitter's location can be estimated by triangulation. In this chapter, some of the mathematics of position determination with a two tower triangulation system will be considered and various problems discussed. The CCS will be examined in greater detail to provide guidance concerning the influence of different variables on the final recorded azimuths (or bearings). This examination involves a consideration of data acquisition problems common to most tracking systems. This discussion will serve to motivate the method of quantifying a triangulation system's performance which is used in the remainder of this thesis. Section 3.2 introduces some basic terminology associated with triangulation systems. Section 3.3 delves into the mathematics of two tower systems. The CCS is used as an example during the discussion of Section 3.4.

The geographical aspects of a two tower triangulation system, such as the Cedar Creek System (CCS), is schematically represented in Figure 3.2.1 on page 20. The logical system aspects were given in Figure 2.2 in Chapter 2. In this thesis the line passing through the two towers will be called the baseline. The angles θ_1 and θ_2 of Figure 3.2.1 are the azimuths or bearings of the fixed transmitter location relative to the tower locations. It is clear on geometrical grounds that knowledge of θ_1 and θ_2 , in conjunction with the separation distance between the towers, is sufficient to determine the triangle formed by the towers and transmitter and thus "fix" the transmitter location. These bearings will always be measured counterclockwise about each tower with zero being the baseline to the right of each tower and they will never be greater than or equal to 2π or negative. The distances from the towers to the fixed transmitter location are called ranges.

It will be convenient to superimpose a Cartesian coordinate system on the triangulation system with the origin placed at the midpoint of the line segment determined by the towers and the positive portion of the x-axis passing through the righthand tower. The towers are located at points $(s,0)$ and $(-s,0)$ and the transmitter at some arbitrary point (x,y) . This is illustrated in Figure 3.2.2 page 20. Vectors will be written as $\underline{x}$, $\underline{w}$, etc. An ordered pair or the transpose of a vector will be written as $\underline{w}'$ or $\underline{x}' = (x,y)$. The Halmos symbol $\square$ will indicate the end of a proof.

This section will discuss certain basic mathematical facts with regard to the functional relationships between bearings and locations. The results presented here are of importance to many of the later sections, however the reader uninterested in mathematics can turn to section 3.4 after noting expressions (3.3.1), (3.3.2), (3.3.3), and (3.3.4) in Theorem 3.3.2 which relate bearings to xy-coordinates; and, Figure 3.3.1 page 24 which graphically indicates those bearings which correspond to (x,y) locations.

Every point in Euclidian 2-space can be represented as an ordered pair of real numbers relative to some Cartesian coordinate system. Each point with coordinates $\underline{x} = (x,y)$ can be associated with a pair of bearings $\underline{\theta} = (\theta_1, \theta_2)$ relative to a tower system located at $(-s,0)$ and $(s,0)$. This correspondence is not invertible because infinitely many $\underline{x}$ on the baseline have the same bearings $\underline{\theta}$. Since the goal of triangulation is to allow determination of a single $\underline{x}$ given $\underline{\theta}$, $(-s,0)$ and $(s,0)$, it becomes necessary to inquire into the nature of the functional relationship relating $\underline{x}$ to $\underline{\theta}$. The following definitions formalize portions of the preceding discussion.

Defn 3.3.1 - R^2 will denote the set of all ordered pairs $\underline{x} = (x,y)$ or real numbers in Euclidian 2-space.

Defn 3.3.2 - $F^2 = \{ (x,y) \in R^2 \mid y \neq 0 \}$.

Defn 3.3.3 - Let $\underline{x}' = (x, y)$ be an arbitrary point in R^2 and let $(z, 0)$ be arbitrary but fixed. The bearing θ_z of $\underline{x}' = (x, y)$ relative to $(z, 0)$ is defined to be one of the following: (1) $\theta_z = 0$ if $y=0$ and $x \geq z$ or (2) π if $y=0$ and $x < z$, or (3) if $\underline{x}' \in F^2$, θ_z is the nonnegative angle less than 2π through which the x-axis must be rotated counterclockwise about $(z, 0)$ to become coincident with the line segment generated by $\underline{x}'$ and $(z, 0)$.

By definition, every element of R^2 has a bearing. Since two points uniquely determine a line, the bearing θ_z of (x, y) to $(z, 0)$ must be unique for these fixed points. In what follows it will be convenient to denote the bearing of an arbitrary point in R^2 about $(-s, 0)$ as θ_1 and about $(s, 0)$ as θ_2 resulting in the pair $\underline{\theta} = (\theta_1, \theta_2)$. This is in accordance with Figures 3.2.1 and 3.2.2 on page 20.

Lemma 3.3.1 - Given $\underline{x}' \in R^2$, the bearings $\underline{\theta}' = (\theta_1, \theta_2)$ of $\underline{x}'$ relative to $(-s, 0)$ and $(s, 0)$ satisfy one of the following conditions provided $y \neq 0$.

- (1) $0 < \theta_1 < \theta_2 < \pi$ if $y > 0$, or
- (2) $0 < \theta_1 < \theta_2 < \pi$ if $y < 0$. If $y=0$,
 $\underline{\theta}' \in \{(0, 0), (\pi, \pi), (\pi, 0)\}$.

Pf - The points $(-s,0)$ and $(s,0)$ and $(x,y) \in \mathbb{R}^2$ form the corners of a triangle or a line segment. If $y > 0$ a triangle is formed and two interior angles are θ_1 and $\pi - \theta_2$ so that $\theta_1 + (\pi - \theta_2) < \pi$ which implies $\theta_1 < \theta_2$. Suppose $y < 0$ forming another triangle with both θ_1 and θ_2 greater than π so that interior angles are $2\pi - \theta_1$ and $\theta_2 - \pi$. Then $(2\pi - \theta_1) - (\theta_2 - \pi) < \pi$ which implies $\theta_2 < \theta_1$. If $y = 0$ then there are three cases: $\underline{\theta} = (\pi, \pi)$, $(\pi, 0)$ or $(0, 0)$ none of which satisfy strict inequalities. $\square$

Lemma 3.3.1 allows the set of all possible bearings to be partitioned into cases corresponding to $\underline{x}' \in \mathbb{F}^2$ and $\underline{x}'$ along the baseline. Hence, if bearings are to be used to uniquely locate or select single locations, they can only do this for locations in $\mathbb{F}^2$ since infinitely many points on the baseline generate identical bearings. On geometric grounds, since a location $(x,y) \in \mathbb{F}^2$ in conjunction with $(-s,0)$ and $(s,0)$ determines a unique pair of lines intersecting at only (x,y) ; and, consequently a pair of bearings - which in turn uniquely determine lines intersecting at (x,y) , it seems that all points of $\mathbb{F}^2$ stand in a one-to-one relationship to bearings satisfying (1) or (2) of Lemma 3.3.1. It remains to clarify exactly what set of bearings corresponds to $\mathbb{F}^2$.

Figure 3.3.1 - The Set β of Bearings for a Two Tower Triangulation System

Defn 3.3.4 - Let $a_u = \{ \underline{\theta} \mid 0 < \theta_1 < \theta_2 < \pi \}$,
 $a_d = \{ \underline{\theta} \mid \pi < \theta_2 < \theta_1 < 2\pi \}$ and $B = a_u \cup a_d$.

Lemma 3.3.2 - Let θ_z be the bearing of $(x,y) \in \mathbb{R}^2$ relative to $(z,0)$, then $\tan \theta_z = y/(x - z)$.

Pf - See Hart (1963) page 13.8

Thm 3.3.1 - If $\underline{\theta} \in a_u$ then a unique point $(x,y) \in \mathbb{R}^2$ with $y > 0$ can be found with $\underline{\theta}$ as bearings. If $\underline{\theta} \in a_d$ then a unique point $(x,y) \in \mathbb{R}^2$ can be found with $y < 0$ and bearings $\underline{\theta}$.

Pf - If (x,y) is any point generating bearing θ_1 , by Lemma 3.3.2, (x,y) must be an element of the line defined by $\tan \theta_1 = y/(x + s)$. Similarly, for θ_2 , if (x,y) has bearing θ_2 then it must be an element of the line defined by $\tan \theta_2 = y/(x - s)$. It now will be shown that these lines must intersect if $\underline{\theta} \in B$ and that the half plane of intersection can be predicted by whether $\underline{\theta} \in a_u$ or a_d . Since $\underline{\theta} \in B$, both $\tan \theta_1$ and $\tan \theta_2$ are nonzero.

Case I - Suppose $\theta_1 \neq \pi/2 \neq \theta_2$. Then $\tan \theta_1$ and $\tan \theta_2$ are defined so that y can be solved for yielding

$$2s = y \left(\frac{1}{\tan \theta_1} - \frac{1}{\tan \theta_2} \right). \text{ Given } \underline{\theta} \in a_u \text{ it can be}$$

seen that $\frac{1}{\tan \theta_1} > \frac{1}{\tan \theta_2}$. Since $2s > 0$ it follows that y can be found uniquely and that $y > 0$. For $\theta_1 \neq 3\pi/2 \neq \theta_2$ it follows likewise that $y < 0$. Once y is found x can be determined from either relationship above. Thus (x,y) is uniquely determined and since only one branch of the tangent is used, the relationship of Lemma 3.3.2 can be inverted given (x,y) to yield (θ_1, θ_2) .

Case III - Suppose $\theta_1 = 3\pi/2$ or $\pi/2$ so that by Lemma 3.3.2, $x = -s$. Then, also by Lemma 3.3.2, $y = -2s \tan \theta_2$ so that if $\pi/2 < \theta_2 < \pi$, $y > 0$ can be found and if $\pi < \theta_2 < 3\pi/2$ then $y < 0$ can be found. In either case given x and y , the relationships can be inverted to yield (θ_1, θ_2) . The argument for $\theta_2 = \pi/2$ or $3\pi/2$ is similar and will not be given. ■

More was established in the proof of the preceding lemma than was contained in its statement. The lemma in conjunction with earlier statements establishes a one-to-one correspondence between all elements of F^2 and $\mathcal{B}$. The proof utilizes those explicit functions which describe the correspondence between these sets. Only the mapping from $\mathcal{B}$ to F^2 will be given below since this direction is more important to the overall discussion. Figure 3.3.1 on page 24 gives a plot of the set containing all and only those bearings which correspond in a one to one manner with F^2 .

Thm 3.3.2 - The transformation $\underline{T} : \mathbb{B} \longrightarrow F^2$ given below is onto, one-to-one and invertible.

If neither θ_1 or θ_2 equals $\pi/2$ or $3\pi/2$ then

$$y = 2s \left(\frac{\tan \theta_2 \tan \theta_1}{\tan \theta_2 - \tan \theta_1} \right) \quad \text{and} \quad (3.3.1)$$

$$x = s \left(\frac{\tan \theta_2 + \tan \theta_1}{\tan \theta_2 - \tan \theta_1} \right). \quad (3.3.2)$$

If $\theta_1 = \pi/2$ or $3\pi/2$ then $x = -s$ and $y = -2s \tan \theta_2$. (3.3.3)

If $\theta_2 = \pi/2$ or $3\pi/2$ then $x = s$ and $y = 2s \tan \theta_1$. (3.3.4)

Pf - See Lemma 3.3.1, 3.3.2, and preceding discussion. ■

Lemma 3.3.4 - The transformation $\underline{T}$ of Lemma 3.3.3 is continuous on the set $\mathbb{B}$.

Pf - $\underline{T}$ has 3 parts, statements (3.3.1) and 3.3.2) jointly, (3.3.3) and (3.3.4). Each part is continuous separately since it is composed of continuous functions. It remains to be shown that there are no discontinuities between parts. Suppose $\theta_2 \neq \pi/2$ or $3\pi/2$, then as

$$\theta_1 \longrightarrow \pi/2 \text{ or } 3\pi/2, \\ \frac{2s (\tan \theta_2 \tan \theta_1)}{(\tan \theta_2 - \tan \theta_1)} = \frac{2s \tan \theta_2}{\left(\frac{\tan \theta_2}{\tan \theta_1} - 1 \right)} \longrightarrow -2s \tan \theta_2,$$

$$\frac{s (\tan \theta_2 + \tan \theta_1)}{(\tan \theta_2 - \tan \theta_1)} = \frac{s \left(1 + \frac{\tan \theta_2}{\tan \theta_1} \right)}{\left(\frac{\tan \theta_2}{\tan \theta_1} - 1 \right)} \longrightarrow -s$$

Similar steps show the functions to coalesce as

$$\theta_2 \longrightarrow \pi/2 \text{ or } 3\pi/2 . \emptyset$$

Although the inverse of $\underline{T}$ will not be given explicitly it can also be shown continuous over F^2 when the sets $\mathcal{B}$ and F^2 are viewed as metric spaces with suitable metrics. The functions $\underline{T}$ and $\underline{T}^{-1}$ establish a topological equivalence between $\mathcal{B}$ and F^2 which ultimately leads to $\mathcal{B}$ and F^2 being homeomorphic under $\underline{T}$ and $\underline{T}^{-1}$ (Mendelson (1962) p. 78 and 106). These topological facts, although interesting, will not be pursued further at this time.

In conclusion, it should be observed that while $\underline{T}$ maps $\mathcal{B}$ onto F^2 , and thus expedites the determination of a location given $\mathcal{Q}$, a two tower radio telemetry system generates observed bearings which differ from the true bearings needed by $\underline{T}$ for perfect estimation. For these reasons it is necessary to examine the acquisition of bearings in greater detail to arrive at an understanding of how the observed bearings relate to the true but unknown bearings in order to create a viable model in this thesis for the CCS and other similar systems.

To understand which factors have an influence on the deviation of observed bearings from the true but unknown bearing for a fixed and unknown location (x,y), it seems informative to follow a signal from an animal-borne transmitter through the environment to the recording of the signal as bearings. Although not all tracking systems will have exactly the same features, many of the same factors influencing the ultimate recorded bearings will be common to most systems. An attempt will be made below to focus on these factors. Later in this chapter the extent of the apparent influence of these factors on the CCS will be discussed. Very detailed descriptions of the CCS and its functioning are to be found in Cochran, Warner and Tester (1964) and Sargeant, Forbes and Warner (1965). An account of radio wave propagation and antennas which is readable by the layman is found in Welch (1958).

Given an animal with a radio transmitter at a fixed location, a signal is generated by the transmitter at a preset and known frequency. Hopefully the signal will reach each tower and their associated rotating directional antennas with sufficient signal power to be detected. Factors which detract from this occurrence include the distance from the transmitter to the receiving antennas, vegetation, soil type, distance from transmitter to ground, terrain variations such as hills, and battery condition at the transmitter. These factors tend to weaken or attenuate the signal's power before it can reach the towers.

There are other factors which mask the transmitter's signal by introducing rapid fluctuations in signal power, an example being movements by the animal which result in shifts of the transmission apparatus on the subject and consequently variations in the signal. In addition to this, electrical noise sometimes makes detection and extraction of the subject's signal difficult. Noise is created by such things as electrical storms, power transmission line breakdown, cosmic static, car generators and is analogous to the radio static with which we are all familiar. When noise is of the same frequency as the signal, it is detected by the directional antennas and "mixed" with the subject's signal in such a way that if the noise is of sufficient power relative to the signal, the signal cannot be extracted from the output of the antennas. Depending on the sensitivity of the receiving antenna, the above factors combine with others and the type of transmitter, frequency of signal and transmitter's output power to determine the effective range of the transmitter. Due to the numerous influences acting to vary the signal, for the purposes of this thesis it is not reasonable to view the effective range of the transmitter as a constant. Later in this thesis it will be treated as a random variable in lieu of a deterministic model combining the above factors.

Before considering other influences on signal quality, a brief qualitative sketch will be given of how a directional Yagi antenna works. Not all directional antennas will be the

same, but a Yagi is a common type. Picture such an antenna fixed and not rotating on top of a tower and assume a transmitter is some distance away sending a signal. Let the transmitter be moved in a circle of constant radius about the antenna. Assume the transmitter's signal remains constant in power at the antenna. Then as the transmitter is moved there is one sector of the circle, perhaps as wide as 30 degrees, in which the directional Yagi antenna is particularly sensitive to the signal - elsewhere the sensitivity is reduced. In the very middle of the sector, the antenna is most sensitive and more responsive than when the signal is propagated from any other part of the sector. Improvements in directional Yagi antennas are aimed at reducing this sector size and increasing sensitivity at the center of the sector. When two signals of equal power reach the antenna from within the sector, the one closest to the center of the sector causes the greatest response of the antenna. Both signals make contributions in this case to the output from the directional antenna, however the signal closest to the center makes the greatest contribution. This feature of combining different signals in differing degrees relative to their location in the sector of greatest antenna sensitivity is what makes it possible to measure bearings with a directional antenna. Supposing the subject is stationary and that the subject's signal propagates in a straight line to the antenna not being refracted or bent, as the antenna rotates the contribution of the subject's signal to the output from the antenna

reaches a maximum and then declines as the middle of the sensitivity sector approaches the line of propagation and then passes it. This also explains how extraneous signals or noise can mask the desired signal and make it impossible to extract. Any signal of the same frequency as the subject's signal will make a contribution with the subject's signal to the output from the antenna - and it is only when the power of the subject's signal is great enough to override this noise that the line of propagation can be detected and a bearing measured.

A system usually has a characteristic value called a signal-to-noise ratio which reflects how strong the signal needs to be at a minimum relative to noise for the signal to be detected for adequate analysis. The radio wave propagation model used in this thesis will depend on a probabilistic model for signal propagation and the signal-to-noise ratio just described.

In summary, when a radio signal is sent through a varied environment it is attenuated by the terrain and other factors; the influences of which are difficult to quantify in a deterministic framework. Movements by the animal will also alter the signal but eventually it will reach both directional antennas. Simultaneously the antennas and towers themselves are subject to different perturbing influences such as wind. The output from the antennas combines both the desired signal and extraneous electrical noise plus the effects of the various perturbing influences. The bearings are determined from this output.

In the absence of all noise, animal movement and undesired influences, one rotation by a Yagi antenna would produce a visual display that would consist of a symmetric curve with three peaks. The true but unknown bearing would in principle fix the location of the maximum point on the middle and largest of the three peaks so that determining the location of this maximum results in a direct estimate of this bearing. In the presence of noise and other factors the shape of this curve changes. Depending on the signal-to-noise ratio, degree of animal movement and severity of other perturbing factors it becomes increasingly difficult to ascertain the location of the maximum point on the middle peak. Thus, the bearing extraction procedure can be important in determining the accuracy of observed bearings. The experience of researchers with the CCS substantiate this point.

At the CCS, the output from the directional antennas is amplified, sent to a central station, sometimes filtered and always visually displayed and photographed to provide a permanent record. In this process, other effects enter some of which would be swayed by the towers in the wind resulting in conceivable misalignment of the antennas, temperature induced equipment noise during amplification, transmission to the station and other manipulations. After the visual peaked signal display is on film a reader selects pairs of images - one member of a pair from each tower both measured at approximately the same time - on the basis of their clarity and degree of

signal definition from those pairs transmitted by the antennas during a 5 to 10 minute period. In so doing, the reader judges the degree of departure of the signal image's shape from the ideal and elects whether or not to read the image to determine a bearing. Thus the film record is read and only the "best" signal pair from each 5 to 10 minute period is converted into bearings for transformation into xy coordinates. Due to the time variation of the influences on signal quality, what is best during one 5 to 10 minute period likewise changes. As a result some periods of time do not produce signal images that are judged to be readable by the technicians or researcher reviewing the film records. When such an occurrence becomes frequent for a given animal the limits of the measurement system have been reached. In any case, the bearings so selected serve as estimates for the bearings of the line of propagation of the subject's signal. Assuming the subject to be stationary during the measurement of the signal, the location is estimated from the pair of bearings so recorded. Quantifying the influences of the above factors is important to this thesis and the decision theoretic model proposed herein. Such a quantification will depend on different parameters, and the specification of values for these parameters amounts to a specification of type (1) information discussed in Chapter 2. Data are available for the CCS that can be used in this thesis to set various parameters.

Early in the functioning life of the CCS, V. B. Kuechle

collected field data providing information about such factors as the environmental attenuation of field strength, background noise, wind and instrumental variation and potential. Table 3.4.1 on page 36 presents a summary of his field data provided by him for this thesis. These data are differences of bearings measured optically with a surveyor's transit and bearings measured with the CCS at different locations. The locations were known to V. B. Kuechle and were selected to cover different types of locations relative to the towers. The signal images were photographed on special film to produce a sharp image for reading. The film was carefully studied in an attempt to reduce reading errors to a minimum and the best signal image on a given location was read and recorded. Summary statistics are given in Table 3.4.2 on page 36 for these data given in Table 3.4.1. Since the antennas are the same for each tower no differentiation is given as to which tower yielded a specific difference. Although there is a difference in tower heights - one tower being seventy feet and the other one hundred feet tall - this difference should not matter here. These tend to suggest the performance of the tracking system is time varying and depends on environmental factors such as wind. In view of the large number of variables influencing signal quality, this seeming time variant nature is not surprising and is of importance not only to the data analyst but also to the model builder. It tends to imply that a single parameter is not adequate to describe the degree of variation in the observed bearings

Table 3.4.1 - Data collected by V. B. Kuechle with CCS during 1964.

<u>Calm Weather</u>	<u>Windy Weather</u>
+1°	+ .8°
+2°	+2.0°
+0°	- .3°
+9°	+ .3°
-1°	+ .1°
-.4°	+ .4°
+1°	+2.0°
+5°	- .3°
-.2°	+ .6°
+0°	+ .4°
-.1°	- .2°
.0°	+ .3°

Table 3.4.2	Average of deviations	Sample variance	Sample standard deviation
Calm	0.092°	.112	.335°
Windy	0.517°	.585	.765°

Confidence Intervals

$$.049 < \sigma_{\text{calm}}^2 < .57 \quad \alpha = .01$$

$$.262 < \sigma_{\text{windy}}^2 < 2.28 \quad \alpha = .01$$

$$.224 < \frac{\sigma_{\text{calm}}}{\sigma_{\text{windy}}} < .817 \quad \alpha = .05$$

from the true but unknown bearings when various environmental factors are changing. Under ideal functioning conditions, there is likely to be some minimum variance for the observed bearings about the true but unknown bearings. However, this component of variation, in light of the preceeding, needs to be augmented by an additional component of variation introduced to account for extraneous influences on the measurement system when the operating conditions are not ideal - the total bearing variation being the sum of the two components.

Kuechle's data, while providing information about the CCS and the influence of variables such as wind, does not provide information concerning the amount of control a reader can exert over the quality of the data selected for later analysis. Careful film reading can be useful in controlling some of the variation in observed bearings by avoiding errors in reading, in transposition and by simply avoiding those film images which are difficult to read and hence conducive to the introduction of large error. It can also serve to reduce the bulk to a high quality subset.

This qualitative selection is also important when the transmitter carrying subject is not stationary since movements by the subject introduce variations in signal quality which must be judged by the reader. In this case the reader may be faced with not only selecting good quality signal images but also with the problem of inferring the animal's activity on the basis of certain characteristic signal variations for each

species. This latter classification problem has been of importance to different studies. Marshall and Kupa (1962) report using signal quality to infer different activities on grouse. Sargeant, Forbes and Warner (1965) have done the same on red fox. Since activity classification can be conducted with only one tower receiving a signal of adequate strength, it is not a primary consideration in the design of a system for tracking purposes and so will not be considered further. Of greater importance to this thesis is the influence of animal movements on signal quality and the difference in bearing estimation by different observers. Sargeant, Forbes and Warner (1965) considered these issues in their study of red fox. Experienced independent observers were used to read the same film record and classify the images as good or bad and also record the bearings for the images after rounding to the nearest degree. Their conclusions were that observer differences did not seem important and that data collected by different observers could be safely compared and used without regard to who extracted the data. They also were concerned with the effect of red fox movement on signal quality and the effect of nonsimultaneous pickup of bearings at the two towers in regard to how close estimated locations would be to the true but unknown locations. Consequently, during the winter of 1964-65 a total of 9.18 miles of radio-tagged fox tracks on fresh snow were mapped and compared with estimated locations from the film record. Along the 9.18 miles of tracks; 32 field documented sites were found where the

fox slept, dug or spent time in a confined area. These sites were independently selected from the film record on the basis of signal quality as being places along the trail where the fox had confined its movements and had not been moving very quickly. On these locations the signal quality varied however all signal images were read and the data was dicotomized as to whether a bearing had been obtained from a good quality signal image or a poor quality signal image. In reviewing their study and analyzing their summarized data (see Appendix I of this thesis) it was determined that the variance of the observed bearings taken on these fixed locations with high quality signal images seems to be comparable to the variance observed by Kuechle (1964) during calm days. Also, the variance with poor quality signal images seems comparable to the data collected on windy days by Kuechle. However, due to the differences involved in the collection of data by Kuechle and Sargeant, Forbes and Warner, certain comparisons can be made which are valuable to this thesis. Summaries of both sets of data are given on page 41 in Tables 3.4.3 and 3.4.4. These tables reflect the fact that the sets of data can be blocked according to different factors. Kuechle's data was extracted by the usual selection procedure using the reader's judgement so that only the best film images were read on calm and windy days. Sargeant, Forbes and Warner collected their data during calm weather following storms which placed fresh snow on the ground and read all film images and grouped their data according to signal image quality. The

state of agreement between the good signal images on calm days for both sets of data may be interpreted in light of the comment made in Appendix I that the results in Table 3.4.4 could be slightly optimistic. For the purposes of this thesis the agreement is adequate to suggest a lower bound for the variance of the bearing error distribution of the CCS. Furthermore the degree of variance reduction indicated in Table 3.4.4 which is attributable to the procedure of differentiating between good and poor quality signal images implies this procedure is effective in controlling the component of variance due to influences attenuating and altering the signal image. And the data of Table 3.4.3 indicates that the variation due to environmental influences such as wind operating on the towers and antennas is not effectively controlled by a reader's judgement. This situation can be described using a components of variance model where the variance of the bearing error distribution is viewed as the sum of a constant σ_0^2 and a random component σ_e^2 , so that $\sigma^2 = \sigma_0^2 + \sigma_e^2$. The constant σ_0^2 can be viewed as the variance component due to inherent variation in the CCS in the absence of perturbing environmental factors which are not controlled by a reader's judgement. The random component σ_e^2 , is due to those factors not controlled by reading. It is treated as a random variable since these factors are time varying and no attempt has been made to predict a value for σ_0^2 on the basis of such factors as wind.

Table 3.4.3 - Standard deviations derived from data collected by V. B. Kuechle.

	<u>Calm Weather</u>	<u>Windy Weather</u>
Good Quality Film Images	.34°	.77°
Poor Quality Film Images		

Table 3.4.4 - Standard deviations derived from analysis of data collected by Sargeant, Forbes and Warner.

	<u>Calm Weather</u>	<u>Windy Weather</u>
Good Quality Film Images	.25°	
Poor Quality Film Images	.50°	

There is an additional reason for proceeding in this fashion which arises out of the discussion in Chapter 2. If the model developed in this thesis is to be useful in the design of tracking systems, it must be adaptable to the amount of information which exists. By treating the components of variation in the fashion proposed above, in lieu of hard data, one still can incorporate into the model the beliefs of engineers or whatever other information is available regarding these components of variation prior to the implementation of a triangulation system by specifying a distribution for the random component of variation. Such an approach could extend the range of situations in which the model might be useful.

In conclusion, an examination of the CCS indicates a number of general features which should be incorporated into the decision theoretic model of this thesis. These features include, (1) a stochastic model for the range of a given radio transmitter, (2) a lower limit to adequate signal power as dictated by the signal-to-noise ratio, and (3) a components of variance model involving one fixed component and one random component to account for the dynamic nature of the variance of observed bearings or to quantify prior beliefs about system performance. These features will be reconsidered in Chapter 5 when they are consolidated into a single model. The data collected by V. B. Kuechle and Sargeant, Forbes and Warner will be helpful when this model is applied to the CCS.

Section 4.1 - Introduction

Chapter 3 has introduced some basic two tower triangulation mathematics and has discussed data acquisition problems. This chapter will consider certain mathematical ramifications of these issues.

A primary consideration in evaluating a tracking system is how well it performs when estimating a subject's unknown location. Generally, the observations obtained with a tracking system are not directly expressed in the same units or on the same coordinate system as the locations about which they provide information. With a two tower triangulation system such as the CCS, the initial observations are bearings which must be converted into locations relative to an xy coordinate system. Such situations - with information regarding measurement errors on a scale different from that of those quantities of prime importance calculated from the observed data - must be scrutinized with care. Transformations of error distributions from one scale to another may or may not introduce undesirable side effects. This chapter examines this topic for two tower triangulation systems under the assumption that signals of adequate strength for the reading of bearings are received by each directional antenna. Later in this thesis this assumption will be enlarged upon.

Section 4.2 will geometrically motivate the problems of this chapter and is intended for the nonmathematically

interested reader. Section 4.3 will discuss the selection of a suitable family of distributions to describe the error distribution of observed bearings. Section 4.4 will investigate certain consequences of mapping observed bearings into xy coordinates. Section 4.5 will apply the concept of asymptotic relative efficiency to the comparison of different two tower triangulation systems.

Section 4.2 - Geometrical Interpretation of Errors in Observed Bearings

By reducing a triangulation system to a set of mathematical definitions, lemmas and theorems for concise and rigorous mathematical analysis and modeling, much of the geometrical charm inherent in the mathematics is lost. This section will consider some of this geometry and motivate analytical results to follow with it.

Given a transmitter at the location (x,y) , two true but unknown bearings θ_1 and θ_2 exist connecting this point to the tower locations. In attempting to measure these bearings a pair $(\hat{\theta}_1, \hat{\theta}_2)$ is obtained which hopefully is "close" to (θ_1, θ_2) . Presuming the bearings from each tower are measured with equal accuracy, one would expect limits to exist so that it would be a virtual certainty to have $\theta_1 - e \leq \hat{\theta}_1 \leq \theta_1 + e$ and $\theta_2 - e \leq \hat{\theta}_2 \leq \theta_2 + e$, for some $e > 0$. If this is the case the error in $\hat{\theta}_i$ as an estimate of θ_i can be written $D(\theta_i, \hat{\theta}_i)$ and treated as a random

variable with a distribution over the interval $[-e, e]$. Later sections will consider the mathematical ramifications of this assumption. In Chapter 3 it was argued that the value of e changes with time. In this thesis, even if e varies, the same value of e will be used simultaneously at each tower. The model used in Appendix I was based on this assumption and did not yield contradictory results when compared with data.

Figure 4.2.1 on page 46 represents the geometrical implications of the error distribution associated with bearing measurement. The distorted diamond centered at (x,y) is the intersection of the two sectors created by $\theta_1 - e$, $\theta_1 + e$ and $\theta_2 - e$, $\theta_2 + e$. Since $\hat{\theta}_1$ and $\hat{\theta}_2$ represent bearings which must fall in these sectors, all observed fixes must fall in this diamond since the diamond contains all possible intersections of observable bearings from (θ_1, θ_2) .

The shape of this distorted diamond associated with (x,y) changes. It tends to be smaller in size when $|\theta_2 - \theta_1|$ is approximately $\pi/2$ than it is elsewhere. As (x,y) is taken close to the baseline or farther away from the towers, the situation becomes more complex. If (x,y) is taken far enough away from the towers, the rays determined by $\theta_1 + e$ and $\theta_2 - e$ will not intersect and the diamond will not exist.

Furthermore, not all observed bearings will intersect yielding a fix. If (x,y) is taken close to the baseline on one side or the other of the two towers the same facts are true. However, if (x,y) is taken between the towers and close to the baseline,

Figure 4.2.1 - The Intersection of Two Sectors Showing One Set of All Possible Fixes Associated With (x, y)

a distorted diamond is formed by the intersecting sectors although it is still true the rays need not intersect. To determine all points of the xy-plane which must always generate intersecting bearings, it is easiest to use the set $\mathcal{B}$, the set of all pairs of bearings that correspond to points of the plane.

Figure 4.2.2 page 48 indicates a subset Λ_e of A_u for a hypothetical value of e which contains all pairs (θ_1, θ_2) which correspond to points (x,y) of the upper xy half-plane which always generate intersecting observed bearings. Since the inequalities $\theta_1 - e \leq \hat{\theta}_1 \leq \theta_1 + e$ and $\theta_2 - e \leq \hat{\theta}_2 \leq \theta_2 + e$ can be thought of as creating a square of width $2e$ centered at (θ_1, θ_2) , Λ_e can be constructed by moving such a square centered at (θ_1, θ_2) over A_u and noting those points (θ_1, θ_2) for which the square is always in A_u . Any (θ_1, θ_2) not in Λ_e will sometimes generate nonintersecting bearings. The analogue of Λ_e in F^2 denoted as $\mathcal{T}(\Lambda_e)$ is given in Figure 4.2.3 page 49. As $e \rightarrow 0$, Λ_e and $\mathcal{T}(\Lambda_e)$ enlarge to all of A_u and the upper-half xy-plane in the limit. The set $\mathcal{T}(\Lambda_e)$ is important in what follows. Since $\mathcal{T}(\Lambda_e)$ contains all points (x,y) of F^2 for which all observable bearings intersect yielding $(\hat{x}, \hat{y})$ in the distorted diamond of Figure 4.2.1, as it turns out $\mathcal{T}(\Lambda_e)$ then has most points (x,y) for which the average of $((\hat{x}-x)^2 + (\hat{y}-y)^2)^{1/2}$ is finite and exists. This is so because of the boundedness of the distorted diamond containing all potential fixes associated

Figure 4.2.2 - Sketch of Λ_e For Hypothetical Value of e

Figure 4.2.3- The Set $\tilde{\mathcal{I}}(\Lambda_e)$ Containing All (x,y) That Always Have Intersecting Observed Bearings Dependent on $e > 0$

with (x,y) . This is not the case when the bearing lines created by $\theta_1 + e$ and $\theta_2 - e$ fail to intersect. Since $\mathbb{T}(\bigwedge_e)$ has a shape such that an animal can be arbitrarily close to either tower and not on the baseline and yet outside of $\mathbb{T}(\bigwedge_e)$ and not have a bounded distorted diamond formed as in Figure 4.2.1 thus lacking the average of $((\hat{x}-x)^2 + (\hat{y}-y)^2)^{\frac{1}{2}}$, the average distance from $\underline{x}$ to $\hat{\underline{x}}$ will not be used to quantify the performance of a triangulation system. A different approach is selected in Chapter 5 and is based on the expected proportion of times fixes will be obtained within a circle of some fixed radius centered at a randomly chosen point (x,y) of $\mathbb{R}^2$. Any reader not interested in the mathematical development of the above issues can turn to Chapter 5 and this probabilistic approach.

Section 4.3 - Bearing Error Distributions

In constructing any mathematical model, eventually a point is reached where it becomes necessary to conjecture or hypothesize something in order to create a model. Part of the essence of modeling a phenomenon or process is then the skillful choice of a hypothesis or set of assumptions during model construction which will least mitigate the usefulness of the model for the purposes which prompted its creation. The present section deals with a situation where hypothesizing seems necessary. It was suggested in Chapter 3 section 4 that the error distribution of the observed bearings at the CCS

was a dynamic entity as evinced by a seeming change in the variance of the observed bearings. To describe such a situation it was proposed that the overall bearing error distribution variance σ^2 , at a randomly chosen time, be a sum of two components - one a fixed constant and one a random variable - so that $\sigma^2 = \sigma_0^2 + \sigma_e^2$. Presuming this to be an adequate description of the performance of the CCS and other tracking systems, then the problem still remains as to the selection of a family of distributions for the use with σ^2 as an index. The choice made in this thesis consists of a family of truncated normal distributions. Before explicitly discussing these distributions, one should consider likely criteria by which a family of distributions may be judged. Various thoughts to keep in mind include (1) the family chosen for investigation should not contradict available data regarding bearing error distributions and should be "realistic", (2) the family should be adaptable to a wide range of situations from the CCS to other tracking systems yielding robust results for incorporation into decision processes, (3) along with (1) above, the family should tend to agree with the opinions of the engineers and users of a system regarding its functioning and capabilities, and (4) the family should be mathematically convenient and should facilitate the development of further quantitative results. In a few paragraphs the family of truncated normal distributions will be presented to be judged by these criteria.

The development of some mathematical notation will serve to extend the discussion. The observations under consideration are the observed bearings viewed as random variables. In Sections 3.2 and 3.3 a notation and terminology were introduced. A bearing by definition must be an element of the half open interval $[0, 2\pi)$. Because of this care must be exercised when subtracting or adding increments to bearings. The following discussion, definitions and lemma provide a measure of distance between bearings and an operation similar to addition for combining bearings or a bearing and an increment.

Defn 4.3.1 - Let z be some real number. Let $\text{sgn}(\cdot)$ be a function of real numbers such that -

$$\text{sgn}(z) = \begin{cases} 0 & \text{if } z = 0 \\ 1 & \text{if } z > 0 \\ -1 & \text{if } z < 0 \end{cases} \quad (4.3.1)$$

Defn 4.3.2 - Let α and β be two bearings in $[0, 2\pi)$.

The distance from α to β , $D(\alpha, \beta)$ is defined as

$$D(\alpha, \beta) = \begin{cases} \beta - \alpha & \text{if } |\alpha - \beta| \leq \pi \\ \text{otherwise,} \\ (-\alpha + \beta) - \text{sgn}(\beta - \alpha) 2\pi. \end{cases} \quad (4.3.2)$$

Geometrically speaking, if the bearings belong to a single tower, $D(\alpha, \beta)$ is the smaller of the two angles between the lines generated by α and β where the counter-clockwise direction is positive and the angle is measured from α to β .

Lemma 4.3.1 - Let α and β be two bearings, then

$$D(\alpha, \beta) \leq \pi.$$

Pf - If $|\alpha - \beta| \leq \pi$, then $|D(\alpha, \beta)| = |\alpha - \beta| \leq \pi$. If $|\alpha - \beta| > \pi$, then $|D(\alpha, \beta)| = |(\beta - \alpha) - \text{sgn}(\beta - \alpha) \cdot 2\pi| = |2\pi - |\beta - \alpha|| < \pi$. $\square$

Lemma 4.3.2 - The absolute value of the measure of distance defined in definition 4.3.2, $|D(\alpha, \beta)|$ is a metric on $[0, 2\pi)$.

Pf - Mendelson (1962) p. 33 gives a definition of a metric the conditions of which now will be verified.

First of all, $|D(\alpha, \beta)| \geq 0$ since it is an absolute value. Second, if $\beta = \alpha$ then $|D(\alpha, \beta)| = 0$ since $|\beta - \alpha| \leq \pi$. If $|D(\alpha, \beta)| = 0$ then either $\beta - \alpha = 0$ or $(\beta - \alpha) = \text{sgn}(\beta - \alpha) \cdot 2\pi$. In the former $\beta = \alpha$ and $\text{sgn}(\beta - \alpha) = 0$. Should $\text{sgn}(\beta - \alpha) = 1$ then $\beta - \alpha > 0$ and $(\beta - \alpha) = \text{sgn}(\beta - \alpha) \cdot 2\pi = 2\pi$ which is a contradiction of

the definition. Similarly if $\text{sgn}(\beta - \alpha) = -1$, then

$$\beta - \alpha < 0 \text{ and } (\beta - \alpha) = \text{sgn}(\beta - \alpha) \cdot 2\pi =$$

-2π which is a contradiction. Third,

$$|D(\beta, \alpha)| = \begin{cases} \alpha - \beta & \text{if } |\alpha - \beta| \leq \pi \\ \text{otherwise} \\ (\alpha - \beta) - \text{sgn}(\alpha - \beta) 2\pi \end{cases}.$$

In the former with $||\alpha - \beta|| \leq \pi$, $|D(\beta, \alpha)| =$

$$|\alpha - \beta| = |\beta - \alpha| = |D(\alpha, \beta)|. \text{ In the latter}$$

$$|D(\beta, \alpha)| = |(\alpha - \beta) - \text{sgn}(\alpha - \beta) 2\pi| =$$

$$|(\beta - \alpha) - \text{sgn}(\beta - \alpha) 2\pi| = |D(\alpha, \beta)|.$$

Lastly, for z a bearing, it needs to be shown that

$$|D(\alpha, \beta)| \leq |D(\alpha, z)| + |D(z, \beta)|.$$

The proof given here is repetitive consisting of 6 cases,

$$\text{namely, } 0 \leq z < \alpha < \beta < 2\pi, \quad 0 \leq \alpha < z < \beta < 2\pi,$$

$$0 \leq \alpha < \beta < z < 2\pi, \quad 0 \leq z < \beta < \alpha < 2\pi,$$

$$0 \leq \beta < z < \alpha < 2\pi \text{ and } 0 \leq \beta < \alpha < z < 2\pi.$$

Only the case $0 \leq \alpha < z < \beta < 2\pi$ will be treated

here the others following similarly. Should $\beta - \alpha \leq$

π then $z - \alpha \leq \pi$ and $\beta - z \leq \pi$. Here the result

follows by the triangle inequality. Should $\beta - \alpha > \pi$

there are 3 further cases to handle.

$$(i) \quad \beta - z \leq \pi, \quad z - \alpha \leq \pi$$

$$\beta - z \leq \pi \text{ iff } 2\pi \leq 2(\beta - \alpha) \text{ iff}$$

$$\alpha - \beta + 2\pi \leq \beta - z + z - \alpha. \text{ Since}$$

$$2\pi > \beta - \alpha > \pi, \quad \alpha - \beta + 2\pi > 0. \text{ So,}$$

$$|\alpha - \beta + 2\pi| \leq |\beta - z| + |z - \alpha|.$$

$$(ii) \quad \beta - z > \pi, z - \alpha \leq \pi$$

$$0 \leq z - \alpha \text{ iff } \alpha - \beta + 2\pi \leq (z - \alpha) +$$

$$(\alpha - \beta - 2\pi) \leq (z - \alpha) + (z - \beta - 2\pi) \text{ iff}$$

$$|\beta - \alpha - 2\pi| \leq |z - \alpha| + |\beta - z - 2\pi|.$$

$$(iii) \quad \beta - z \leq \pi \text{ and } z - \alpha \geq \pi.$$

$$0 \leq \beta \text{ iff } \alpha - \beta + 2\pi \leq \alpha - z + 2\pi + \beta - z$$

$$\text{iff } |\beta - \alpha - 2\pi| \leq |\alpha - z - 2\pi| + |\beta - z|.$$

This completes the case $0 \leq \alpha < z < \beta < 2\pi$.

Once the five other cases are verified the result follows. ■

Thus Defn 4.3.2 and Lemma 4.3.2 give a measure of the directed distance between two bearings. These results will be useful now in discussing error distributions for bearings.

Defn 4.3.2 - Let the symbol $\hat{\theta}_k$ denote an observed bearing from tower k when the true but unknown bearing it estimates is θ_k , ($k=1,2$). Then $D(\theta_k, \hat{\theta}_k)$ denotes the error in $\hat{\theta}_k$ as an estimate of θ_k with $D(\cdot, \cdot)$ defined in Defn 4.3.2. For brevity let $D_k = D(\theta_k, \hat{\theta}_k)$, ($k=1,2$) and define $\Delta \hat{\theta}' = (D_1, D_2)$, the vector of errors.

In view of Lemma 4.3.1, it is clear the errors under such a definition are bounded and that any selected distribution for the errors must be truncated over the closed interval $[-\pi, \pi]$. One further mathematical point should be

mentioned. When adding an increment such as D_k to θ_k to recover $\hat{\theta}_k$, it is necessary in certain cases to adjust

$\theta_k + D_k$ to remain in the interval $[0, 2\pi)$. In other words addition in this case must be modulo 2π . For example let

$\theta_k = 2\pi - \epsilon$, for some small $\epsilon > 0$, and let $D_k = 2\epsilon$. Then $(\theta_k + D_k) \bmod 2\pi = (2\pi - \epsilon + 2\epsilon) \bmod 2\pi = \epsilon$. This is a minor but nevertheless important computational step in view of the boundedness of error and bearings.

The components of variance model proposed in Section 3.4 can now be joined with the preceding development. For the sake of simplicity, it will be assumed that the family of distributions to be selected and indexed by σ^2 will be such that (1) the expectation $E(D(\theta_k, \hat{\theta}_k)) = 0$, and (2) the conditional expectation $E(D^2(\theta_k, \hat{\theta}_k) | \sigma^2)$ approximately equals σ^2 for $k=1,2$. Thus, it is going to be assumed that $\hat{\theta}$ is unbiased as an estimate of θ and that the two components of $\hat{\theta}$ have identical variances at any given instance, although this variance may change with time. A third assumption can be added, (3) $\hat{\theta}_1$ and $\hat{\theta}_2$ are independent. This means the final recorded bearings are independent and identically distributed, however the indexing parameter σ^2 is allowed to change with time since it is construed as a random component of variance. Qualitatively, this means that while there are enough factors such as noise and terrain operating on the observations individually to warrant treating

them as independent observations other factors such as movements by the subject and various environmental factors such as wind serve to influence the observations similarly enough to warrant treating a pair of bearings θ_1 and θ_2 as having the same distribution during the period of time they were both measured. More intricate assumptions than this are possible but the above seem reasonable and have the potential to be robust.

To this point only a measure of distance has been given to measure bearing errors along with various general requirements for a family of distributions to satisfy before being seriously considered for investigation. The truncated normal distribution family would seem a reasonable choice for this family since as a family it is adaptable to the conditions just placed on $\Delta \hat{\theta}$. Being a simple variation of the familiar normal distribution, one would hope it would retain some of the utility shown over the years by the univariate and bivariate normal distributions. And, as Appendix I demonstrates, it yields reasonable results when compared with data collected by Sargeant, Forbes and Warner (1965).

Defn 4.3.4 - The distribution function of a standard normal random variable z will be denoted as $N(\lambda) = P[z \leq \lambda]$. The distribution function of a truncated standard normal random variable z_t with parameter L will be defined as $N_L(\lambda)$ or

$$P[z_t \leq \lambda] = \begin{cases} 0 & \text{if } \lambda < -L \\ \frac{N(\lambda) - N(-L)}{2N(L) - 1} & \text{if } -L \leq \lambda \leq L \\ 1 & \text{if } \lambda > L \end{cases} \quad (4.3.3)$$

for some constant $L > 0$.

For the present the value of L will be left unspecified except that in view of Lemma 4.3.1, $\pi > L\sigma > 0$. For a given value of σ^2 both D_1/σ and D_2/σ will be treated as truncated standard normal random variables where D_1 and D_2 denote errors in the bearings. When σ^2 is treated as a random variable the distributions will become conditional on σ^2 .

Lemma 4.3.3 - As $L \rightarrow +\infty$, $N_L(\lambda) \rightarrow N(\lambda)$ so that z_t converges weakly to z in Defn 4.3.4 as $L \rightarrow +\infty$.

Pf - As $L \rightarrow +\infty$, $N(L) \rightarrow 1$ and $N(-L) \rightarrow 0$ so that $N_L(\lambda) \rightarrow N(\lambda)$. The result follows.

See Lamperti (1966) page 52, Thm 2.3

Lemma 4.3.4 - Let z_t be a truncated standard normal random variable, then (1) $Ez_t = 0$, and (2) $\text{Var}(z_t) = 1 - \frac{2 \exp(-L^2/2)}{2(2N(L) - 1)}$.

$$\text{Pf} - (1) \ E z_t = \frac{1}{\sqrt{2\pi} (2N(L) - 1)} \int_{-L}^L z_t \exp(-z_t^2/2) dz_t = 0$$

$$(2) \ E(z_t)^2 = \frac{1}{\sqrt{2\pi} (2N(L) - 1)} \int_{-L}^L z_t^2 \exp(-z_t^2/2) dz_t$$

$$= 1 - \frac{2 \exp(-L^2/2)}{2(2N(L) - 1)}$$

Since $E z_t = 0$ then $\text{Var}(E z_t) = E z_t^2$.

Defn 4.3.4 can be extended for the bivariate case of two independent truncated standard normal random variables. This must be done due to the bivariate nature of $\Delta \hat{\theta}$. For a bivariate random vector $\Delta \hat{\theta}$, the bivariate distribution and density functions follow as products of the function given in expression (4.3.3) due to the hypothesized independence of the components. Similarly to Lemma 4.3.3, as $L \rightarrow +\infty$ the truncated bivariate standard normal distribution function converges weakly to the untruncated bivariate standard normal distribution function. Since L will be moderately large in this thesis, normal theory statistics will be used for estimation since the aforementioned weak convergence should serve to make them quite robust. The variance of the truncated standard normal distribution given in Lemma 4.3.4 is virtually unity for $L \geq 4$, which gives some indication of the adequacy of the approximation.

Section 4.4 - Loss Function Construction Using Mean Squared Error

As stated in the introduction to this chapter, a primary consideration in evaluating a tracking system is how well the tracking system performs when estimating a subject's location as time passes. One natural way to judge two different systems with respect to tower allocation (TA) is to compare the sequences of locations generated by each and declare one system as superior to the other if it succeeds in producing estimates which are "better". Using statistical decision theory, such a quantification of performance and attendant comparison, can be based on a loss function which measures the severity of departures from some optimum state. If such an approach is acceptable, the problem remains as to the properties of an adequate loss function. A loss function which has been used in estimation problems by statisticians is a quadratic loss function of the form $\lambda(\hat{c}, c) = A(c) (\hat{c} - c)^2$, in the case of a single parameter c . In the estimation problem at hand, given a subject stationary for the time it takes to obtain two signals of adequate power relative to the noise (as described in Section 3.4), there are two parameters to

estimate, namely - both coordinates of the subject's location $\underline{x} = (x, y)$. A quadratic loss function for this 2 parameter problem could be $\ell(\hat{\underline{x}}, \underline{x}) = (\hat{x} - x)^2 + (\hat{y} - y)^2$ with $A(\underline{x}) = 1$. This loss function measures the squared distance from $\hat{\underline{x}}$ to $\underline{x}$ - the observed point to the true but unknown point in Euclidean 2-space. Since the point $\underline{x}$ is unknown and since $\hat{\underline{x}}$ can be considered a random variable, it is convenient to study the expectation of $\ell(\hat{\underline{x}}, \underline{x})$ referred to as the expected mean squared error (EMSE) by statisticians. The distribution used to calculate $E[\ell(\hat{\underline{x}}, \underline{x})]$ is induced by the error distribution of the bearings $\hat{\theta}$ about θ . Unfortunately, as the following demonstrates, the expected mean squared error (EMSE) fails to exist in different parts of F^2 irrespective of the particular truncated continuous distribution selected for $\hat{\theta}$ about θ . The nonexistence of the EMSE in different part of F^2 will follow as a direct result of mathematical properties of $T: \mathcal{B} \rightarrow F^2$. Since it is undesirable to base an analysis on an optimality criterion which does not always exist, the search for a suitable loss function will be taken up in the next chapter. A discussion of estimation from a decision theoretic point of view is given in Lindgren (1968) p. 266. After considering some of the following discussion pertaining to Figures 4.4.1, 4.4.2 and 4.4.3, the nonmathematically inclined reader can ignore the remainder of this section.

The central proof of this section will be based on a particular class of distributions to which the truncated standard normal distribution of Defn 4.3.4 belongs. This class of distributions is defined below, and certain comments regarding the use of this class are made prior to proving the main results of this section on pages 72 - 79.

Defn 4.4.1 - A distribution function $F(x)$ is defined as continuous if $F(x)$ is everywhere continuous and if $F(x)$ has a density function continuous for all values of x except on a set of measure zero.

Defn 4.4.2 - A distribution function $F(.)$ will be called continuous, truncated and nonzero if (1) there exists an interval (a, b) over which $F(.)$ is continuous, and (2) $F(.)$ is truncated such that $F(a) = 1 - F(b) = 0$ but $0 < F(x) < 1$ for all $x \in (a, b)$, and (3) for any $x, y \in [a, b]$ with $y < x$ the difference $F(x) - F(y) > 0$.

Defn 4.4.3 - The class of continuous, truncated and nonzero distributions over the interval $[-L, L]$, $L > 0$ will be written $C[-L, L]$.

Lemma 4.4.1 - $N_L(.)$, the truncated standard normal distribution with parameter $L > 0$ is a member of $C[-L, L]$.

Pf - Continuity over $[-L, L]$ follows from Defn 4.3.4 and the continuity of the standard normal distribution function. By Defn 4.3.4, $N_L(-L) = 0$ and $N_L(L) = 1$.

$$\text{If } -L < \lambda < L, N_L(\lambda) = \frac{N(\lambda) - N(-L)}{2N(L) - 1}.$$

Since $N(-L) < N(\lambda) < N(L)$ it follows in this case that

$$0 < N_L(\lambda) < 1. \text{ Also, } N_L(x) - N_L(y) = \frac{N(x) - N(y)}{2N(L) - 1} > 0$$

if $x > y$. $\square$

Given a member $G(\cdot)$ of $C[-L, L]$ with $\pi > \sigma_L > 0$ that is suitable as a distribution function for the bearing errors at either tower, a joint distribution function for $\Delta \hat{\theta}$ can be constructed. Due to tower independence the joint distribution $H(\cdot, \cdot)$ is the product of $G(\cdot)$ with itself.

$$H(x, y) = G(x) \times G(y) \quad (4.4.1)$$

Geometrically speaking, this means that given a location in F^2 with bearings (θ_1, θ_2) , a tracking system generates estimates $(\hat{\theta}_1, \hat{\theta}_2)$ of these bearings such that all these estimates have differences $(D(\theta_1, \hat{\theta}_1)/\sigma, D(\theta_2, \hat{\theta}_2)/\sigma)$ which fall in a square of length $2L$ centered at the origin.

Defn 4.4.4 - Let $\underline{\theta} \in \mathcal{B}$ (see Figure 3.3.3) be given.

The sample space associated with $\underline{\theta}$ will be denoted as $S_L(\underline{\theta}, \sigma)$ and consists of all observable bearings $\hat{\underline{\theta}}$ for which $\left| \frac{D(\underline{\theta}_1, \hat{\underline{\theta}}_1)}{\sigma} \right| \leq L$ and

$$\left| \frac{D(\underline{\theta}_2, \hat{\underline{\theta}}_2)}{\sigma} \right| \leq L \text{ with probability one such that}$$

for $0 < \epsilon < L$ there is a positive probability for at least one of $\left| \frac{D(\underline{\theta}_1, \underline{\theta}_1)}{\sigma} \right|$ or $\left| \frac{D(\underline{\theta}_2, \underline{\theta}_2)}{\sigma} \right|$ to exceed ϵ .

If one is willing to dispense with the modular adjustment implied by $D(\cdot, \cdot)$, the square of length $2L$ mentioned above, which symbolized the sample space associated with each $\underline{\theta} \in \mathcal{B}$, can be superimposed on $\mathcal{B}$ centered at any $\underline{\theta}$. By so doing it becomes geometrically clear what points $\underline{\theta}$ have the potential of generating estimates which do not belong to $\mathcal{B}$. Some of the possibilities are given in Figure 4.4.1 on page 65. Whenever the square of length $2L$ is centered at some $\underline{\theta} \in \mathcal{B}$ and extends outside of $\mathcal{B}$, there is the possibility of some observations not being transformable which means these observations do not correspond to points of F^2 .

There is a problem that arises when the sample space $S_L(\underline{\theta}, \sigma)$ is treated as a square superimposed on

Figure 4.4.1 - Superimposition of Various Sample Spaces on a_u for Some L and σ

$\theta \in \mathcal{B}$. Some of the square will contain points outside of $\mathcal{B}$ which after addition or subtraction of 2π would be points of $\mathcal{B}$. These points do in fact correspond to points of F^2 . The cases to be concerned with are indicated in Figure 4.4.1 on page 65.

Since the parameters L and σ delineate the boundaries of $S_L(\theta, \sigma)$ variations in the magnitude of L have an influence on which observations will be obtained. If $\sigma L = \pi/4$, only the points $(\pi/4, 3\pi/4)$ and $(7\pi/4, 5\pi/4)$ have sample spaces contained within $\mathcal{B}$. If $\sigma L > \pi/4$ every point of $\mathcal{B}$ has the potential of generating observations outside $\mathcal{B}$ which cannot be transformed to F^2 . For $\sigma L \leq \pi/4$, there will always be a maximal set in $\mathcal{B}$ which contains the bearings of all locations which will always generate transformable observations. The portion of this set in $\mathcal{B}$ has been given in Section 4.2 in Figure 4.2.2. Its analogue in F^2 is given in Figure 4.2.3 .

The fact that some observed bearings measured on certain points θ of $\mathcal{B}$ will not be transformable to F^2 prompts a search for some means of selecting a point $\theta^* \in \mathcal{B}$, given $\hat{\theta} \notin \mathcal{B}$, to serve as an estimate of θ and also provide a point estimate $T(\theta^*)$ of $T(\theta)$ in F^2 . (A confidence region for θ given $\hat{\theta} \notin \mathcal{B}$ can always be created.) Natural statistical procedures to use in devising selection rules are estimation procedures based on the maximum likelihood or on the least squares principle. The next lemma shows

the least squares principle to be insufficient in this situation.

Lemma 4.4.2 - Let $\hat{\theta} \in S_L(\theta, \sigma)$ for some unknown $\theta \in \mathcal{B}$ and assume $\hat{\theta} \notin \mathcal{B}$. Then no point of $\mathcal{B}$ can be selected as a least squares estimate of θ given $\hat{\theta}$.

Pf - Suppose θ^* is an element of $\mathcal{B}$ and also a least squared estimate of θ given the observation $\hat{\theta}$. Since $\mathcal{B}$ is an open set, there exists an open disk centered at θ^* wholly inside $\mathcal{B}$. The line segment from θ^* to $\hat{\theta}$ must cross the boundary of this disk, hence there must be another point of $\mathcal{B}$ between θ^* and $\hat{\theta}$ closer to $\hat{\theta}$ than θ^* . This contradicts the assumption that θ^* is a least squares estimate of θ minimizing the distance from $\hat{\theta}$ to $\mathcal{B}$.

Lemma 4.4.2 indicates that maximum likelihood estimation with truncated standard normal distributions is not possible. In fact any distribution of $C[-L, L]$ which depends on the minimization of the quantity $(D(\theta_1^*, \theta_1))^2 + (D(\theta_2^*, \theta_2))^2$ for the selection of a maximum likelihood estimator θ^* will not provide such estimators in this case. The possibility is open that a distribution asymmetric with respect to the origin may yield maximum likelihood estimators, however such a possibility will not be investigated here.

One further point needs to be mentioned. The transformation from the set $S_L(\underline{\theta}, \sigma) \cap \mathcal{B}$ into F^2 will induce a distribution in F^2 which will be improper should part of $S_L(\underline{\theta}, \sigma)$ with nonzero probability extend outside of $\mathcal{B}$. This distribution can be normalized by $P[\hat{\underline{\theta}} \in S_L(\underline{\theta}, \sigma) \cap \mathcal{B}]$.

In what follows it will be sufficient to restrict attention to the lower half of $\mathcal{B}$, or the set $\mathcal{A}_u$ defined in Chapter 3, and distinguish between different cases. Results true for $\mathcal{A}_u$ will follow by symmetry for the remainder of $\mathcal{B}$. Certain topological concepts will be useful in the discussion that follows. Definitions for the concepts of a set's interior, boundary and closure can be found in Mendel-
sen (1962) p. 100-101. His notation will be followed here in that given a set A the closure of A will be denoted $c(A)$ and the interior as $\text{int}(A)$. Two cases are considered before showing that the EMSE fails to exist for certain points of F^2 .

Case I - $0 < L \sigma < \pi/4$

$$E_1 = \{\underline{\theta} \mid S_L(\underline{\theta}, \sigma) \cap \mathcal{A}_u = S_L(\underline{\theta}, \sigma)\} \quad (4.4.2)$$

$$E_2 = \{\underline{\theta} \mid S_L(\underline{\theta}, \sigma) \cap \mathcal{A}_u = S_L(\underline{\theta}, \sigma) \text{ and } (\theta_1 + L, \theta_2 - L) \in \mathcal{B}\} \quad (4.4.3)$$

$$E_3 = \{\underline{\theta} \mid S_L(\underline{\theta}, \sigma) \cap \mathcal{A}_u = S_L(\underline{\theta}, \sigma) \text{ and } (\theta_1 + L, \theta_2 - L) \notin \mathcal{B}\} \quad (4.4.4)$$

Figure 4.4.2 on page 69 gives the sets E_1 , E_2 and E_3 for some

Figure 4.4.2 - The Sets E_1 , E_2 and E_3

(Solid lines indicate boundary possession by a set. Dotted lines are not in E_2 or E_3 .)

Figure 4.4.3 - The Sets $\mathcal{I}(E_1)$, $\mathcal{I}(E_2)$
and $\mathcal{I}(E_3)$

All points on and outside this dotted line belong to the set $\mathcal{I}(E_2)$

hypothetical value of $L\sigma$. On geometric grounds it is clear

$A_u = E_1 \cup E_2 \cup E_3$ and that these sets are mutually exclusive. In Chapter 2 it was shown that $\mathcal{T} : \mathcal{B} \rightarrow F^2$ and its inverse establish a homeomorphism between $\mathcal{B}$ and F^2 .

Consequently, any closed, bounded and connected set in $\mathcal{B}$ corresponds to a closed, bounded and connected set in F^2 .

(Mendelson(1962) p. 119 and 167) The set E_1 is open but $c(E_1)$ is closed and also a connected, bounded and proper subset of

$\mathcal{B}$. Hence $\mathcal{T}(c(E_1))$ is closed, bounded and connected in F^2 .

Let $\underline{x}$ be a point of $\mathcal{T}(E_1)$ so that $\mathcal{T}^{-1}(\underline{x})$ is in E_1 . Then the sample space is a closed, bounded, connected and proper subset

of A_u which means the distribution induced by $\mathcal{T}$ for each

$\underline{x} \in \mathcal{T}(E_1)$ is bounded and has moments of all orders. See

Figure 4.4.3 on page 70 for the set $\mathcal{T}(E_1)$.

Different reasoning shows $\mathcal{T}(E_2)$ to be a bounded set. By its definition in expression 4.4.3, the corner point $(\theta_1 + L,$

$\theta_2 - L) \in \mathcal{B}$. This means all points of $S_L(\underline{\theta}, \sigma)$,

$\underline{\theta} \in E_2$ are bounded away from the line $\theta_1 = \theta_2$. Hence

$|\tan \theta_2 - \tan \theta_1| \geq \epsilon > 0$, ϵ some constant depending on $\underline{\theta}$.

Thus even when $\tan \theta_1$ or $\tan \theta_2$ is undefined, examination

of (3.3.3) and (3.3.4) shows bounded expressions exist for

x and y when $\underline{\theta} \in S_L(\underline{\theta}, \sigma) \cap A_u$. Therefore, the improper induced distributions for points $\underline{x}$ of F^2 with

$\underline{x} \in \mathcal{T}(E_2)$ have moments of all orders after normalization.

The case of $\underline{x} \in \mathcal{T}(E_3)$ requires special attention for it

is here that the EMSE fails to exist. The set E_3 is closed, but the proof in Thm 4.4.1 and Cor 4.4.1 below deals only with the interior of E_3 , $\text{int}(E_3)$. The question of the existence of the EMSE along this boundary is still open.

Case II - $\pi/4 \leq L\sigma \leq \pi$

The details of this case are not important to this thesis. The set E_1 of (4.3.2) is void. E_3 always exists and so the EMSE fails to exist in different part of F^2 depending on L and σ .

The following theorem and corollary establish the chief results of this section dealing with the EMSE.

Thm 4.4.1 - Let $0 < L\sigma < \pi/4$, define E_3 as in expression (4.4.4) and let $\underline{x} \in \text{int}(\underline{T}(E_3))$. Assume the distribution of the standardized bearing errors $D(\theta_1, \theta_1)/\sigma$ and $D(\theta_2, \theta_2)/\sigma$ is a member of the class $C[-L, L]$. Then the distribution induced in F^2 about $\underline{x}$ by $\underline{T}$ from $S_L(\underline{T}^{-1}(\underline{x}), \sigma) \cap \mathcal{B}$ is such that $\lim_{c \rightarrow +\infty} P[|\hat{\underline{x}} - \underline{x}| \geq c] > 0$ with $\underline{x}' = (x, y)$ and $\hat{\underline{x}} = \underline{T}(\hat{\underline{\theta}})$.

Pf - The set E_3 is that portion of $\mathcal{A}_u$ strictly above the line $\theta_1 = \theta_2$ and on or below the line $\theta_2 =$

$\theta_1 - 2L$. In this proof only the interior of E_3 is considered and only these points are in $\mathcal{A}_u$ and satisfy

$$\theta_1 < \theta_2 < \theta_1 - 2L \text{ if } \theta < \theta_1 < \pi - 2L$$

$$\theta_1 < \theta_2 < \pi \text{ if } \pi - 2L \leq \theta_1 < \pi \quad (4.4.5)$$

In all cases $0 < \theta_2 - \theta_1 < 2L$. Since this last inequality implies $\theta_2 - L < \theta_1 + L$ or that the corner point $(\theta_1 - L, \theta_2 - L)$ is below the line $\theta_1 = \theta_2$, it

follows that all points of $\text{int}(E_3)$ have associated sample spaces intersecting the line $\theta_1 = \theta_2$ at more than one point. Thus there exist points of any such

$S_L(\underline{\theta}, \sigma)$ arbitrarily close to the line $\theta_1 = \theta_2$ which implies that θ can generate bearings $\hat{\theta}$ such that one or both components of $\underline{T}(\hat{\theta})$ have arbitrarily large absolute values. Since $S_L(\underline{\theta}, \sigma) \cap \mathcal{B} \neq S_L(\underline{\theta}, \sigma)$, it is important to recognize that not all elements of $S_L(\underline{\theta}, \sigma)$ can be transformed by $\underline{T}$ to F^2 .

Restrict attention to the first quadrant of F^2 .

The following results will apply to the remainder of F^2 by symmetry. Let $\underline{x} \in \text{int}(\underline{T}(E_3))$. To complete the proof it will be necessary to construct a set $\Lambda \subseteq$

$S_L(\underline{T}^{-1}(\underline{x}), \sigma)$ for which the absolute value of the error in the x-coordinate of $\underline{T}(\hat{\theta})$ exceeds an arbitrary but fixed constant $c > 0$. Define e_1 and e_2 as in expression (4.4.6) below.

Figure 4.4.4 - Three Possible Strips Used in Defining the Set Λ of Thm 4.4.1

Comment - For definition of e_1 and e_2 see expression (4.4.6).

$$e_1 = \begin{cases} L & \text{if } \theta_1 + L \leq \pi/2 \\ \pi/2 - \theta_1 & \text{if } \pi/2 + L > \theta_1 + L > \pi/2 \end{cases} \quad (4.4.6)$$

$$e_2 = \begin{cases} L & \text{if } \theta_2 > L \\ \theta_2/2 & \text{if } \theta_2 \leq L \end{cases}$$

and require $\theta_1 + e_1 \geq \hat{\theta}_2 \geq \theta_2 - e_2$. Geometrically, this defines a horizontal strip of positive measure across the sample space of $\underline{\theta}$. See Figure 4.4.4 on page 74 for sketches of three possible cases. The strip always intersects the line $\theta_1 = \theta_2$ so it is possible to find points in this strip which make y and x arbitrarily large. Given θ_2 require $\hat{\theta}_2 > \hat{\theta}_1 \geq \theta_1^*$ where $\theta_1^* \in S_L(T^{-1}(x), \sigma) \cap A_u$ and is the smallest value of $\hat{\theta}_1$ which as a bearing from tower one will intersect $\hat{\theta}_2$ as a bearing from tower two so that the error $\hat{x} - x \geq c$. It can be shown in quadrant one that

$$\theta_1^* = \text{Arctan}((\tan \hat{\theta}_2) (\frac{x+c-s}{x+c+s})). \quad (4.4.7)$$

As $c \rightarrow +\infty$, $\theta_1^* \rightarrow \hat{\theta}_2$. Thus given $c > 0$ and $\underline{\theta}$ a set Λ has been defined. All points of Λ are in $S_L(\underline{\theta}, \sigma) \cap A_u$ and yield $|\hat{x} - x| \geq c$. As $c \rightarrow \infty$ Λ collapses to the intersection of $S_L(\underline{\theta}, \sigma)$ with the line $\theta_1 = \theta_2$. Hence, as $c \rightarrow \infty$ points of

generate fewer small values of $|\hat{x} - x|$. Taking $h(.,.)$ as the density of the distribution defined in expression (4.4.1),

$$P[|\hat{x} - x| \geq c] \geq \int_{\theta_2 - e_2}^{\theta_1 + e_1} \int_{\theta_1^*}^{\hat{\theta}_2} h(u, \hat{\theta}_2) du d\hat{\theta}_2 \quad (4.4.8)$$

The density $h(.,.)$ is defined continuous and nonzero almost everywhere along $\theta_1 = \theta_2$. By construction of this density function, it is known to be integrable over all sample spaces that are proper subsets of $\mathcal{B}$. It follows Fubini's Theorem and Fatou's Lemma (Chung(1968) pages 41 and 57) that

$$\lim_{c \rightarrow \infty} cP[|\hat{x} - x| \geq c] \geq \int_{\theta_2 - e_2}^{\theta_1 + e_1} \lim_{c \rightarrow \infty} (c \int_{\theta_1^*}^{\hat{\theta}_2} h(u, \hat{\theta}_2) du) d\hat{\theta}_2$$

But, as $c \rightarrow \infty$, $\theta_1^* \rightarrow \hat{\theta}_2$ so that the integral

$$\int_{\theta_1^*}^{\hat{\theta}_2} c h(u, \hat{\theta}_2) du \rightarrow ch(\hat{\theta}_2, \hat{\theta}_2) (\hat{\theta}_2 - \theta_1^*)$$

$$\text{But, } \tan(\hat{\theta}_2 - \theta_1^*) = \frac{\tan \hat{\theta}_2 - \tan \theta_1^*}{1 + \tan \hat{\theta}_2 \tan \theta_1^*} =$$

$$\frac{2s \tan \hat{\theta}_2}{c \left(\left(1 + \frac{x+s}{c}\right) + \tan^2 \hat{\theta}_2 \left(1 + \frac{x-s}{c}\right) \right)} \quad \text{by (4.4.7).}$$

$$\therefore \hat{\theta}_2 - \theta_1^* =$$

$$\text{Arctan} \left(\frac{2s \tan \hat{\theta}_2}{c \left[\left(1 + \frac{x+s}{c}\right) + \tan^2 \hat{\theta}_2 \left(1 + \frac{x-s}{c}\right) \right]} \right)$$

As $c \rightarrow +\infty$, the first term of the Taylor series expansion becomes dominant so that

$$\lim_{c \rightarrow +\infty} c (\hat{\theta}_2 - \theta_1^*) = \frac{2s \tan \hat{\theta}_2}{1 + \tan^2 \hat{\theta}_2}.$$

$$\text{Hence, } \lim_{c \rightarrow +\infty} cP[|\hat{x} - x| \geq c] \geq$$

$$\int_{\theta_2 - e_2}^{\theta_1 + e_1} h(\hat{\theta}_2, \hat{\theta}_2) \left(\frac{2s \tan \hat{\theta}_2}{1 + \tan^2 \hat{\theta}_2} \right) d\hat{\theta}_2 > 0,$$

since $h(\hat{\theta}_2, \hat{\theta}_2) > 0$ a.e. and $\tan \hat{\theta}_2 > 0$ due to the definition of $\wedge$ and restriction to the first quadrant of F^2 . $\square$

A similar result can be shown for $\left| \hat{y} - y \right|$. Theorem 4.4.1 is important in that it allows conclusions to be drawn about various moments such as in the following corollary.

Cor 4.4.1 - Under exactly the same assumptions and conditions

stated in Thm 4.4.1, it follows that

$$E \left[(x-x)^2 + (y-y)^2 \right], E \left[(x-x)^2 + (y-y)^2 \right]^{\frac{1}{2}}, \\ E \left| \hat{x}-x \right| \text{ and } E \left| \hat{y}-y \right| \text{ fail to exist for} \\ \underline{x} \in \text{int}(\underline{T}(E_3)).$$

Pf - The proof for $E \left| \hat{y}-y \right|$ follows as for $E \left| \hat{x}-x \right|$.

Using a result given in Chang(1968) page 41, it follows

$$\text{that when } E \left| \hat{x}-x \right| < +\infty, E \left| \hat{x}-x \right| \geq \\ \sum_{c=1}^{+\infty} P \left[\left| \hat{x}-x \right| \geq c \right]. \text{ So, let } \epsilon > 0 \text{ be arbitrary} \\ c=1$$

but fixed. Then there exists an even integer $N > 0$ large enough so that

$$\epsilon/2 \geq \sum_{c=N/2}^N P \left[\left| \hat{x}-x \right| \geq c \right].$$

Since $P \left[\left| \hat{x}-x \right| \geq c \right] \geq P \left[\left| \hat{x}-x \right| \geq c+1 \right]$ for every c , it follows that $(\epsilon/2) \geq (N/2) \cdot (P \left[\left| \hat{x}-x \right| \geq N \right])$.

Therefore $\lim_{c \rightarrow +\infty} c P \left[\left| \hat{x}-x \right| \geq c \right] = 0$ when $E \left| \hat{x}-x \right| < +\infty$.

By Thm 4.4.1 it follows that $E \left| \hat{x}-x \right|$ does not exist.

Since $E \left[(\hat{x}-x)^2 + (\hat{y}-y)^2 \right] \geq E \left[(\hat{x}-x)^2 + (\hat{y}-y)^2 \right]^{\frac{1}{2}} \geq E \left| \hat{x}-x \right|$ these higher moments also fail to exist.

Thus the expected distance as well as the expected mean squared error fail to exist for points of $\text{int}(\underline{T}(E_3))$.

The importance of these results will be discussed further in Section 4.5 .

Section 4.5 - Asymptotic Uses of Second Moments

Theorem 4.4.1 and Corollary 4.4.1 have broader implications than those considered in Section 4.4. Assuming different tracking systems are to be compared as to the quality or "accuracy" of the estimated locations they produce, this comparison revolves around the selection of some optimality criterion which quantifies in one way or another the degree of dispersion and bias if any in the distributions of position estimates. There are different approaches to this quantification one being the EMSE and another, yet to be discussed, is the generalized variance (GV). Some of these different approaches need to be reconsidered in light of Corollary 4.4.1 which established under certain conditions for the class of truncated, continuous and nonzero distributions of Defn 4.4.2 that $E|x - \hat{x}|$ does not exist (and so the EMSE does not exist) for every point of F^2 . A similar argument could be given to show $E|y - \hat{y}|$ does not exist for all $\underline{x} \in F^2$. This means that if one wants to quantify the degree of dispersion and bias of sampling distributions prior to comparing tracking systems and have the criterion used in this quantification exist for

all $\underline{x} \in F^2$, no criterion based on the second moments of the sampling distributions can be used. This tends to mitigate the usefulness of the EMSE and generalized variance (GV) to this thesis and provides the motivation for the approach of chapter five; however, it is possible to devise certain asymptotic optimality measures starting with the EMSE and the generalized variance (GV) which will yield informative results. This section will deal with such asymptotic results. An application of one result will be made to the comparison of collinear two tower systems. This application will help substantiate the nature of the trade offs involved in finding an optimum tower allocation. The qualitative discussion surrounding Figure 4.5.1 and 4.5.2 pages 91 and 92 may be of more interest to some readers than the mathematics involved. The discussion is recommended to such readers as a prelude to chapter five and is found at the very end of this section.

Defn 4.5.1 - Let $\underline{x}$ be a n -dimensional random vector with a distribution such that all second order moments exist.

Let $\underline{\mu} = E \underline{x}$. Then the $n \times n$ variance-covariance matrix associated with the distribution of $\hat{\underline{x}}$ is $M = E((\hat{\underline{x}} - \underline{\mu})(\hat{\underline{x}} - \underline{\mu})')$ and the generalized variance (GV) is the determinant of M , $\det(M)$.

The GV was first defined by S.S. Wilks and serves as a measure of the dispersion of a distribution (Fisz(1963) p.90).

Cramer (1964 p. 301) recognizes the GV as being proportional to the square of the volume of his ellipsoid of concentration and that in a univariate situation it is the distribution's variance and in a two dimensional distribution it is

$\sigma_1^2 \sigma_2^2 (1 - \rho^2)$ for ρ = correlation coefficient and σ_k^2 = variance of k^{th} random variable ($k=1,2$). For the sake of completeness the next definition is given.

Defn 4.5.2 - Let $\hat{\underline{x}}$ be an n-dimensional random vector. The expected mean squared error (EMSE) for the distribution of $\hat{\underline{x}}$ about $\underline{x}$ is $E((\hat{\underline{x}} - \underline{x})' (\hat{\underline{x}} - \underline{x}))$, the squared Pythagorean distance from $\hat{\underline{x}}$ to $\underline{x}$.

Keifer (1959) in discussing optimal experimental designs recognizes several optimality criteria of which at least three do not depend on normal distributions. His A-optimum criterion is based on an EMSE and his D-optimum depends on a GV. Keifer points out different properties of importance for each criterion and notes of all the optimality criteria in his paper only D-optimum experimental designs are invariant under non-singular linear transformations of the estimable parametric functions. One conclusion he draws from his work is that, "A precise statement of losses will obviously entail the use of any of a large number of designs not always D-optimum design. However, the discussion of the previous paragraphs should give some workers a good justification for using

D-optimum designs in many settings." Although Keifer was concerned with optimal experimental designs as opposed to the optimal tower allocation for a tracking system, his comments still carry weight. The quotation given above is interesting because it summarizes the present situation. The GV criterion has several good points recommending its use however it does not have as direct an interpretation in physical reality as the EMSE criterion based on an average squared distance from $\hat{\underline{x}}$ to $\underline{x}$.

The next lemma provides a basis for the asymptotic results to follow by showing that any optimality criterion functionally dependent only on moments can be evaluated at any $\underline{x}$ of F^2 if σ^2 is small enough for that $\underline{x}$. One value of σ^2 will not serve this purpose for all $\underline{x}$ of F^2 . This applies to both the EMSE and the GV.

Lemma 4.5.1 - Let L be fixed and $\underline{x}$ an arbitrary but fixed point of F^2 . For sufficiently small σ^2 , the error distribution induced by $\underline{T}$ for the observations made on $\underline{x}$ will have moments of all orders.

Pf - Let $\underline{\theta} = \underline{T}^{-1}(\underline{x})$ with sample space $S_L(\underline{\theta}, \sigma)$ for some fixed σ . By Defn 4.4.4 if σ is small enough depending on $\underline{\theta}$ then $S_L(\underline{\theta}, \sigma) \subset \mathcal{B}$. This implies $\underline{x} \in \underline{T}(E_1)$, defined in expression (4.3.1),

or the analogue of E_1 in the other half of $\textcircled{B}$. This is sufficient to guarantee the existence of moments of all orders. $\square$

Defn 4.5.3 - Let L be given and fixed. For each $\underline{x} \in F^2$ let

$$\Theta_A(\underline{x}) = \lim_{\sigma^2 \rightarrow 0} \frac{E(\hat{\underline{x}} - \underline{x})(\hat{\underline{x}} - \underline{x})'}{\sigma^2} \quad (4.5.1)$$

and,

$$\Theta_D(\underline{x}) = \lim_{\sigma^2 \rightarrow 0} \frac{\det(E(\hat{\underline{x}} - \underline{x})(\hat{\underline{x}} - \underline{x})')}{\sigma^4} \quad (4.5.2)$$

whenever the limits exist. As defined both

$\Theta_A(\cdot)$ and $\Theta_D(\cdot)$ are functions of $\underline{x} \in F^2$.

The discussion to this point has not dealt with the mechanics of comparing two different triangulation systems. To make such a comparison it will be convenient to divorce the baseline of a tracking system from the x -axis by requiring R^2 to have a fixed Cartesian coordinate system with the towers of different tracking systems being expressed in terms of this fixed coordinate system. Then any number of tracking systems can be simultaneously compared at each point of R^2 . Suppose two systems are given simultaneously in R^2 . Each system has its own baseline and with it a new coordinate system relative to the fixed Cartesian coordinate system of R^2 . The two tracking systems can be compared at any given point $\underline{w}$ of R^2 by first transforming $\underline{w}$ into each tracking system's individual

coordinate system and then evaluating one of (4.5.1) and (4.5.2) after the transformations. Consequently $\underline{w}$ has associated with it two values of a single optimality function one for each tracking system. That system yielding the best value of the optimality function is superior at $\underline{w}$. One common way of comparing such values is to use their ratio as a measure of relative efficiency. In the situation at hand, as $\underline{w}$ changes so will this ratio thus generating a surface over R^2 . Variations in this surface above and below unity give an indication where each tracking system is superior to the other.

Defn 4.5.4 - Let a coordinate system be given and fixed in R^2 and let two distinct tracking systems be given and fixed. Let $\underline{\Psi}_1(.)$ and $\underline{\Psi}_2(.)$ denote linear transformations from the coordinate system of R^2 to the respective coordinate system of each tracking system. Let the asymptotic relative efficiency of system 1 relative to system 2 with (4.5.1) and (4.5.2) at $\underline{w}$ of R^2 be respectively:

$$e_{12}^A(\underline{w}) = \frac{\Theta_A(\underline{\Psi}_1(\underline{w}))}{\Theta_A(\underline{\Psi}_2(\underline{w}))} \quad (4.5.3)$$

$$e_{12}^D(\underline{w}) = \frac{\Theta_D(\underline{\Psi}_1(\underline{w}))}{\Theta_D(\underline{\Psi}_2(\underline{w}))} \quad (4.5.4)$$

Where the right hand sides are based on limits
as $\sigma_1^2 \rightarrow 0$ and $\sigma_2^2 \rightarrow 0$ from Defn 4.5.3.

To illustrate the nature of the computations involved, one asymptotic moment will be derived. The computation of the other needed moments follows similarly. Assume $\underline{w}$ of R^2 has been given and transformed into the coordinate system of a fixed two tower system. In terms of this latter tracking and coordinate system, the transformation $\underline{T}$ of (3.3.1) and (3.3.2) applies. $\underline{T}$ has continuous second order partials existing over $\mathcal{B}$ so Taylor's formula can be applied to $\underline{T}$ (Fulks (1961) p. 230). Let $f_{x_k}^{(n)}$ denote the n^{th} order partial of x in (3.3.2) with respect to θ_k , $k=1,2$. Let $f_{y_k}^{(n)}$ be defined similarly. Recall D_1 and D_2 of Definition 4.2.3, $D_k = D(\theta_k, \hat{\theta}_k)$, ($k=1,2$). Then

$$\hat{y} = y + f_{y_1}^{(1)} D_1 + f_{y_2}^{(1)} D_2 + R_y \quad (4.5.5)$$

$$\hat{x} = x + f_{x_1}^{(1)} D_1 + f_{x_2}^{(1)} D_2 + R_x \quad (4.5.6)$$

where the partials are evaluated at $\underline{\theta} = \underline{T}^{-1}(\underline{x})$ and $\hat{\underline{x}} = \underline{T}(\hat{\underline{\theta}})$ and the remainders R_y and R_x adjust for the differences between the random variables $\hat{x}$ and $\hat{y}$ and the first three terms of each expansion. Since D_1 and D_2 are bounded random variables and the partials exist, the remainders are bounded provided σ^2 is sufficiently small to guarantee that all points of $S_L(\underline{\theta}, \sigma)$, the sample space associated with

ϱ , are in the bearing space. Taylor's formula guarantees under these conditions that there exist constants M_x and M_y depending on ϱ such that $|R_y| \leq M_y (L \sigma)^2$ and $|R_x| \leq M_x (L \sigma)^2$. Therefore,

$$\begin{aligned} E(\hat{y}-y)^2 &= E(f_{y_1}^{(1)} D_1 + f_{y_2}^{(1)} D_2)^2 + \\ &E(R_y(f_{y_1}^{(1)} D_1 + f_{y_2}^{(1)} D_2)) + E((f_{y_1}^{(1)} D_1 + f_{y_2}^{(1)} D_2)R_y) + ER_y^2 \end{aligned} \quad (4.5.7)$$

and similarly for $E(\hat{x}-x)^2$. The last three terms of (4.5.7) have upper bound due to the boundedness of R_y .

$$\begin{aligned} |E(R_y(f_{y_1}^{(1)} D_1 + f_{y_2}^{(1)} D_2))| &\leq E|R_y| |f_{y_1}^{(1)} D_1 + f_{y_2}^{(1)} D_2| \\ &\leq M_y (L \sigma)^3 (|f_{y_1}^{(1)}| + |f_{y_2}^{(1)}|). \end{aligned} \quad \text{Similarly,}$$

$$|E(f_{y_1}^{(1)} D_1 + f_{y_2}^{(1)} D_2)R_y| \leq M_y (L \sigma)^3 (|f_{y_1}^{(1)}| + |f_{y_2}^{(1)}|) \text{ and}$$

$$|ER_y^2| \leq M_y^2 (L \sigma)^4. \quad \text{Moreover, since } |E(D_1, D_2)| = 0,$$

$$E(f_{y_1}^{(1)} D_1 + f_{y_2}^{(1)} D_2)^2 = ((f_{y_1}^{(1)})^2 + (f_{y_2}^{(1)})^2) \sigma^2. \quad (4.5.8)$$

$$\text{Hence, } \lim_{\sigma^2 \rightarrow 0} \frac{E(\hat{y}-y)^2}{\sigma^2} = (f_{y_1}^{(1)})^2 + (f_{y_2}^{(1)})^2 \quad (4.5.9)$$

Similarly it can be shown,

$$\lim_{\sigma^2 \rightarrow 0} \frac{E(\hat{x}-x)^2}{\sigma^2} = (f_{x_1}^{(1)})^2 + (f_{x_2}^{(1)})^2 \quad (4.5.10)$$

$$\lim_{\sigma^2 \rightarrow 0} \frac{E(\hat{x}-x)(\hat{y}-y)}{\sigma^2} = (f_{x_1}^{(1)} f_{y_1}^{(1)}) + (f_{x_2}^{(1)} f_{y_2}^{(1)}) \quad (4.5.11)$$

The sum of (4.5.10) and (4.5.9) is (4.5.1). The product of (4.5.9) and (4.5.10) minus the square of (4.5.11) is (4.5.2). Evaluating these partial derivatives and carrying out the necessary algebraic calculations yields (4.5.12) and (4.5.13) for $\underline{x} = \underline{T}(\underline{\theta})$.

$$\begin{aligned} \theta_A(\underline{x}) = \\ \frac{4s^2(\sec^2 \theta_1 \sec^2 \theta_2)(\tan^2 \theta_2 \sec^2 \theta_1 \tan^2 \theta_1 \sec^2 \theta_2)}{(\tan \theta_2 - \tan \theta_1)^4} \end{aligned} \quad (4.5.12)$$

$$\begin{aligned} \theta_D(\underline{x}) = \\ \frac{16s^4 \tan^2 \theta_1 \tan^2 \theta_2 \sec^4 \theta_1 \sec^4 \theta_2}{(\tan \theta_2 - \tan \theta_1)^6} \end{aligned} \quad (4.5.13)$$

These expressions can be simplified by substituting algebraic expressions for the trigonometric expressions. Let r_i = the range from tower i to $\underline{x}$ ($i=1,2$). Then $\tan \theta_k = y/(x+s)$, $\tan \theta_2 = y/(x-s)$, $\sec \theta_1 = r_1/(x+s)$ and $\sec \theta_2 = r_2/(x-s)$. Substitution and simplification yields (4.5.14) and (4.5.15).

$$\theta_A(\underline{x}) = \frac{r_1^4 r_2^4}{4s^2 y^2} \left(\frac{1}{r_1^2} + \frac{1}{r_2^2} \right) \quad (4.5.14)$$

$$\Theta_D(x) = \frac{r_1^4 r_2^4}{4s^2 y^2} \quad (4.5.15)$$

Both functions show similar behavior being unbounded at the baseline and for infinitely long ranges. They are symmetric with respect to the origin and are proportionate to each other along constant contours of the quantity $(1/r_1^2 + 1/r_2^2)$.

This last quantity has an interesting interpretation in terms of electric charges. Given two point charges q_1 and q_2 and a test charge q_0 a distance r_1 from point charge q_1 and r_2 from point charge q_2 , the electric field strength contributed by q_i at the site of the test charge q_0 is $E_i = \frac{q_i}{4\pi\epsilon_0 r_i^2}$

$i=1,2$. To calculate the resultant field strength acting at q_0 the contributions of E_1 and E_2 are vectorially added.

If just the magnitudes of the contributions are added the total magnitude contributed is a scalar and is proportionate to

$(1/r_1^2 + 1/r_2^2)$. (Halliday and Resnick(1962) p. 578-579)

Drawing from this analogy, one can intuitively conceive of each tower of a two tower tracking system measuring a bearing with a level of accuracy which provides a certain amount of information about the bearing. This information becomes less sufficient for estimating the unknown location as the range from tower to location is increased. If this deterioration follows an inverse square law, those locations with

$(1/r_1^2 + 1/r_2^2)$ constant have the same total amount of bearing information available for estimation. Hence, optimality functions measuring quantities related to the total contributed information should be proportionate. Moreover, the points of F^2 forming contours where $(1/r_1^2 + 1/r_2^2)$ is constant should form curves similar to but not identical to the lines of force for equal but opposite point charges. The reason they are not identical is that electrical lines of force represent constant contours of the magnitude of the vectoral sum of E_1 and E_2 rather than the sum of the magnitudes E_1 and E_2 . This is intuitively reasonable too since the information offered by a tower about a bearing is not a directed quantity and so vectoral addition is not appropriate hence the result.

To illustrate how the asymptotic optimality functions can be used to compare tracking systems, a special application of expression (4.5.4) $e_{12}^D(w)$ will be made. Let two tracking systems be given with the same baseline, only let one system have a tower separation of s_1 and the other s_2 , $s_1 \neq s_2$. For simplicity assume the x-axis of the fixed coordinate system of R^2 corresponds with the common baseline. Then (4.5.4) becomes (4.5.16) for some $w' = (x, y)$ of F^2 .

$$e_{12}^D(w) = \left(\frac{s_2}{s_1} \right)^2 \frac{[(x+s_1)^2 + y^2]^2 [(x-s_1)^2 + y^2]^2}{[(x+s_2)^2 + y^2]^2 [(x-s_2)^2 + y^2]^2} \quad (4.5.16)$$

Plots of $e_{12}^D(w)$ in the first quadrant of F^2 for $s_1 = .25$,

$s_2 = 1.0$ and $s_1 = .25$, $s_2 = .5$ are given in Figure 4.5.1 and 4.5.2 pages 91 and 92 respectively. In comparing these figures, one sees in both cases the system with $s_1 = .25$ is superior in the neighborhood of the origin. Moreover, this region of superiority increases as s_2 is increased. This illustrates the nature of the tradeoffs involved in tower allocation. For an animal that roams widely relative to the towers, greater asymptotic efficiency can be obtained over a larger region by separating the towers more than is desirable if the system is to monitor an animal which confines itself to a small region such as the neighborhood of the origin. If such a system is to monitor both types of animals simultaneously the question of the best or optimum tower placement is open.

Figure 4.5.1 - Plot of e (·)
with $S = .25$ and $S = 1.0$

Figure 4.5.2 - Plot of $e_{12}^D(.)$ With $S_1 = .25$ and $S_2 = .5$

Section 5.1 - Introduction

This chapter crystallizes the preceding development and presents a decision theoretic model that can be used to investigate the issues raised in the introduction of this thesis - most notable the tower allocation problem. This model incorporates the information deemed necessary in Chapter 2 for a realistic solution of the tower allocation problem. In addition it is amenable to a stochastic treatment of radiowave propagation in a heterogeneous environment, and a components of variance model for the bearing error distribution's variance - both of which were discussed in Chapter 3. Furthermore, this decision theoretic model uses an optimality function which does not depend on the existence of second moments in accordance with the discussion of Chapter 4.

Section 5.2 deals with the selection of a loss function to serve as an optimality criterion. The adaptability of this loss function to all the above issues is discussed in Section 5.4. Section 5.3 contains the derivation of a stochastic model for radio wave propagation.

Section 5.2 - Probabilistic Loss Functions

In devising a loss function to serve as an optimality criterion for differentiating between different tower sites, it is necessary to relate the performance of a tracking system placed at certain sites to those factors influencing this performance. At this point no direct attempt has been made to relate the influences of the environment, the distances between transmitter and antennas and other factors to the resultant transmitter signal power at each tower and the bearings measured on the basis of this signal. The various influences acting upon the signal were discussed in Section 3.4 but were not used in Chapter 4 as it was assumed that signals of adequate power relative to the noise could always be obtained so that bearings could always be measured and treated as observed values of random variables with a particular distribution about the true but unknown bearings. This assumption now will be enlarged upon.

The procedure followed at the CCS in extracting bearings is to review the signal image pairs for an animal (a pair of images being collected on approximately simultaneous rotations of the antennas) photographed during a 5-10 minute period and select the pair with the best signal image clarity for reading or bearing determination. If during this period no pair is judged adequate on the basis of the clarity of signal images no bearings are determined.

In the absence of animal movement, signal image clarity reflects directly the signal-to-noise ratio so that in selecting good quality film images the reader is selecting those signals with higher signal-to-noise ratios. When no images are selected for reading, the signal-to-noise ratio is low. So, let r_1 denote the level above which the signal-to-noise ratio must be if a reader is to select an image for reading. A signal image pair is read only if both signals satisfy this requirement and this depends on the location of the transmitter and the type of transmitter. Since different animals have different transmitters with different transmission ranges it is necessary to differentiate with respect to species. Let $E_i(\underline{x})$ denote the event that a signal pair obtained from location $\underline{x}$ on species i has acceptable clarity. Assuming consecutive signal pairs to be independent and that the radio-tagged animal is stationary during the time it takes to measure n consecutive signal pairs, all pairs are reviewed and the probability of at least one pair being acceptable and so yielding a bearing pair is

$$P_i^n(\underline{x}) = 1 - (1 - P(E_i(\underline{x})))^n \quad (5.2.1)$$

where $P(E_i(\underline{x}))$ equals the probability of occurrence of the event $E_i(\underline{x})$. This raises an important point. Since it is not a certainty that n consecutive signal pairs will yield an observed pair $\hat{\theta}$, the distribution used for the bearing errors needs to be viewed as a conditional distribution - conditional on the event $\hat{\theta}$ is read. This conditional

viewpoint will be important as the discussion progresses.

Allowing for animal movement it will be useful to let

$E(\hat{\theta}, n)$ denote the generalized event that $\hat{\theta}$ is obtained from n consecutive signal images on an individual of species i . In general $P[E(\hat{\theta}, n)] \neq P[E_i(\underline{x})]$, since $E_i(\underline{x})$ requires a stationary animal at point $\underline{x}$ and $E(\hat{\theta}, n)$ does not.

To motivate the loss function which will be investigated further in this thesis, it will be convenient to reconsider the quadratic loss function discussed in Section 4.3. This loss function has a very appealing geometric interpretation, being the squared Pythagorean distance between two points in Euclidian 2-space, which recommends it for use in studying estimation problems of triangulation systems. As pointed out in Chapter 4 the expectation of this function (EMSE) does not exist for all distributions of observations induced by $\underline{T}$ in F^2 for a particular family of bearing error distributions. However, for those locations in F^2 when the EMSE exists, it offers information not only of a metric nature but also of a probabilistic nature. Chebyshev's inequality forges a link between the two. In the case at hand assuming the event $E(\hat{\theta}, n)$ has occurred, σ^2 fixed,

$$\begin{aligned} P[(\hat{x}-x)^2 + (\hat{y}-y)^2 \geq c^2 \mid E(\hat{\theta}, n)] \\ \leq \frac{E[(\hat{x}-x)^2 + (\hat{y}-y)^2 \mid E_i(\hat{\theta}, n)]}{c^2} \end{aligned} \quad (5.2.2)$$

The quantity on the left hand side of (5.2.2) is the probability of an estimated location being outside a circle of radius c centered at $\underline{x}$. This probability always exists (given c , $\underline{x}$ and any distribution for $\hat{\underline{x}}$ about $\underline{x}$) which suggests that it be used in place of the EMSE if it can serve as a measure of loss. The fact it is related (by (5.2.2)) to the EMSE suggests it can serve, however its precise meaning will be considered. The constant $c \geq 0$ either defines a single point or a circle of radius c centered at $\underline{x}$ and the probability given in (5.2.2) measures the relative frequency with which observed bearings on an individual of species i can be transformed and result in an estimated location on or outside of the circle of radius c . If the constant c is chosen to reflect the maximum distance a fix x can be away from $\hat{\underline{x}}$ and still be moderately informative for some specified research purpose, then this probability measures the frequency with which transformed bearings result in unsuitable estimates. Such an occurrence is a loss in the sense that a repeated bearing can never be measured on $\underline{x}$ under exactly the same conditions and the observation $\hat{\underline{x}}$ has a reduced value relative to c in providing information about $\underline{x}$. It is desirable to reduce the frequency of this occurrence to as low a level as possible.

One thing not taken into account (by 5.2.2) is the possibility that the observed bearings may not be in $\mathcal{B}$ as a pair and so cannot be associated with a location of F^2 by the

transformation T . This occurrence is also a loss in that not obtaining an estimate $\hat{x}$ about x from $\hat{\theta}$ is intuitively analogous to obtaining one infinitely far away. It is interesting to note that whenever the EMSE exists such a possibility never arises, since $S_L(T^{-1}(x), \sigma)$ is wholly in $\mathcal{B}$ when the EMSE exists, so all recorded bearings can be transformed. It is when the EMSE fails to exist that there is a positive probability of the above occurrence. Consequently, define the conditional loss function

$$\begin{aligned} l_i(x, c | E_i(x)) = & P[(\hat{x}-x)^2 + (\hat{y}-y)^2 \geq c^2 | E(\hat{\theta}, n)] \\ & + P[\hat{\theta} \notin S_L(T^{-1}(x), \sigma) \cap \mathcal{B} | E(\hat{\theta}, n)]. \end{aligned} \quad (5.2.3)$$

If $S_L(T^{-1}(x), \sigma) \subset \mathcal{B}$, $l_i(x, c | E(\hat{\theta}, n))$ is identical to the probability of (5.2.2). When this is not the case, it measures the overall frequency of both losses. The quantity $1 - l_i(x, c | E(\hat{\theta}, n))$ could be called the gain in information about x relative to c and conditional on $E(\hat{\theta}, n)$.

$$1 - l_i(x, c | E_i(x)) = P[(\hat{x}-x)^2 + (\hat{y}-y)^2 < c^2 | E(\hat{\theta}, n)] \quad (5.2.4)$$

This is the conditional probability given recordable signals that bearings will be measured which are transformable and yield an estimate $\hat{x}$ within the circle of radius c .

Thus it seems $l_i(x) = l_i(x, c | E(\hat{\theta}, n)) P(E(\hat{\theta}, n))$ can serve as a loss function. In section 5.3, $P(E(\hat{\theta}, n))$

will be considered in certain cases and in Section 5.4 uses of this loss function will be elaborated upon. Methods for handling variations in σ^2 will be discussed.

Section 5.3 - A Stochastic Model for Radio Wave Propagation

In this section a model will be developed for the calculation of $P(E_i(x))$ defined in expression (5.2.1). This model will be species specific and will require a certain amount of field information to be applicable to the CCS. Fortunately, all of this information has been either published or is available from researchers using the CCS.

Ideally, given a stationary radio tagged animal at $\underline{x}$, one could evaluate the power output of the transmitter and its height above the ground, the nature of the intervening terrain between $\underline{x}$ and the two antennas, type of vegetation, frequency of signal, the height of the two antennas, noise level and other factors and arrive at a prediction of the signal-to-noise ratio at each antenna and so conclude if the signal image will have sufficient clarity to be read. However, some of these factors are time variant, so this calculation would be valid only for a short time provided it could be started in the first place. Indications are that such a calculation is presently impossible. There have been recent theoretical advances in the study of radio-wave propagation in forest environments and over uneven terrain. (Tamir(1967,

Wait(1967)) However, these advances tend to provide theoretical justification for various empirical results observed by experimentalists rather than extend the prediction capabilities of the theory. As Pienkowski (1965) page 117 observed, detailed path-loss calculations are very complicated. For these reasons they will not be attempted in this thesis in a deterministic framework. Rather, a stochastic model will be utilized.

Fortunately, the transmitter frequencies used in animal tracking studies are in the VHF band from 30 - 300 Mc and this corresponds to the frequencies used by various television stations. Much field research has been carried on by such companies and by the FCC in monitoring television service areas, quantifying the quality of service to the users and in developing models for forecasting the service fields. (Head (1960), Head and Prestholdt(1960), LaGrone(1960), Kirby (1957) and others) Besides, there has been research on radio wave propagation in jungles and over irregular terrain which provides some useful empirical results for this thesis. (Burrows(1966), Herbstreit and Crichlow(1964), Egli(1957) and others)

A relationship which has had reasonable empirical success in the papers cited above and which has been given theoretical justification in a paper by Tamir (1967) is-

$$\frac{P_r}{P_t} = \frac{9h_1^2 h_2^2}{4d^4} \left(\frac{R_o^2}{R^2} \right) F_j^2 F_s^2. \quad (5.3.1)$$

R_o is the radiation resistance of the dipole antenna in free space.

R is its value as affected by the proximity of the ground and foliage.

F_s is the shadow factor that accounts for the curvature of the earth.

F_j is the factor that accounts for the effect of the foliage.

d is the distance from the transmitter to receiving antenna.

h_1 is the height of one antenna and h_2 is the height of the other.

P_r is the received signal power at the receiver's antenna.

P_t is the transmitted signal's power at the transmitter.

At the CCS the transmitter frequency most commonly used is approximately 53 Mc. For this frequency R_o/R can be taken as unity and $(F_s F_j)^2$ as approximately .581 over a wide range of terrain and habitat including what is found at the Cedar Creek Natural History Area. (Egli(1957), Burrows (1966)) In the absence of attenuating factors expression (5.3.1)

becomes

$$\frac{P_r}{P_t} = \frac{9h_1^2 h_2^2}{4 d^4}. \quad (5.3.2)$$

This is sometimes referred to as the free space relationship and is needed for later developments.

As indicated in the preceding section, the major issue determining whether a signal pair is read is whether or not both signals have acceptable signal-to-noise ratios. Cochran and Lord (1963) and Bullington (1957) indicate for the 20 - 80 Mc range in rural areas that the noise level is relatively constant with exceptions being electrical storms, car generators and other factors which are of relatively short duration. Thus, the major variable at the CCS in determining the signal-to-noise ratio is the signal itself. Kirby and Capps (1956) have shown for frequencies in the VHF range that fluctuations in received signal power follow a lognormal distribution over farm land and heavily wooded land. They also demonstrate that there seems to be little if any correlation between fluctuations in signals passing over different terrain or lines of propagation to the receiver. This means $P(E_i(x)) = P(E_i^1(x)) P(E_i^2(x))$ where $E_i^k(x)$ is the event the signal image quality for the signal reaching tower k is acceptable, ($k=1,2$). To calculate these latter probabilities some additional definitions are necessary. Let N_1 denote the average noise level at the CCS. From Section 5.2, r_1 is the minimum signal-to-noise level for reading to occur. Let d_k equal the distance from x to tower k , ($k=1,2$) and let the power transmitted by a transmitter placed on species i be

$P_{t,i}$. Then,

$$P[E_i^k(\underline{x})] = P[S_i^k(\underline{x}) \geq r_1 N_1] = P[\log(S_i^k(\underline{x})) \geq \log r_1 N_1] =$$

$$P\left[\log\left(\frac{S_i^k(\underline{x}) d_k^4}{1.31 P_{t,i} h_1^2 h_2^2}\right) \geq \log\left(\frac{r_1 N_1 d_k^4}{1.31 P_{t,i} h_1^2 h_2^2}\right)\right] \quad (5.3.3)$$

where $S_i^k(\underline{x})$ denotes the received signal power from location $\underline{x}$ on species i at tower k ($k=1,2$). This last probability can be expressed as a standard normal probability if a standard deviation can be found to normalize the differences involved. Kirby and Capps (1956) have observed values on terrain similar to the Cedar Creek Natural History Area ranging from 5.28 to 7.18. A value of 14.39 will be used in this thesis to account for the additional variability introduced by shifts of the transmitter on the animal. This value has units of decibels, an electrical engineering concept. Before 14.39 can be used with base 10 logarithms (as those above) it must be divided by 10.0. Then after division with 1.439 the left hand side of (5.3.3) can be treated as a standard normal deviate following Kirby and Capps (1956).

$$P(E_i^k(\underline{x})) = 1 - N\left(\frac{1}{1.439} \log\left(\frac{r_1 N_1 d_k^4}{1.31 h_1^2 h_2^2 P_{t,i}}\right)\right) \quad (5.3.4)$$

It is not necessary to know all the parameters in (5.3.4).

Based on field experiences the value $r_1 N_1$ corresponds to some

range r_i such that on the average

$$r_1 N_1 \doteq 2.25 P_{t,i} h_1^2 h_2^2 / r_i^4 \quad (5.3.5)$$

Substituting into (5.3.4) and simplifying yields (5.3.6).

$$P(E_i^k(x)) = N(\log \left(\frac{.581 r_i^4}{d_k^4} \right) / 1.439) \quad (5.3.6)$$

This expression should give an adequate approximation for the probability $P(E_i^k(x))$ in the case of an isotropic transmitter that radiates power equally in all directions. A whip type transmitter used on some animals can be placed in this category. However, as Cochran and Lord (1963) point out a loop type transmitter produces a directive field so that the amount of power directed to a receiver depends on the orientation of the collar. Descriptions of this directive field are given by Cochran and Lord. One can think of the plane of the loop as being perpendicular to the plane of the earth. No power is radiated in a line parallel to the earth passing perpendicularly through the center of the loop and maximum power radiation is in the plane of the loop. Relative radiated power contours of the field about the loop would look somewhat like a donut with the loop inside the donut and center of the loop at the center of the donut. As an approximation, assume the loop is vertical and let β be the angle between the line of signal propagation from the center of the loop to a receiver and the line perpendicular to the plane of the loop passing

perpendicularly through the center of the loop. Assume these two lines define a plane parallel to the earth. For

$\pi/2$ or $3\pi/2$, a maximum amount of power is propagated to the receiver. Assume r_i in (5.3.5) corresponds to this power level and call this power output $P_{t_o,i}$. Then following

Cochran and Lord, $P_{t,i}$ equals $P_{t_o,i} \cos^2 \beta$. As a result,

$$P(E_i^k(x)) \doteq \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{\log(.581(r_i/d_k)^4 \cos^2 \beta)}{1.439} \right) d\beta \quad (5.3.7)$$

on the average assuming all rotations of the loop relative to the receiver are equally likely.

Using (5.3.6) or (5.3.7) depending on transmitter type should give an adequate approximation for the likelihood of receiving a readable signal image. This development has been given largely for the CCS. It could serve for other rural locations, but if conditions were to change significantly a different formulation might be more appropriate. A more general formulation than the one given above is beyond the scope of this thesis.

Section 5.4 - Multivariate Extensions of the Probabilistic Loss Function

The situation at present is the following - a loss function has been proposed in Section 5.2 but it applies only to a single stationary individual at a single point $\underline{x}$. It is conditional on the event $E(\hat{\theta}, n)$ of Section 5.2 and it assumes a constant bearing error variance σ^2 despite the fact that σ^2 is conjectured to be the sum of a fixed and a random component. This section will deal with multivariate extension of this loss function. Procedures for handling variation in σ^2 will be discussed. The dependence on a single point $\underline{x}$ will be removed when a tracking system's overall performance is quantified. This quantification of performance will be limited to the case of stationary subjects, and will be adequate for providing a solution to the tower allocation problem. A discussion of decision theory is given in Lindgren (1968).

Let a tracking system be specified so that the tower locations are known and fixed. Assume σ^2 is fixed for the moment and assume there are I different species to be tracked by the system. Also assume it is known what types of transmitter are used on the different species and that the field performance characteristics of the transmitters as indicated in Section 5.3 are known.

Defn 5.4.1 - Define the multivariate loss at the point

$\underline{x} \in \mathbb{R}^2$ as

$$\underline{L}'(\underline{x}) = (l_1(\underline{x}), \dots, l_I(\underline{x})) \quad (5.4.1)$$

To evaluate a tracking system it is necessary to devise a measure of the overall performance of the system on the region of scientific concern and research activity. In the case of the CCS this region of concern is the locality on and about the Cedar Creek Natural History Area. One measure to consider in this endeavor is the expectation of $l_i(\underline{x})$ over the study area for each species. However such an expectation necessitates the specification of a distribution over the study region so that each point $\underline{x}$ can be weighted according to its relative importance to other locations. The specification of such a distribution was discussed in Chapter 2 and can be as empirical or as subjective as is desired. It can be made to reflect the relative importance of different regions to different investigators or the geographic distribution of the study animals at some given time during the year. In this thesis it will be convenient to express various results in terms of the density function of such a distribution.

Defn 5.4.2 - Let $f_j(\cdot)$ $j=1, \dots, I$ denote a density function associated with species j which weights locations of a study region relative to each other in their importance to the research mission of the scientists

using a tracking system in their scientific investigations on the study region. This function is assumed to have sufficiently well behaved properties to be integrable.

Defn 5.4.3 - Let σ^2 be fixed and let $l_j(\underline{x})$ and $f_j(\underline{x})$ ($j=1, \dots, I$) be given and fixed. Assume a tracking system with fixed tower locations is given. Then the overall risk associated with σ^2 and this tracking system on the study region Ω in tracking an individual of species j with density $f_j(\underline{x})$ is R_j defined below. The multivariate risk is $\underline{R}' = (R_1, \dots, R_I)$.

$$R_j = \iint_{\Omega} l_j(\underline{x}) f_j(\underline{x}) d\underline{x} \quad (5.4.2)$$

Defn 5.4.4 - Let M_{Ω} denote the set of feasible tower allocations for the study region Ω , as specified by the users of Ω . Let m_s denote an element of this set.

Given σ^2 , I species on Ω and $f_j(\cdot)$ all fixed, every m_s of the set M_{Ω} has associated with it a vector of risks $\underline{R}$. As discussed in Chapters 2 and 4, the selection of a tower allocation in M_{Ω} as an optimum allocation depends on the goals of the system users. Different tower allocations

are better for certain species than for others and in the case where $I > 1$ and the species inhabit different locations of Ω the problem can be unresolvable without some indications as to how the tracking capabilities of the system should be allocated across the different species. The importance on this point was demonstrated in Figures 4.5.1 and 4.5.3 of Chapter 4 and will be considered again in Chapter 7.

To apply decision theory to the tower allocation problem, it is necessary to associate a single number with each m_s of M_Ω on the basis of the risk vector calculated for m_s . No one procedure for doing this is considered best by statisticians and so the choice is open to the users of Ω . Various possibilities are given below.

Case I - σ^2 fixed and known

- (1) In the absence of a ranking of the species or a set of preference weights, it is possible to apply a minimax type procedure to the components of $\underline{R}$ and select the largest component to represent m_s . Then, since σ^2 is known and fixed the set M_Ω can be searched for that member which minimizes this maximum component. In the future this procedure will be denoted by $\min_{M_\Omega} (\max_j R_j)$. The chief virtue of this procedure is that it requires no preference weights and it guards against any one species being overly ignored by the selected m_s . Any unselected

system will monitor at least one species less well than the species monitored the worst by the selected m_s .

- (2) Presuming preference information can be given in the form of weights $\underline{w}' = (w_1, \dots, w_I)$ where

$$\sum_{j=1}^I w_j = 1. \text{ Then the inner product } \underline{w}' \underline{R} \text{ reflects}$$

a center of mass or balance point for the vector of risks relative to the weights. It also serves to order the members of M_Ω according to the expressed research priorities of the system. By searching the set M_Ω a system can be found which lowers this average risk to a minimum. With this procedure, there is the possibility that gains may be obtained in monitoring one species at the expense of greatly increasing the risks on other species.

- (3) The procedure in (2) can be modified to avoid this last problem. If bounds can be given which reflect maximum desirable risks on each species, or component of $\underline{R}$, then those members of M_Ω which satisfy the bounds on the components can be searched as a group and a solution minimizing $\underline{w}' \underline{R}$ over this restricted set of systems can be found if the set is not void. No solution with this procedure can be found if the bounds are made too small.

There are other procedures which can be used in assigning a value to each m_s of M_Ω on the basis of R . Whatever procedure is used, when σ^2 is known and fixed once a value is assigned to each m_s of interest, the problem reduces to a search for that system with the minimum assigned value. When σ^2 is not known and fixed - as in the components of variance model proposed earlier - different methods need to be used. However, these methods can be based on the preceding three procedures.

Case II - σ^2 a random variable

- (1) If no distribution is specified for σ^2 and only a range of values is given, the minimax principle can be applied. For each σ^2 in the specified range, the set M_Ω can be searched for that m_s with the largest ordering value (as determined by some procedure such as the three above). This is done for each value of σ^2 in the specified range and that system yielding the minimum maximum ordering value over the range of σ^2 is selected as "best".
- (2) If a distribution is proposed for σ^2 the minimax principle described above can be used, however a less conservative procedure would be to calculate the expected value for a fixed tracking system of the ordering values treated as functions of the

random variable σ^2 . The "best" in this case would be that tracking system with the smallest expectation.

These procedures are discussed in Lindgren (1968) pages 200 - 211, consequently the interested reader can review them and others elsewhere. No more time will be spent describing such procedures in this thesis. The procedures used for investigating the tower allocation problem are included above. In conclusion, it needs to be recognized that different selection procedures can result in different tower allocations and only the user can determine what seems the appropriate procedure for his situation.

Section 6.1 - Introduction

This chapter contains extensions of various results presented previously for two tower systems to the case of a three tower system with the towers forming a triangle. Section 6.2 extends the topological result of Section 3.2, and Section 6.3 contains a description of $\mathcal{B}^3$ the three tower analogue of $\mathcal{B}$ in Section 3.2. Section 6.4 contains various results from an investigation into the problems of estimation with three towers. Mention is made of possibilities for future research.

Section 6.2 - Basic Mathematics for Three Tower Systems

There are two basic types of arrangements for three tower tracking systems - the towers can be arranged in a line or they can form a triangle. This section deals with the latter arrangement since much of the mathematics for a linear three tower system follows similar lines and since a triangular arrangement will be seen to be superior in one aspect in this section and so is more versatile. Some of the notation and terminology from Sections 3.2 and 3.3 will be useful. Figure 6.2.1 illustrates a three tower system and shows how coordinate systems will be defined in this chapter. Bearings will still be measured in a counterclockwise direction but since a triangular arrangement has no common baseline, bearings will start with the angle bisector of each vertex

Figure 6.2.1— Triangular Three Tower System Showing Terminology and Coordinate Construction

taken as zero. The following definitions formalize this discussion.

Defn 6.2.1 - Let (x_i, y_i) denote the fixed coordinates of each tower of a nonlinear system ($i=1,2,3$) where the origin $(0,0)$ is taken as a point of intersection of the vertex bisectors. Let (x,y) be an arbitrary point of R^2 and define the bearing θ_i of (x,y) to (x_i, y_i) as (1) 0 if (x,y) equals (x_i, y_i) , or as (2) that nonnegative angle less than 2π through which the line segment defined by (x_i, y_i) and $(0,0)$ must be rotated in a counterclockwise direction to become coincident with the line defined by (x_i, y_i) and (x,y) .

Each point of R^2 has a bearing to each tower so that associated with $\underline{x}$ is a triple $\underline{\theta}' = (\theta_1, \theta_2, \theta_3)$. Let F^3 denote R^2 without the coordinates of the towers and let the set of all triples be $\mathcal{B}_3$. Since $\underline{x}$ uniquely determines three line segments - one segment to each tower - $\underline{x}$ uniquely determines a triple $\underline{\theta}$ and $\underline{\theta}$ in turn uniquely determines three lines with one point of intersection at $\underline{x}$. This is in contrast to two tower and linear three tower systems which associate the same bearings with infinitely many points on the baseline. In this sense, triangular three tower systems are superior to linear systems. Thus the sets F^3 and $\mathcal{B}_3$

stand in a one-to-one correspondence. Since $\mathcal{B}_3$ only contains triples associated with points of F^3 , the points of F^3 are mapped onto $\mathcal{B}_3$. Let this mapping be denoted as $\underline{T}(\cdot)$. Since $\underline{T}(\cdot)$ is known to be onto and one-to-one there exists a function $\underline{H}(\cdot)$ mapping from $\mathcal{B}_3$ to F^3 such that $\underline{T}(\cdot)$ and $\underline{H}(\cdot)$ are inverses. The lemma given below allows the derivation of analytic expressions for $\underline{T}$ and $\underline{T}^{-1}$, the inverse of $\underline{T}$.

Lemma 6.2.1 - Let $\underline{x}' \in F^3$ and the triple $\underline{\theta}$ associated with $\underline{x}$ be given and fixed. Let the tower coordinates be $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) . For $i=1,2,3$ it follows that

(1) if $\theta_i \neq \pi/2$ or $3\pi/2$,

$$\tan \theta_i = \frac{yx_i - xy_i}{x_i(x-x_i) + y_i(y-y_i)}, \text{ or}$$

(2) if $\theta_i = \pi/2$ or $3\pi/2$, $0 = x_i(x-x_i) + y_i(y-y_i)$.

Pf - See Hart (1963) page 30, and Defn 6.2.1.

Defn 6.2.2 - Let $\underline{u}' = (u_1, u_2)$ and $\underline{v}' = (v_1, v_2)$ be two points of F^3 . Define $\rho(\underline{u}, \underline{v}) = (u_1 - v_1)^2 + (u_2 - v_2)^2$ which is the squared Pythagorean distance from $\underline{u}$ to $\underline{v}$.

Defn 6.2.3 - Let the metric $D(\cdot, \cdot)$ on $[0, 2\pi)$ be defined as in Section 4.2. Let $\underline{\theta}' = (\theta_1, \theta_2, \theta_3)$ and $\underline{w}' = (w_1, w_2, w_3)$ be points of $\mathcal{B}_3$ and define

$$\tau(\underline{\theta}, \underline{w}) = \sum_{i=1}^3 D^2(\theta_i, w_i).$$

The functions ρ and τ defined above are metrics on the sets F^3 and $\mathcal{B}_3$ respectively. Thus the pair (F^3, ρ) and $(\mathcal{B}_3, \tau)$ are metric spaces.

Lemma 6.2.2 - $\underline{T}: F^3 \rightarrow \mathcal{B}_3$ is continuous with respect to the metrics ρ and τ defined above.

Pf - Let (x_i, y_i) be a tower location. The lines $xy_i - yx_i = 0$ and $x_i(x - x_i) + y_i(y - y_i) = 0$ determine perpendicular axes of a new coordinate system centered at (x_i, y_i) . Let (x, y) be an arbitrary point of F^3 . The bearing of (x, y) to (x_i, y_i) for $i=1, 2, 3$ on the basis of Lemma 6.2.1 is then -

$$\theta_i = \text{Arctan}(u/v) \text{ if } \begin{cases} u = x_i y - y_i x \leq 0 \\ v = x_i(x - x_i) + y_i(y - y_i) < 0 \end{cases} \quad (6.2.1)$$

$$\theta_i = \pi + \text{Arctan}(u/v) \text{ if } v = x_i(x - x_i) + y_i(y - y_i) \geq 0 \quad (6.2.2)$$

$$\theta_i = 2\pi + \text{Arctan}(u/v) \text{ if } \begin{cases} u = x_i y - y_i x > 0 \\ v = x_i(x - x_i) + y_i(y - y_i) < 0 \end{cases} \quad (6.2.3)$$

This defines the function $\underline{T}(\cdot)$ and yields the triple $\underline{\theta}$ associated with $\underline{x}$. Continuity of $\underline{T}$ with respect to

$\tau(\cdot, \cdot)$ will follow if each component of $\underline{T}(\cdot)$ is continuous with respect to $D(\cdot, \cdot)$. Consider the bearing

θ_i of x to tower i separately. If $\underline{x}$ strictly satisfies any set of inequalities in (6.2.1), (6.2.2) or (6.2.3) so

that $\underline{x}$ is not on one of the lines $xy_i - yx_i = 0 = x_i(x - x_i) + y_i(y - y_i)$, then an open set can be found around $\underline{x}$ such that no points of this open set are on the lines or across the lines from $\underline{x}$. Within this open set θ_i varies continuously with $\underline{x}$ because of the continuity of the Arctan(.). Continuity of variation in θ_i is retained as $\underline{x}$ crosses the line $x_i(x - x_i) + y_i(y - y_i) = 0$ by adding π or 2π to the Arctan(.) as in the expressions above. For $\underline{x}$ such that $x_i y - y_i x = 0$ with $x_i(x - x_i) + y_i(y - y_i) < 0$, an open set about $\underline{x}$ contains points of F^3 with bearings close to 0 and some with bearings close to 2π . The metric $D(.,.)$ adjusts for this disparity. $\square$

In summary, it has been noted that Defn. 6.2.1 establishes a one-to-one and onto correspondence between points of F^3 and bearing triples of $\mathcal{B}_3$ which must be invertible as a consequence. Lemma 6.2.1 reflects this fact through analytic geometry and allows quantitative expressions to be developed for calculation of $\underline{\theta}$ from $\underline{x}$. The mathematical expressions in (6.2.1), (6.2.2) and (6.2.3) which describe $\underline{T}: F^3 \rightarrow \mathcal{B}_3$ facilitate the proof that $\underline{T}$ is continuous. Since it is known that an inverse for $\underline{T}$, $\underline{T}^{-1}: \mathcal{B}_3 \rightarrow F^3$ must exist, one would expect the expressions of Lemma 6.2.1 to allow calculation of $\underline{x}$ given $\underline{\theta}$ and that this determination would only depend on two components of $\underline{\theta}$ which specify nondegenerate lines

intersecting at $\underline{x}$. Also, one would expect the bearings to be implicit functions of each other.

Theorem 6.2.1 - Let $\underline{\theta} \in \mathcal{B}_3$ be arbitrary but fixed. Then there exist at least two linear expressions derived from (1) and (2) of Lemma 6.2.1 which allow determination of $\underline{x}$ from $\underline{\theta}$.

Pf - The expressions (1) and (2) of Lemma 6.2.1 result in the description of $\underline{T}$ given in (6.2.1), (6.2.2) and (6.2.3). Let W_i ($i = 1, 2, 3$) be defined as

$$W_i = \begin{cases} \theta_i & \text{if (6.2.1)} \\ \theta_i - \pi & \text{if (6.2.2)} \\ \theta_i - 2\pi & \text{if (6.2.3)} \end{cases} .$$

W_i merely adjusts the bearings to the range $-\pi/2$ to $\pi/2$. Define

$$f_i = W_i - \text{Arctan} \left(\frac{y x_i - x y_i}{x_i(x - x_i) + y_i(y - y_i)} \right), \quad (i=1, 2, 3) .$$

Then $0 = f_1 = f_2 = f_3$ constitutes a simultaneous set of implicit equations involving x, y and the bearings. If one pair of the three pairs of functions f_1 and f_2 , f_1 and f_3 or f_2 and f_3 satisfies certain conditions it will follow that this pair embodies an inverse which can be used to determine $\underline{x}$ as a function of the two bearings used by the pair. The necessary conditions are given

by Taylor (1955) p. 249 and are that f_i and f_j must be continuous, have continuous first partials existing in a neighborhood of $(\underline{\theta}', \underline{x}')$, both functions must vanish when evaluated at $(\underline{\theta}', \underline{x}')$ and that the Jacobian does not vanish at $(\underline{\theta}', \underline{x}')$. By assumption there exists an $\underline{x}$ such that $\underline{\theta} = \underline{T}(\underline{x})$ so that for $(\underline{\theta}', \underline{x}')$, $f_1 = f_2 = f_3 = 0$. Lemma 6.2.2 has shown these functions continuous since W_i serves the purpose of the metric $D(.,.)$. So it remains to show the partials continuous in a neighborhood of $(\underline{\theta}', \underline{x}')$ and the Jacobian not zero at $(\underline{\theta}', \underline{x}')$.

$$\frac{df_i}{dW_i} = 1; \quad \frac{df_i}{dW_j} = 0 \quad (j \neq i)$$

$$\frac{df_i}{dx} = \frac{(y_i - y)(x_i^2 + y_i^2)}{(yx_i - xy_i)^2 + (x_i(x - x_i) + y_i(y - y_i))^2} = \frac{(y - y_i)r_i^2}{U_i}$$

$$\frac{df_i}{dy} = \frac{(x_i - x)(x_i^2 + y_i^2)}{(yx_i - xy_i)^2 + (x_i(x - x_i) + y_i(y - y_i))^2} = \frac{(x_i - x)r_i^2}{U_i}$$

Hence,

$$\det \begin{pmatrix} \frac{(y - y_i)r_i^2}{U_i} & \frac{(x_i - x)r_i^2}{U_i} \\ \frac{(y - y_j)r_j^2}{U_j} & \frac{(x_j - x)r_j^2}{U_j} \end{pmatrix} = \frac{((y - y_i)(x_j - x) - (x_i - x)(y - y_j)) r_i^2 r_j^2}{U_i U_j}.$$

$\frac{df_i}{dx}$ and $\frac{df_i}{dy}$ fail to exist only at (x_i, y_i)

($i=1,2,3$) which are not in F^3 . The expression $(y-y_i)(x_j-x) - (x_i-x)(y-y_j)$ vanishes only for (x,y) on the line determined by the towers (x_i, y_i) and (x_j, y_j) . There are three such lines with no simultaneous intersection but 3 pairwise intersections - one at each tower. Consequently, if (x,y) is along one of these lines, it is always possible to find a pair from f_1, f_2, f_3 satisfying the requirements needed to establish the existence of a continuous local inverse about $\underline{\theta}$. Returning to (6.2.1), (6.2.2) and (6.2.3) and hence to Lemma 6.2.1 it follows that two of the three linear expressions must yield $\underline{x}$ when solved simultaneously given $\underline{\theta} \in \mathcal{B}_3$.

For brevity the specific cases will not be considered since they must follow directly in view of the existence theorem above.

Lemma 6.2.3 - $\underline{T}^{-1} : \mathcal{B}_3 \rightarrow F^3$ is continuous with respect to ρ and τ .

Pf- The determination of (x,y) in cases (1) and (2) of Lemma 6.2.1 depends on $\underline{\theta}$ solely through the values of the tangent function. Fix $\underline{\theta}$. Continuity of $\underline{T}(\cdot)$ guarantees the existence of an open spheroid N_δ

(with $\delta > 0$) equal to the set $\{\hat{\theta} \mid \tau(\underline{\theta}, \hat{\theta}) < \delta\}$ containing $\underline{\theta}$ and also containing infinitely many points of $\mathcal{B}_3$. So as $\tau(\underline{\theta}, \underline{\theta}_i) \rightarrow 0$, for $i=1,2,3$ $D(\underline{\theta}_i, \hat{\theta}_i) \rightarrow 0$. By Defn 4.2.2 this means $|\theta_i - \hat{\theta}_i|$ tends to 0 or 2π . In either case $\tan \hat{\theta}_i \rightarrow \tan \theta_i$ whether the limit is finite or infinite. Let $(\hat{\theta}_n)$ be any sequence of values of $\mathcal{B}_3$ such that $\lim_{n \rightarrow \infty} D(\hat{\theta}_n, \underline{\theta}) \rightarrow 0$. Then the equations mentioned in cases (1) and (2) above which yield $\hat{x}_n = T^{-1}(\hat{\theta}_n)$ tend as $n \rightarrow \infty$ to a set of equations which will yield $\underline{x}$ so that for any $\epsilon > 0$ an integer M_ϵ can be found so that $\rho(\underline{x}, \underline{x}_n) < \epsilon \quad \forall n \geq M_\epsilon$. $\square$

Theorem 6.2.2 - The metric spaces (F^3, ρ) and $(\mathcal{B}_3, \tau)$ are topologically equivalent under the mapping

$$T : F^3 \rightarrow \mathcal{B}_3.$$

Pf - See the preceding discussion and Mendelson (1962) page 78. $\square$

Discarding the metric of a metric space and retaining the open sets generated by the metric yields a different mathematical object called a topological space. Doing this to both (F^3, ρ) and $(\mathcal{B}_3, \tau)$ yields two topological spaces which are homeomorphic due to Theorem 6.2.2. Whether F^3 and $\mathcal{B}_3$ are viewed as topologically equivalent metric spaces under

ρ and τ respectively or as homeomorphic topological spaces, certain facts remain true.

- (1) There is a one-to-one correspondence between open sets of each space.
- (2) Since F^3 is arcwise connected so is $\mathcal{B}_3$ and vice versa.
- (3) Since F^3 is not compact, $\mathcal{B}_3$ is not compact and vice versa.

Item (3) is interesting since $\mathcal{B}_3$ means $\mathcal{B}_3$ is not closed since as a set $\mathcal{B}_3$ is bounded. Item (2) implies that the set $\mathcal{B}_3$ is simply connected.

Section 6.3 - Description of the Set of Triples

One reason for considering the preceding is that it has not been possible to arrive at as simple a description and graphical representation of $\mathcal{B}_3$ as it was of $\mathcal{B}$ in Chapter 3. By proving the metric spaces to be topologically equivalent certain facts are then known about $\mathcal{B}_3$ since they are true in F^3 . In pursuing the preceding results it was mentioned that the bearings should satisfy some constraint. Establishing an analytic expression for this constraint will help to describe the nature of $\mathcal{B}_3$ more fully.

Lemma 6.3.1 - Let $\underline{x} \in F^3$ be given with the triple $\underline{\theta}$ where $\underline{x}$ is not equal to the coordinates of any tower. Let $2\alpha_i$ denote the angle of the vertex formed by tower i for $i=1,2,3$. Then the components of $\underline{\theta}$ satisfy (6.2.4).

$$\begin{aligned} 0 = & \sin 2\alpha_1 \sin(-\theta_1 + \theta_2 + \theta_3) + \\ & \sin 2\alpha_2 \sin(\theta_1 - \theta_2 + \theta_3) + \\ & \sin 2\alpha_3 \sin(\theta_1 + \theta_2 - \theta_3) \end{aligned} \quad (6.3.1)$$

Pf - The angles α_1, α_2 , and α_3 are given in Figure 6.2.1 as are the ranges r_1, r_2 and r_3 . By the law of sines (Spiegel (1968) page 19)

$$\begin{aligned} r_3 \sin(\theta_3 + \alpha_3) &= -r_1 \sin(\theta_1 - \alpha_1) \\ r_1 \sin(\theta_1 + \alpha_1) &= -r_2 \sin(\theta_2 - \alpha_2) \\ r_2 \sin(\theta_2 + \alpha_2) &= -r_3 \sin(\theta_3 - \alpha_3) \end{aligned}$$

Eliminating the ranges yields -

$$\prod_{i=1}^3 \sin(\theta_i + \alpha_i) = - \prod_{i=1}^3 \sin(\theta_i - \alpha_i)$$

By various trigonometric identities it follows that,

$$\begin{aligned} 0 = & \sin(-\theta_1 + \theta_2 + \theta_3) \cos(\alpha_1 - \alpha_2 - \alpha_3) \\ & + \sin(\theta_1 - \theta_2 + \theta_3) \cos(-\alpha_1 + \alpha_2 - \alpha_3) \\ & + \sin(-\theta_1 - \theta_2 + \theta_3) \cos(\alpha_1 + \alpha_2 - \alpha_3) \\ & - \sin(\theta_1 + \theta_2 + \theta_3) \cos(-\pi/2). \end{aligned}$$

Introducing cofunctions yields expression (6.3.1) and completes the proof. $\square$

Due to Lemma 6.3.1, it could be hoped that an adequate description of $\mathcal{B}_3$ might be obtained by projecting $\mathcal{B}_3$ perpendicularly onto the square $[0, 2\pi) \times [0, 2\pi)$ containing only two bearings - say θ_1 and θ_2 - and then obtaining θ_3 by solving the constraint given θ_1 and θ_2 . Part of the proof of Theorem 6.2.1 indicates $\underline{x}$ cannot always be obtained from θ_1 and θ_2 and so θ_1 and θ_2 do not necessarily fix the value of θ_3 . This indicates such a projection will not adequately describe $\mathcal{B}_3$. The following lemma serves to substantiate this view.

Lemma 6.3.2 - The constraint in (6.3.1) determines θ_{i_3} given

θ_{i_1} and θ_{i_2} only up to one of two possibilities α or β where $|\alpha - \beta| = \pi$.

Pf - Only one case will be considered since the others are similar. Let $w_i = \sin(2\alpha_i)$ $i=1,2,3$. It also is true that

$$\sin(-\theta_1 + \theta_2 + \theta_3) = \sin \theta_3 \cos(\theta_2 - \theta_1) + \sin(\theta_2 - \theta_1) \cos \theta_3$$

$$\sin(\theta_1 - \theta_2 + \theta_3) = \sin \theta_3 \cos(\theta_1 - \theta_2) + \cos \theta_3 \sin(\theta_1 - \theta_2)$$

$$\text{and, } \sin(\theta_1 + \theta_2 - \theta_3) = \sin(-\theta_3) \cos(\theta_1 + \theta_2) + \cos(-\theta_3) \sin(\theta_1 + \theta_2).$$

By expression (6.3.1), if $\underline{\theta}$ is a triple,

$$0 = \sin \theta_3 (w_1 \cos(\theta_2 - \theta_1) + w_2 \cos(\theta_1 - \theta_2) - w_3 \cos(\theta_1 + \theta_2)) + \cos \theta_3 (w_1 \sin(\theta_2 - \theta_1) + w_2 \sin(\theta_1 - \theta_2) + w_3 \sin(\theta_1 + \theta_2)).$$

If $w_1 \cos(\theta_2 - \theta_1) + w_2 \cos(\theta_1 - \theta_2) - w_3 \cos(\theta_1 + \theta_2)$ and $w_1 \sin(\theta_2 - \theta_1) + w_2 \sin(\theta_1 - \theta_2) + w_3 \sin(\theta_1 + \theta_2)$ are not both zero, knowledge of θ_1 and θ_2 fixes the value of $\tan \theta_3$ so that θ_3 must be one of the two bearings in $[0, 2\pi)$ yielding this value of the tangent. Since the period of the tangent is π , these values are π apart.

In those cases when the former two quantities are both zero, θ_3 is undefined. ■

Thus, to calculate θ_{i_3} from θ_{i_1} and θ_{i_2} it is necessary to know what portion of $\mathcal{B}_3$ the point $\underline{\theta}$ is in in order to discriminate between the possibilities given above. In some cases it will be impossible to discriminate. It can be proved using Theorem I given by Taylor(1955) page 241 that there exists a neighborhood of any fixed $\underline{\theta}$ in which at least one component of $\underline{\theta}$ can be viewed as a continuous function of the others. The particular functional relationship may change if $\underline{\theta}$ is changed. Despite its limitations, a projection of $\mathcal{B}_3$ is given in Figure 6.3.1. It can be applied to any two components of $\underline{\theta}$, provided that a convention is followed in labeling the axes of the figure. Let (x_i, y_i) and (x_j, y_j) be the coordinates of two towers. Let l_i be the line segment

Figure 6.3.1 — The Set $\mathcal{B}_3$ Projected Onto $[0, 2\pi] \times [0, 2\pi]$

defined by $(0,0)$ and (x_i, y_i) and let l_j be defined similarly. One of l_i and l_j when rotated in a clockwise direction about $(0,0)$ through an angle of $\pi - \alpha_i - \alpha_j$ becomes coincident with the other line. Suppose the rotated line to be l_i . Then the bearings measured at this tower, denoted as θ_i , must be placed on the x-axis of Figure 6.3.1. If this convention is not followed a reflection about $y=x$ of the desired projection will be obtained. In Figure 6.3.1 solid lines and points are included in the projection of $\mathcal{B}_3$. Open circles and dotted lines are not included. It is interesting to note that rearranging the pieces results in a close copy of the set $\mathcal{B}$ in Chapter 3, the chief difference being the inclusion of certain corner points in $\mathcal{B}_3$ that are not included in $\mathcal{B}$. They are $(\alpha_i, \pi - \alpha_j)$, $(\alpha_i, 2\pi - \alpha_j)$ and $(\pi + \alpha_i, 2\pi - \alpha_j)$. They are included because of an infinite number of points of $\mathcal{B}_3$ vertically suspended above them.

Section 6.4 - Some Estimation Theory and Results

The assumptions made in Chapter 4 concerning bearing error distributions are viable for use with three tower systems. If $\hat{\theta}$ is an observed triple of bearings, the components of $\hat{\theta}$ will be considered as independent random variables with a common distribution from the family $C[-L, L]$, for some $L \geq 4$.

Then the expectation $E(\hat{\underline{\theta}}) = \underline{\theta}$ and the variance-covariance matrix $E(\hat{\underline{\theta}} - \underline{\theta})(\hat{\underline{\theta}} - \underline{\theta})' = \sigma^2 I_3$, I_3 the 3x3 identity matrix. The sample space $S_L^3(\underline{\theta}, \sigma)$ associated with $\underline{\theta}$ consists of all triples in Euclidean 3-space such that

$$\left| \frac{D(\underline{\theta}_i, \hat{\underline{\theta}}_i)}{\sigma} \right| \leq L, \text{ for } i=1,2,3. \text{ The union of all such}$$

spaces generates a set of positive volume completely containing

$\mathcal{B}_3$. In comparison, $\mathcal{B}_3$ has no volume so that the probability of observing $\hat{\underline{\theta}} \in \mathcal{B}_3$ is zero. Consequently, with probability one, $\hat{\underline{\theta}}$ will not be transformable into a single point of F^3 , but rather the bearings will determine three lines intersecting at three, two or no points of F^3 . One could attempt to meld whatever intersections are formed into a single point of F^3 ignoring the cases when no intersection occurs. Besides being conditional on the number of observed intersections such an analysis would be complicated by the nature of the induced error distributions in F^3 . Consequently, this estimation problem will be considered in the context of the initial observations and their sample spaces. The set $\mathcal{B}_3$ becomes the parameter space and the space of observations $S_L^3(\sigma) = \bigcup_{\underline{\theta} \in \mathcal{B}_3} S_L^3(\underline{\theta}, \sigma)$. The problem

can then be stated as one of finding a procedure that will map or "project" $\hat{\underline{\theta}} \in S_L^3(\sigma)$ into the subspace $\mathcal{B}_3$ in an optimum way or in such a way that the image point will be in some sense close to $\underline{\theta}$. On general procedure which has

been used in such "projection" problems is the method of least squares. In the problem at hand, a least squares estimator (if it exists) will select the closest point of B_3 as the image of $\hat{\theta}$. Such a procedure deserves consideration here. Since B_3 is difficult to describe, projection to the surface created by the constraint given in expression (6.3.1) will be studied. Since B_3 is embedded in this surface, projection usually will be into B_3 because of the proximity of $\hat{\theta}$, however the possibility exists that a projected point can be outside of B_3 . One other reason for considering the least squares approach is that for truncated normal distributions a least squares estimator is also a maximum likelihood estimator. Thus such estimators have several desirable properties of value in triangulation problems. The least squares estimation problem will be approached as a Lagrangian multiplier problem after certain preparations are made.

Defn 6.4.1 - Assume n observed triples $\hat{\theta}_i$, $i=1, \dots, n$ are available on a single location $\underline{x}$. Using the metric $D(.,.)$ of Defn 4.2.2 define $d_{ji} = D(\hat{\theta}_{ji}, \hat{\theta}_i)$, for $i=1, \dots, n$ and $j=1, 2, 3$ and

$$\bar{d}_j = \frac{1}{n} \sum_{i=1}^n d_{ji} . \text{ Then the average of the}$$

components ($j=1, 2, 3$) is $\bar{\theta}_j = (\hat{\theta}_{j1} + \bar{d}_j) \bmod 2\pi$. Also, let $\bar{\theta}' = (\bar{\theta}_1, \bar{\theta}_2, \bar{\theta}_3)$.

Consequently $D(\theta_j, \hat{\theta}_{ji}) = D(\theta_j, \bar{\theta}_j) + D(\bar{\theta}_j, \hat{\theta}_{ji})$ so

the terms in (6.2.1) can be expressed as :

$$\begin{aligned} \sin(-\theta_1 + \theta_2 + \theta_3) &= \\ \sin(-\theta_{1i} + \theta_{2i} + \theta_{3i} + D(\theta_1, \theta_{1i}) - D(\theta_2, \theta_{2i}) - D(\theta_3, \theta_{3i})) &= \\ = \sin(\underbrace{-\bar{\theta}_1 + \bar{\theta}_2 + \bar{\theta}_3}_{m_1} + \underbrace{D(\theta_1, \bar{\theta}_1)}_{D_1} - \underbrace{D(\theta_2, \bar{\theta}_2)}_{D_2} - \underbrace{D(\theta_3, \bar{\theta}_3)}_{D_3}) &= \\ = \sin(m_1 + D_1 - D_2 - D_3). \end{aligned} \quad (6.4.1)$$

And if $m_2 = \theta_1 - \theta_2 + \theta_3$ and $m_3 = \theta_1 + \theta_2 - \theta_3$ then

$$\sin(\theta_1 - \theta_2 + \theta_3) = \sin(m_2 - D_1 + D_2 - D_3) \quad (6.4.2)$$

$$\sin(\theta_1 + \theta_2 - \theta_3) = \sin(m_3 - D_1 - D_2 + D_3). \quad (6.4.3)$$

Let $w_j = \sin 2\alpha_j$, ($j=1,2,3$) and λ be some undetermined constant. Following Taylor (1955) page 198, the constrained least squares minimization problem amounts to finding the critical points of the function U .

$$\begin{aligned} U = \sum_{j=1}^3 D_j^2 + \lambda [&w_1 \sin(m_1 + D_1 - D_2 - D_3) + w_2 \sin(m_2 - D_1 + D_2 - D_3) \\ &+ w_3 \sin(m_3 - D_1 - D_2 + D_3)] \end{aligned} \quad (6.4.4)$$

as a function of D_1 , D_2 and D_3 .

As the problem now stands, after partial differentiation the desired estimates will be expressed as implicit functions of $\bar{\theta}$ in a set of transcendental equations which cannot be explicitly solved. An iterative routine would yield the solutions given values for $\bar{\theta}$, but this could take

computational time. Alternatives to just this situation have been developed by Neyman (1949) and others. Neyman (1949) created a class of estimators called Best Asymptotically Normal (BAN) which have the same asymptotic properties as maximum likelihood estimators and which sometimes are easier to compute along with sometimes possessing better small sample properties than maximum likelihood estimators. Ferguson (1958) gives a review of known methods for generating BAN estimators. The specific procedure to be used here was first used by Neyman (1949) in studying multinomial populations and has been extended by Barankin and Gurland (1951). The procedure is based on a linearization of the side conditions placed on the parameters of interest. In the problem at hand, estimators of D_1 , D_2 and D_3 are needed given $\bar{\theta}$. Presumably $\bar{\theta}$ is relatively close to θ so the magnitudes of D_1 , D_2 and D_3 should be small enough so that (6.3.1) can be satisfactorily approximated with a truncated Taylor's series.

$$\sin(m_1 + D_1 - D_2 - D_3) \doteq \sin m_1 + \cos m_1 (D_1 - D_2 - D_3)$$

$$\sin(m_2 - D_1 + D_2 - D_3) \doteq \sin m_2 + \cos m_2 (-D_1 + D_2 - D_3)$$

$$\sin(m_3 - D_1 - D_2 + D_3) \doteq \sin m_3 + \cos m_3 (-D_1 - D_2 + D_3)$$

$$\text{Let } \delta_0 = \sum_{i=1}^3 w_i \sin m_i, \delta = w_1 \cos m_1 + w_2 \cos m_2 + w_3 \cos m_3,$$

$$\delta_1 = -\delta + 2w_1 \cos m_1, \quad \delta_2 = -\delta + 2w_2 \cos m_2 \text{ and}$$

$$\delta_3 = -\delta + 2w_3 \cos m_3. \text{ Then,}$$

$$U^* = \sum_{j=1}^3 D_j^2 + \lambda (\delta_0 + D_1 \delta_1 + D_2 \delta_2 + D_3 \delta_3) \quad (6.4.5)$$

From this point the procedure is continued as though no linearization has occurred.

$$\frac{dU^*}{dD_j} = 2D_j + \lambda \delta_j \quad (j=1,2,3) \quad (6.4.6)$$

This yields three simultaneous equations plus the linearized constraint. Solving gives an estimator D_j^* in (6.4.7).

$$D_j^* = \frac{-\delta_0 \delta_j}{\sum_{k=1}^3 \delta_k^2} \quad (6.4.7)$$

Hence an estimator of θ is θ^* where $j=1,2,3$

$$\theta_j^* = \left(\bar{\theta}_j + \frac{\delta_0 \delta_j}{\sum_{k=1}^3 \delta_k^2} \right) \bmod 2\pi \quad (6.4.8)$$

Various properties of (6.4.8) will be demonstrated now and eventually it will be shown to be BAN.

Lemma 6.4.1 - With δ_1 , δ_2 , and δ_3 as defined in (6.4.5)

The probability is zero that

$$\sum_{i=1}^3 \delta_i^2 = 0.$$

$$\text{Pf} - \sum_{i=1}^3 \delta_i^2 = 0 \text{ if and only if } \delta_1 = \delta_2 = \delta_3 = 0.$$

The latter occurs if and only if

$$\begin{pmatrix} -w_1 & w_2 & w_3 \\ w_1 & -w_2 & w_3 \\ w_1 & w_2 & -w_3 \end{pmatrix} \begin{pmatrix} \cos m_1 \\ \cos m_2 \\ \cos m_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This matrix has a nonzero determinant in general which

means $\sum \delta_i^2 = 0$ if and only if $\cos m_1 = \cos m_2 = \cos m_3 = 0$. Hence, for some odd integers n_1, n_2 and n_3

$$\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} \bar{\theta}_1 \\ \bar{\theta}_2 \\ \bar{\theta}_3 \end{pmatrix} = \pi/2 \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}.$$

Inverting gives

$$\begin{pmatrix} \bar{\theta}_1 \\ \bar{\theta}_2 \\ \bar{\theta}_3 \end{pmatrix} = \pi/2 \begin{pmatrix} \frac{n_1 + n_2}{2} \\ \frac{n_1 + n_3}{2} \\ \frac{n_3 + n_2}{2} \end{pmatrix}$$

Therefore, only a finite number of points give

$$\sum_{i=1}^3 \delta_i^2 = 0.$$

It can be shown also that those triples yielding $\sum_{i=1}^3 \delta_i^2 = 0$

do not belong to the closure of the set $\mathcal{B}_3$. This implies that for sufficiently small σ^2 , the set $S_L^3(\sigma)$ will not

include any of the critical triples. It will be important later to have an estimate of the size needed for $L\sigma$ to guarantee that the set $S_L^3(\sigma)$ does not contain any of these critical points. Since $n_i = 1$ or 3 for each i , the critical bearings for the set $[\pi/2, \pi, 3\pi/2]$. The set of triples formed from this set by sampling with replacement includes all cases of importance. By geometric considerations using Figure 6.2.1 it is clear that as long as $\pi/2 - L\sigma > k_1, k_2, k_3$ (defined in this chapter) no point of $\mathcal{B}_3$ exists with a sample space which contains any of these critical triples. This condition is not satisfied if $2k_i \geq \pi - 2L\sigma$ for some i . If this happens this means the tower arrangement is close to linearity relative to the value of $L\sigma$. For the purposes of this thesis it will be a reasonable requirement to have $\pi/2 - L\sigma$ greater than k_i for all i since the practical range for $L\sigma$ is quite small and close to zero. This condition points out that linear three tower arrangements must be considered as a separate case since the estimators devised for triangular three tower arrangements will not always be directly adaptable to the linear case.

Lemma 6.4.2 - If $D(\theta_j, \bar{\theta}_j)$ converges in probability to 0, ($j=1,2,3$), then δ_0 converges in probability to zero. Also, the value of δ_0 is zero when $\bar{\theta}_j = \theta_j$ ($j=1,2,3$).

Pf - By definition $\delta_0 = w_1 \sin m_1 + w_2 \sin m_2 + w_3 \sin m_3$.

Note the left hand side of δ_0 is (5.3.1) if $\bar{\theta}_j = \theta_j$

for $(j=1,2,3)$ which renders $\delta_0 = 0$. Subtracting

(6.3.1) from δ_0 and applying the triangle inequality

yields

$$\begin{aligned} |\delta_0| \leq & w_1 |\sin m_1 - \sin(-\theta_1 + \theta_2 + \theta_3)| + \\ & w_2 |\sin m_2 - \sin(+\theta_1 - \theta_2 + \theta_3)| + \\ & w_3 |\sin m_3 - \sin(+\theta_1 + \theta_2 - \theta_3)|. \end{aligned}$$

Since $|\sin \alpha - \sin \beta| = 2 |\cos \frac{1}{2}(\alpha + \beta) \sin \frac{1}{2}(\alpha - \beta)|$
 $\leq |\sin \frac{1}{2}(\alpha - \beta)|$ it follows that,

$$\begin{aligned} \frac{1}{2} |\delta_0| \leq & w_1 |\sin \frac{1}{2}(m_1 - (-\theta_1 + \theta_2 + \theta_3))| \\ & w_2 |\sin \frac{1}{2}(m_2 - (+\theta_1 - \theta_2 + \theta_3))| \\ & w_3 |\sin \frac{1}{2}(m_3 - (\theta_1 + \theta_2 - \theta_3))| \leq \\ & w_1 |\sin \frac{1}{2}(-D(\theta_1, \bar{\theta}_1) + D(\theta_2, \bar{\theta}_2) + D(\theta_3, \bar{\theta}_3))| + \\ & w_2 |\sin \frac{1}{2}(D(\theta_1, \bar{\theta}_1) - D(\theta_2, \bar{\theta}_2) + D(\theta_3, \bar{\theta}_3))| + \\ & w_3 |\sin \frac{1}{2}(D(\theta_1, \bar{\theta}_1) + D(\theta_2, \bar{\theta}_2) - D(\theta_3, \bar{\theta}_3))|. \end{aligned}$$

Always, $|\sin u| \leq |u|$ and $|w_i| \leq 1$ ($i=1,2,3$). Hence,

$$|\delta_0| \leq 3(|D(\theta_1, \bar{\theta}_1)| + |D(\theta_2, \bar{\theta}_2)| + |D(\theta_3, \bar{\theta}_3)|).$$

By assumption $|D(\theta_i, \bar{\theta}_i)| \xrightarrow{p} 0$ for $i=1,2,3$ so

$$|\delta_0| \xrightarrow{p} 0. \blacksquare$$

The distributions which have been investigated for use in this thesis are such that the hypothesis of Lemma 6.4.2 can always be demonstrated as true. This is verified by Theorem 6.4.1 which concludes more concerning the estimator in (6.4.8).

Theorem 6.4.1 - Let $D(\theta_j, \hat{\theta}_{ji})$ ($i=1, \dots, n$), ($j=1, 2, 3$) be a sequence of independent random variables with a common distribution from the family $C[-L, L]$ with $0 < L\sigma < \pi/2$, then for each $j=1, 2, 3$ the estimator θ_j^* converges almost surely to θ_j as $n \rightarrow \infty$ in the metric $D(.,.)$ i.e., $D(\theta_j^*, \theta_j) \rightarrow 0$ a.s. .

Pf - Since $0 < L\sigma < \pi/2$, $D(\hat{\theta}_{jl}, \hat{\theta}_{ji}) = D(\theta_j, \hat{\theta}_{ji}) + D(\hat{\theta}_{jl}, \theta_j)$. Following Defn 6.4.1 it is true that

$$\bar{d}_{j.} = \frac{1}{n} \sum_{i=1}^n (D(\theta_j, \hat{\theta}_{ji}) + D(\hat{\theta}_{jl}, \theta_j)),$$

and

$$\begin{aligned} \bar{\theta}_{j.} &= (\hat{\theta}_{jl} + \bar{d}_{j.}) \bmod 2\pi = \\ &= (\theta_j + \frac{1}{n} \sum_{i=1}^n D(\theta_j, \hat{\theta}_{ji})) \bmod 2\pi. \end{aligned}$$

Since $D(\theta_j, \theta_{jl}), \dots, D(\theta_j, \theta_{jn})$ for each $j=1, 2, 3$ is a sequence of independent random variables with a common bounded distribution with an expectation of zero the average converges to zero almost surely. Hence $D(\theta_j, \bar{\theta}_{j.}) \rightarrow 0$ almost surely. Recalling expression

(6.4.8) it is immediate that $D(\theta_j^*, \theta_j) = D(\bar{\theta}_j, \theta_j)$

$$+ D\left(\frac{\delta_0 \delta_j}{\sum_{k=1}^3 \delta_k^2}, 0\right). \text{ In light of the proof of}$$

Lemma 6.4.2 which is valid for almost sure convergence it follows that $D(\theta_j^*, \theta_j) \rightarrow 0$ almost surely. $\square$

The discussion from Lemma 6.4.1 to Theorem 6.4.1 concerns various properties of expression (6.4.8) particularly as the number of observed bearings on a particular location is increased to an arbitrarily large value. The following text extends the discussion and concludes (6.4.8) is BAN.

Since θ_1 , θ_2 , and θ_3 are composed of independent random variables, as a group they are independent. Moreover, by a version of the central limit theorem (Lamperti(1966) page 65)

$$\frac{D(\theta_j, \theta_j)}{\sigma\sqrt{n}} = \frac{\frac{1}{n} \sum_{i=1}^n D(\theta_j, \hat{\theta}_{ji})}{\sigma\sqrt{n}} \sim N(0,1) \quad (6.4.9)$$

By independence it follows that $\Delta' = \frac{n}{\sigma} (D_1, D_2, D_3)$ has an asymptotic trivariate normal distribution with mean zero and an identity matrix for the variance-covariance matrix. This is an important observation which allows the discovery of an asymptotic distribution for the vector of differences $(D(\theta_1, \theta_1^*), D(\theta_2, \theta_2^*), D(\theta_3, \theta_3^*))$.

Let $v_j = -D_j^*$ from expression (6.4.7). Suppose that the tower arrangement is such that $\pi/2 - k_i > L\sigma$ for $i=1,2,3$, then v_j is defined at each point of $S_L^3(\sigma)$ and has continuous second order partials existing at each point. Let $\underline{\theta}$ be an element of $\mathcal{B}_3$ and fixed and let $\bar{\underline{\theta}}$ be as in Defn 6.4.1. Then $\bar{\underline{\theta}} = S_L^3(\sigma)$ and the value of v_j at $\bar{\underline{\theta}}$ can be expressed in terms of a Taylor Series with remainder term.

$$\text{Let } f_{jk} = \left. \frac{d v_j}{d \bar{\theta}_k} \right|_{\bar{\theta}_k = \theta_k} \quad (j=1,2,3) \text{ and } (k=1,2,3).$$

Then,

$$v_j = \sum_{k=1}^3 f_{jk} D_k + R_j \quad (6.4.10)$$

Since the second order partials of v_j are continuous and bounded over the set $S_L^3(\sigma)$ by the assumptions above, it can be shown that the expectation $ER_j^2 = O(\sigma^2/n)$. Hence as $\sigma^2/n \rightarrow 0$, $R_j \rightarrow 0$ in probability. Following Cramér (1946) page 254, it follows that $n^{1/2} v_j / \sigma$ has the same asymptotic distribution as the sum of random variables

$$(n^{1/2}/\sigma) \sum_{k=1}^3 f_{jk} D_k \text{ which is } N(0, \sum_{k=1}^3 f_{jk}^2). \text{ From (6.4.7) it}$$

follows that $D(\theta_j, \theta_j^*) = D_j + v_j$ so that

$$\frac{n^{1/2}}{\sigma} \begin{pmatrix} D(\theta_1, \theta_1^*) \\ D(\theta_2, \theta_2^*) \\ D(\theta_3, \theta_3^*) \end{pmatrix} = \frac{n^{1/2}}{\sigma} \begin{pmatrix} D_1 + v_1 \\ D_2 + v_2 \\ D_3 + v_3 \end{pmatrix} = \frac{n^{1/2}}{\sigma} M \begin{pmatrix} D_1 \\ D_2 \\ D_3 \end{pmatrix} + \frac{n^{1/2}}{\sigma} \begin{pmatrix} R_1 \\ R_2 \\ R_3 \end{pmatrix} \quad (6.4.11)$$

where the matrix M is

$$M = \begin{pmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{pmatrix}$$

with elements $f_{ij} = \frac{-\delta_i \delta_j}{\sum \delta_k^2} \Big|_{\bar{\theta} = \theta}$.

Furthermore by (6.4.9) it follows that $\frac{1}{\sigma} M \begin{pmatrix} D_1 \\ D_2 \\ D_3 \end{pmatrix}$ is

asymptotically multivariate normal with mean $\mu' = (0,0,0)$ and singular variance-covariance matrix MM' where

$$MM' = \frac{1}{\sum \delta_k^2} \begin{pmatrix} \delta_2^2 + \delta_3^2 & -\delta_1 \times \delta_2 & -\delta_1 \times \delta_3 \\ -\delta_1 \times \delta_2 & \delta_1^2 + \delta_3^2 & -\delta_2 \times \delta_3 \\ -\delta_1 \times \delta_3 & -\delta_2 \times \delta_3 & \delta_1^2 + \delta_2^2 \end{pmatrix}. \quad (6.4.12)$$

Expression (6.4.11) in conjunction with the fact that $R_j \rightarrow 0$ in probability with $\sigma^2/n \rightarrow 0$ allows the conclusion to be reached that the left hand side of (6.4.11) has an asymptotic multivariate normal distribution as implied earlier. The singularity of MM' can be interpreted in terms of the linear constraint the estimates must satisfy. It can also be shown after long and tedious algebra involving implicit differentiation that least squares estimators are defined by (6.4.4)

and exist in some neighborhood of any $\theta \in \mathcal{B}$ and that these estimates have asymptotic moments the same as those given in (6.4.11). When (6.4.4) is the likelihood function for some distribution such as the multivariate generalization of the truncated normal distribution the least squares estimators are also maximum likelihood estimators. In this case (6.4.8) is BAN in addition to being asymptotically normal.

The question remains open as to which of (6.4.4) or (6.4.8) has superior small sample properties and computational efficiency. One procedure may not be uniformly superior compared to the other over the entire set $\mathcal{B}_3$. There may be locations of $\mathcal{B}_3$ with sample spaces containing points which are not projectable into $\mathcal{B}_3$. This would follow in analogy to the result of Lemma 4.4.2 on page 67. Thus there seem to be certain issues to be resolved before this estimation theory can be applied in the resolution of the three tower allocation problem for triangular systems.

Section 7.1 - Introduction

The contents of Section 7.2 include a description of the computational procedure used to estimate the risks defined in Chapter 5. An upper bound is presented for the variance of weighted averages of these estimated risks. This computational procedure forms the basis of the computer program in Appendix 3 which is used extensively in the present chapter.

Sections 7.3 and 7.4 contain a study and discussion of the tower allocation problem on the Cedar Creek Natural History Area as well as considerations related to alternative improvements which can be made to enhance the functioning of the current monitoring system. Those readers solely interested in these two sections are urged to skim Section 7.2 in order to develop familiarity with the notation of this chapter.

Section 7.2 - Calculation Procedure

To apply the decision theoretic procedures of Chapter 5, a region of the natural history area containing all feasible tower sites must be designated so that M_{Ω} , the set of all feasible pairs of tower sites, can be defined. Then given m_s from M_{Ω} and a fixed σ^2 the integral in expression (5.4.2) needs to be evaluated for each species. The

integrand of this integral depends on a prior density. The specification of this continuous density function for each point of the study region Ω and each species is difficult in practice. As a consequence, the decision theoretic procedures of Chapter 5 will have to be applied to a quantity related to (5.4.2) which is more easily calculated than (5.4.2). The derivation of such a quantity is given in this section along with a description of the calculation procedure used in Appendix 3 to evaluate this quantity.

The evaluation of (5.4.2) for each species generates a vector of risks $\underline{R}' = (R_1, \dots, R_I)$. Given such a vector for each element of M_Ω , various ways of ordering the elements of M_Ω can be investigated in the search for a "best" tower allocation. Given m_s and σ^2 , estimates of $\underline{R}$ can be calculated in different ways. The computational procedure used in this thesis is suggested by the usual mathematical definition of a double integral, has been adaptable to the type of information a biologist can provide regarding the prior distributions required of each species, and has turned out to be computationally efficient. The calculations involve a combination of numerical integration and simulation. This computational method assumes the study region or natural history area to be bounded and contained within a rectangle of length wn_x and height wn_y . Without any difficulty this rectangle can be chosen so that it can be decomposed into

$n_x n_y$ squares of area w^2 where the size w^2 and the number of $n_x n_y$ depends in part on the opinion of the triangulation system users as to what size squares constitute a suitable unit for habitat evaluation per species.

After choosing a rectangle containing Ω which can be partitioned into a checkerboard of $n_x n_y$ squares of size w^2 , let A_{ik} ($i=1, \dots, n_x$) and ($k=1, \dots, n_y$) denote a square of size w^2 located in the rectangle. The numbering can be taken so that i increases from left to right and k from bottom to top. Then recalling (5.4.2),

$$R_j = \iint_{\Omega} l_j(\underline{x}) f_j(\underline{x}) d\underline{x} = \sum_i \sum_k \iint_{A_{ik}} f_j(\underline{x}) l_j(\underline{x}) d\underline{x} \quad (7.2.1)$$

By the mean value theorem for multiple integrals (Apostol (1957) page 269) there exists $\underline{x}_{ik} \in A_{ik}$ such that

$$R_j = \sum_i \sum_k l_j(\underline{x}_{ik}) \iint_{A_{ik}} f_j(\underline{x}) d\underline{x} \quad (7.2.2)$$

Besides being suggested by the usual definition of double integrals of continuous functions, this summation is dependent on certain prior probabilities

$$P_{jik} = \iint_{A_{ik}} f_j(\underline{x}) d\underline{x}, \quad j=1, \dots, I. \quad (7.2.3)$$

These probabilities can be specified by the scientists studying Ω provided A_{ik} is chosen appropriately. In practice this

means the size w^2 cannot become arbitrarily small for if it did few scientists could specify all the probabilities needed. However, it is necessary to make A_{ik} moderately small relative to the study area so that the variation in $l_j(x)$ on each A_{ik} will be slight enough so that

$$l_j(x_{ik}) \approx \frac{1}{w^2} \iint_{A_{ik}} l_j(x) dx = \mu_{jik}.$$

By Duhamel's principle (Taylor (1955) p. 515) it is clear the degree of this approximation must improve as $w^2 \rightarrow 0$ for any square A_{ik} . Thus, for moderately small w^2 ,

$$R_j \approx \sum_i \sum_k \mu_{jik} P_{jik} = R_j^* \quad (7.2.4)$$

Define the vector $\underline{R}^* = (R_1^*, \dots, R_I^*)$. On the basis of the foregoing practical matters this vector will be used in lieu of $\underline{R}$ for decision making. As a result, the decisions that are made may depend to one degree or another on the partitioning of Ω , and consequently some consideration of the effects of variation in n_x and n_y should be made by the decision maker. In general, A_{ik} should be as small as is practical for habitat evaluation.

In essence the partitioning of Ω described above amounts to a stratification of Ω into sets small enough so that the variation in $l_j(x)$ is reduced over each set. Estimates of the stratum means μ_{jik} and hence R_j^* can be obtained by simulating the performance of m_s at randomly chosen points

of A_{ik} . Since all components of $\underline{R}^*$ are needed, computational time can be saved by simultaneously estimating μ_{jik} , $j=1, \dots, I$, at each randomly chosen point in A_{ik} . All such computations were conducted for this thesis using the computer program given in Appendix 3. The specific application of this program to the CCS will be considered in the next sections. Here only an outline of the computational procedure will be given.

In view of (7.2.4), given estimates of $\mu_{1ik}, \dots, \mu_{Iik}$ ($i=1, \dots, n_x$ and $j=1, \dots, n_y$) an estimate of $\underline{R}^*$ can be constructed. Thus, the computational procedure reduces mainly to the method of estimating $\mu_{1ik}, \dots, \mu_{Iik}$ for each i and k . By changing points at random in each A_{ik} and determining the losses $l_1(\underline{x}), \dots, l_I(\underline{x})$ at each such point in an unbiased fashion, unbiased estimates of the stratum means and hence of $R_1^*, \dots, R_I^*$ are available. (It will be assumed that the prior probabilities P_{jik} ($j=1, \dots, I$, $i=1, \dots, n_x$ and $k=1, \dots, n_y$) are fixed and not random.) Simultaneously unbiased estimation of the losses $l_1(\underline{x}), \dots, l_I(\underline{x})$ at each randomly chosen point of A_{ik} depends to a large extent on the definition of this loss. Each species has an associated disk of some radius r_j , $j=1, \dots, I$. Since duplication in these radii values is possible, assume of the I radii that n_I are distinct and when given in order of increasing size they are $d_1 < d_2 < d_3 < \dots < d_{n_I}$. Given $\underline{x}$ chosen at

random from A_{ik} a set of concentric circles centered at $\underline{x}$ is defined by these distinct radii. Let the event $E_j(\underline{x})$ be defined as in Section 5.3 and let

$$\begin{aligned} p_0 &= P[\hat{\underline{x}} \text{ cannot be determined from } \hat{\theta} \mid E_j(\underline{x})] \\ p_1 &= P[(\hat{x}-x)^2 + (\hat{y}-y)^2 < d_1^2 \mid E_j(\underline{x})] \\ &\vdots \\ p_t &= P[d_{t-1}^2 \leq (\hat{x}-x)^2 + (\hat{y}-y)^2 < d_t^2 \mid E_j(\underline{x})] \\ &\vdots \\ p_{n_I+1} &= P[d_t^2 \leq (\hat{x}-x)^2 + (\hat{y}-y)^2 \mid E_j(\underline{x})] \end{aligned} \quad (7.2.5)$$

for $t=0,1,\dots, n_I+1$.

The reader may recall the discussion of Section 5.2 regarding the fact that at the CCS several photographs of signal image pairs are reviewed before a pair is selected for bearing determination. In specifying $E_j(\underline{x})$ here, the simplest situation is assumed namely, each pair of signal images is read whenever the images are of acceptable quality. The event $E_j(\underline{x})$ could be defined more broadly.

The probabilities $p_0, \dots, p_{n_I+1}$ are conditional on $E_j(\underline{x})$ and hence are the same for all species, since the performance of the tracking system is uniform with regard to bearing determination. To estimate these probabilities, a measurement system m_s , σ^2 and a location $\underline{x}$ must be specified. Then the true bearings of $\underline{x}$ relative to m_s can be

determined and in conjunction with σ^2 they define a sample space containing all observable bearing pairs that could be measured by m_s on the location $\underline{x}$. Let n bearing pairs be chosen from this sample space independently and at random. Each pair of bearings must satisfy one and only one of the events used to define the conditional probabilities $p_0, p_1, \dots, p_{n_I+1}$. Let the random variable X_t = the number of times

the event defining p_t occurs in n trials so that $n = \sum_{t=0}^{n_I+1} X_t$.

Then the random variables $X_0, \dots, X_{n_I+1}$ have a multinomial distribution with probabilities $p_0, \dots, p_{n_I+1}$. Define $\hat{p}_t = (X_t / n)$, $t=0, \dots, n_I+1$. Suppose species j has a disk of radius r_j which is the m_j^{th} member of the increasing sequence $d_1, \dots, d_I$. Then

$$\hat{\lambda}_j(\underline{x}) = 1 - \left(\sum_{t=1}^{m_j} \hat{p}_t \right) P[E_j(\underline{x})] . \quad (7.2.6)$$

Since $P[E_j(\underline{x})]$ is fixed given $\underline{x}$ and $\hat{p}_t$ is an unbiased estimate of p_t , $E(\hat{\lambda}_j(\underline{x})) = \lambda_j(\underline{x})$. Furthermore,

$$\text{Var}(\hat{\lambda}_j(\underline{x})) = P^2[E_j(\underline{x})] \left(\sum_{t=1}^{m_j} p_t \right) \left(1 - \sum_{t=1}^{m_j} p_t \right) / n .$$

Because of the simultaneous nature of the estimation of $\lambda_1(\underline{x}), \dots, \lambda_I(\underline{x})$, the estimates $\hat{\lambda}_j(\underline{x})$ and $\hat{\lambda}_h(\underline{x})$ have a positive covariance.

Lemma 7.2.1 - Let $\hat{\lambda}_j(\underline{x})$ and $\hat{\lambda}_h(\underline{x})$ be obtained as above.

Suppose species j has a disk of radius d_{m_j} and

species h has a disk of radius d_{m_h} with $d_{m_h} \geq d_{m_j}$.

Then

$$\text{Cov}(\hat{\lambda}_j, \hat{\lambda}_h) = \frac{1}{n_A} \left(\sum_{t=1}^{m_j} p_t \right) \left(1 - \sum_{t=1}^{m_h} p_t \right) > 0, \quad (7.2.7)$$

$\underline{x}$ understood and a total of n_A such $\underline{x}$ independently and randomly chosen in each A_{ik} .

$$\text{Pf} - \text{Cov}(\hat{\lambda}_j, \hat{\lambda}_h) = E((\hat{\lambda}_j - E(\hat{\lambda}_j))(\hat{\lambda}_h - E(\hat{\lambda}_h))).$$

$$\begin{aligned} \hat{\lambda}_j - E \hat{\lambda}_j &= \left(1 - \sum_{t=1}^{m_j} \hat{p}_t P_j \right) - \left(1 - \sum_{t=1}^{m_j} p_t P_j \right) \\ &= -P_j \left(\sum_{t=1}^{m_j} (\hat{p}_t - p_t) \right). \end{aligned}$$

Suppose $m_h \geq m_j$ (equivalent to $d_{m_h} \geq d_{m_j}$).

$$\text{Cov}(\hat{\lambda}_j, \hat{\lambda}_h) = P_h P_j E \left(\sum_{t=1}^{m_j} (\hat{p}_t - p_t) \times \sum_{t=1}^{m_h} (\hat{p}_t - p_t) \right)$$

$$= P_h P_j E \left(\left(\sum_{t=1}^{m_j} (\hat{p}_t - p_t) \right)^2 + \left(\sum_{t=1}^{m_j} (\hat{p}_t - p_t) \times \sum_{t=1}^{m_h} (\hat{p}_t - p_t) \right) \right)$$

$$\begin{aligned}
&= \frac{P_h P_j}{n_A} \left(\left(\sum_1^{m_j} p_t \right) \left(1 - \sum_1^{m_j} p_t \right) - \left(\sum_1^{m_j} p_t \right) \left(\sum_{m_j+1}^{m_h} p_t \right) \right) \\
&= \frac{P_h P_j}{n_A} \left(\sum_1^{m_j} p_t \right) \left(1 - \sum_1^{m_h} p_t \right) > 0.
\end{aligned}$$

As one would expect when $m_j = m_h$, $\text{Cov}(\hat{\lambda}_j, \hat{\lambda}_h) = \text{Var}(\hat{\lambda}_j) = \text{Var}(\hat{\lambda}_h)$ so that the correlation between estimates is plus one. Let n_A points be randomly chosen in each A_{ik} . Then define

$$\hat{\mu}_{jik} = \frac{1}{n_A} \sum_{\alpha=1}^{n_A} \hat{\lambda}_j(\tilde{x}_\alpha), \text{ all } \tilde{x} \in A_{ik}.$$

It follows that an unbiased estimate of R_j^* is

$$\hat{R}_j^* = \sum_{i=1}^{n_x} \sum_{j=1}^{n_y} \left(\frac{1}{n_A} \sum_{\alpha=1}^{n_A} \hat{\lambda}_j(\tilde{x}_\alpha) \right) P_{jik}. \quad (7.2.8)$$

In this thesis most attention will be given to weighted

averages of the form $\sum_{j=1}^I w_j R_j^*$ estimated by $\sum_{j=1}^I w_j \hat{R}_j^*$

for non-negative weights $w_1, \dots, w_I$, $\sum_{i=1}^I w_i = 1$, which can be

chosen to reflect the research interest of those using the

monitoring system. Consequently, it is important to have an estimate or upper bound to the variance of

$$\sum_{j=1}^I w_j \hat{R}_j^* = \sum_{i=1}^{n_x} \sum_{k=1}^{n_y} \left(\sum_{j=1}^I w_j \hat{\mu}_{jik} P_{jik} \right). \quad (7.2.9)$$

Thm 7.2.1 - Suppose all quantities are as defined above,

then $\text{Var} \left(\sum_{j=1}^I w_j \hat{R}_j^* \right)$ is less than or equal to

$$\left(\frac{\sum_{j=1}^I w_j^2}{4n_A} \right) \sum_{j=1}^I \left(\sum_{i=1}^{n_x} \sum_{k=1}^{n_y} P_{jik}^2 \right). \quad (7.2.10)$$

Pf - Since the estimation in a square A_{ik} is independent of the estimation in every other square,

$$\begin{aligned} \text{Var} \left(\sum_{j=1}^I w_j \hat{R}_j^* \right) &= \sum_{i=1}^{n_x} \sum_{k=1}^{n_y} \text{Var} \left(\sum_{j=1}^I w_j \hat{\mu}_{jik} P_{jik} \right) \\ &\leq \sum_{i=1}^{n_x} \sum_{k=1}^{n_y} \left(\sum_{j=1}^I w_j P_{jik} \text{Var}^{\frac{1}{2}} \hat{\mu}_{jik} \right)^2 \end{aligned}$$

by replacing the covariance terms with $(\text{Var} \hat{\mu}_{jik} \text{Var} \hat{\mu}_{hik})^{\frac{1}{2}}$ using the Cauchy-Schwarz Inequality. It follows that

$$\text{Var} \hat{\mu}_{jik} \leq \max_{x \in A_{ik}} \left(P^2[E_j(x)] \frac{\left(\sum_{t=1}^{m_j} p_t \right) \left(1 - \sum_{t=1}^{m_j} p_t \right)}{n_A} \right) \leq \frac{1}{4n_A}.$$

By this inequality and an additional application of the Cauchy-Schwarz Inequality

$$\begin{aligned} \text{Var} \left(\sum_{j=1}^I w_j R_j^* \right) &\leq \sum_{i=1}^{n_x} \sum_{k=1}^{n_y} \frac{1}{4n_A} \left(\sum_{j=1}^I w_j P_{jik} \right)^2 \\ &\leq \left(\frac{\sum_{j=1}^I w_j^2}{4n_A} \right) \sum_{j=1}^I \left(\sum_{i=1}^{n_x} \sum_{k=1}^{n_y} P_{jik}^2 \right). \quad \square \end{aligned}$$

This upper bound will be useful later. One would expect it to be conservative because of the inequalities above, however in view of Lemma 7.2.1, the substitution of standard deviations for the covariance terms should not be that harsh an approximation. This upper bound is used in Appendix 2 in a study of the variance of certain estimates used in the study of the CCS.

In summary, the decision procedures applied to the CCS are based on $\underline{R}^*$ in place of $\underline{R}$. The computer program in Appendix 3 which calculates $\underline{R}^*$ and various weighted averages does so by first randomly selecting points in each A_{ik} and then simulates the performance of a given m_s (with σ^2 fixed at each of these points. The results are averaged in (7.2.8) yielding $\hat{R}_j^*$ and so $\hat{\underline{R}}^*$.

Section 7.3 - Performance Characteristics of the Present CCS

One goal for the mathematical model developed in this thesis is that it should aid in evaluating a tracking system and so shed light on areas where further work is necessary in the pursuit of tracking system improvement. To indicate how this might be done, the CCS will be evaluated in this section preliminary to considering the tower allocation problem in Section 7.4 .

To apply the mathematical model of this thesis, certain information must be available. (This information was discussed in Chapter 2.) In the case of the radio transmitters, if the results of Section 5.3 are to be used to calculate $P[E_i(\underline{x})]$ for each species, the type of transmitter must be known and also the maximum range at which on the average this transmitter produces the minimum acceptable signal-to-noise ratio for signal reading. The former information is more easily obtained than the latter in practice. To obtain this information for the CCS, various individuals were contacted who have had extensive experience with the CCS and the animals studied with the system. On the basis of these discussions the values of Table 7.3.1 were established. In this chapter these values will be fixed and taken as exact and accurate estimates. One future extension of this thesis would be to treat them as random variables in the decision process.

Using Table 7.3.1 and the results of Section 5.3, it is possible to compute $P[E_i^k(\underline{x})]$ as a function of the distance

Table 7.3.1 - Transmitter characteristics.

Species	Type of Antenna	Maximum Range for Bearing Determination (miles)
Grouse	whip	2
Squirrel	loop	1 1/8
Hawk	whip	2
Raccoon	loop	2 1/4
Skunk	loop	1 3/4
Deer	loop	3 1/2
Badger	loop	2 1/2
Hare	loop	1 1/4
Red Fox	loop	2
Muskrat	loop	1 1/2

from $\underline{x}$ to tower k . It is assumed $P[E_i^1(\underline{x})] = P[E_i^2(\underline{w})]$ if $\underline{x}$ and $\underline{w}$ are at the same distance from towers 1 and 2 respectively. Graphs of these probabilities are given in Figures 7.3.1 and 7.3.2 on pages 158 and 157. Using these values, it is possible to calculate $P[E_i(\underline{x})] = P[E_i^1(\underline{x})] \times P[E_i^2(\underline{x})]$ for each species as a function of location. The trend present in Figures 7.3.1 and 7.3.2 is indicative of the fact that larger animals tend to carry larger stronger transmitters and so have greater ranges. Grouse and hawks have superior ranges since they have whip transmitters and will also fly above the treetops or perch above the ground thus enhancing the range of their transmitters.

In Section 5.2 and 7.2 the importance of $P[E_i(\underline{x})]$ was pointed out. To use $P[E_i(\underline{x})]$ in the computation of $\hat{R}^*$, it is necessary to have prior probabilities specified as discussed in Section 7.2. Preparatory to specifying these probabilities a square four miles in width and length was defined on a map so that the portion of the Cedar Creek Natural History Area to be monitored was contained inside. (See Figure 7.3.5) Dick Huempfner, the resident biologist at the Cedar Creek Natural History Area, was asked to specify distributions over this square for the different study animals. Squares A_{ik} covering areas a quarter section in extent were agreed upon as a convenient unit for such an evaluation of the natural history area and values for P_{jik} ($j=1, \dots, 10$;

$i=1,\dots,8; k=1,\dots,8$) were given. To do this for a given species j , each quarter section square A_{ik} was ranked as 0,1,2,3,4, or 5 with 0 going to those squares receiving the lowest level of utilization by individuals of species j and 5 going to those squares receiving the highest level of utilization. The other squares were rated accordingly between these extremes.

Such a utilization rating is subjective and depends on a judgement by the biologist of a compendium of things such as relative population levels on different parts of the natural history area, seasonal shifts in the geographic location of different species and the differing expenditures of time on the quarter section squares by individuals of a given species. This information may never be available in other than a subjective sense. Fortunately, the most important thing is to obtain indications of the general variation in habitat usage over the region to be monitored, and the informed opinions of biologists familiar with the study area should suffice for this purpose. Diagrams showing the partitioning of the Cedar Creek Natural History Area and the ratings given by D. Huempfner are given in Section 7.4.

These ratings after normalization can be used as prior probabilities in conjunction with $P[E_j(\underline{x})]$, ($j=1,\dots,10$) to calculate values for R_j^* corresponding to the CCS. Figures 7.3.3 and 7.3.4 give values for R_j^* ($j=1,\dots,10$) for

Figure 7.3.1 — Probabilities of Receiving Signals of Adequate Strength At a Fixed Distance For Different Species

Figure 7.3.2— Probabilities of Receiving Signals of Adequate Strength At a Fixed Distance For Different Species

the CCS over a range of standard deviations. Each curve corresponds to a single species j and is based on a disk with a diameter that reflects what is considered to be acceptable accuracy for those research projects dealing with the species. The disk diameters used are given in Figures 7.3.3 and 7.3.4. These curves show how improvements in the quality of bearing determination will improve the performance of the system as measured by R_j^* . The computer program in Appendix 3 was used in this calculation and executed so that the estimation error associated with the plotted points can be ignored.

These two plots have interesting interpretations. As $\sigma^2 \rightarrow 0$, the accuracy of bearing determination increases to such an extent that no estimation errors are made provided signals of adequate strength are obtained at each tower. In other words, irrespective of the disk diameter associated with each species, the probability of all location estimates being in a circle of any fixed radius at any x must go to one as $\sigma^2 \rightarrow 0$. Hence, $\hat{f}_j(x) \rightarrow 1 - P[E_j(x)]$ as $\sigma^2 \rightarrow 0$

$$\text{so that } R_j^* = \sum_{i=1}^{n_x} \sum_{k=1}^{n_y} \left(\frac{1}{n_A} \sum_{\alpha=1}^{n_A} \hat{f}_j(x_\alpha) \right)^{P_{jik}}$$

$$\text{tends to, } \sum_{i=1}^{n_x} \sum_{k=1}^{n_y} \left(\frac{1}{n_A} \sum_{\alpha=1}^{n_A} (1 - P[E_j(x_\alpha)]) \right)^{P_{jik}}$$

$$\approx \iint_{\Omega} (1 - P[E_j(x)]) f_j(x) dx.$$

Figure 7.3.3 – Theoretical Performance of CCS As a Function of σ

Figure 7.3.4 – Theoretical Performance of CCS As a Function of σ

1^o Tower 1

2^o Tower 2

1 mile

Figure 7.3.5 — That Portion of the Cedar Creek Natural History Area to be Monitored by an Automatic Tracking System

This is the expected probability of not receiving a pair of readable signals at both towers from a randomly selected individual of species j that is radio-tagged and free roaming.

In general these values form asymptotic lower bounds for R_j^* for all $\sigma^2 > 0$ and thus represent the limits of system improvement from solely a bearing determination point of view. These values also provide a way of testing the veracity of the radio propagation model of Section 5.3.

However, accepting this model as an accurate representation, the values of $\lim_{\sigma^2 \rightarrow 0} R_j^*$ indicate that improvements in

bearing determination at the CCS are most helpful for those species carrying relatively strong transmitters which are also located most of the time in range of the monitoring system. Improvements in bearing determination are not effective in improving the performance of the monitoring system in tracking those species for which R_j^* does not decrease appreciably as $\sigma^2 \rightarrow 0$. If such species must be studied with the CCS, improvements in their radio transmitters would be most helpful. Short of this, either the individuals chosen for study must be close to the towers or the towers must be moved into a better location relative to the areas frequented by such species. Thus the tower allocation problem enters the picture once more.

The vertical dotted lines on Figures 7.3.3 and 7.3.4 indicate the range of variances within which the present CCS

operates. Over this range the curves almost form a set of parallel lines. This is important since it means that as σ^2 varies (whether by design or otherwise) it will almost be the case that the change in risks $R_1^*, \dots, R_{10}^*$ is the same for each species. Because of this behavior the research mission of the CCS need not be contingent on the value of σ^2 but rather only on the comparative magnitudes of $R_1^*, \dots, R_{10}^*$.

In summary, it seems that improvements in the CCS must be considered in terms of which species are to be researched most intensively since different improvements are needed by different species. Assuming no significant improvements are possible in bearing determination or the performance of the radio transmitters without investment of considerable time and money, optimum tower allocation may serve in place of these other improvements as a means of enhancing data quality. The next section considers this possibility for the Cedar Creek Natural History Area.

Section 7.4 - The Tower Allocation Problem on the Cedar Creek Natural History Area

The tower allocation problem on the Cedar Creek Natural History Area will be considered in this section. Preference information was made available concerning the present research mission of the CCS by D. Siniff. This information permits procedure (2) on page 110 to be applied which amounts to an

investigation of various weighted averages of the form

$$\sum_{j=1}^{10} w_j R_j^* \quad \text{for nonnegative constants } w_1, \dots, w_{10} \text{ adding to}$$

to unity. The disk diameters of Figures 7.3.3 and 7.3.4 will be used to reflect what is considered to be acceptable accuracy.

Before the tower allocation problem can be discussed the set M_Ω mentioned in Section 7.2 must be defined. By defining this set appropriately various cost factors can be controlled. On the Cedar Creek Natural History Area, there is a laboratory where the display screens and photographic equipment are located. This laboratory cannot be easily moved so the towers need to be located relatively close to it. West of the laboratory is a bog which is unsuitable for tower location. North of the laboratory is a paved road which would be costly to tunnel under if a cable were to be laid from the laboratory to a tower north of this road. Bringing these facts together, it was decided that all feasible tower sites were in the set given in Figure 7.4.1. It is impossible to investigate all possible pairs of tower sites in this set. Consequently a lattice of points was selected from this set with the vertical and horizontal spacing of the closest lattice points being 0.2 miles. The set M_Ω was defined as the set of all pairs of lattice points no closer together than 0.25 miles or farther apart than 1.0 miles. This restriction of the problem

Figure 7.4.1 — Lattice of Potential Tower Sites to Be Investigated

Present Towers = +
Lattice Point = ●

0.5 mile

yields 294 possible tower site pairs to investigate with the computer program in Appendix 3. Approximately two hours of computer time on a CDC 3300 were required to calculate values of $\underline{R}^*$ for each tower site pair at each of three standard deviations for the bearing error distribution: $1/4^\circ$, $1/2^\circ$, and $3/4^\circ$.

It is felt that the results from the analysis of a lattice will be extendable in a robust fashion to the infinite set of all possible tower pair sites since the vector of risks $\underline{R}^*$ seems to vary slowly as the tower site pair is changed. This will become evident as the computer results are analyzed.

As just stated, three standard deviations were studied. This discussion will be based solely on results computed for a standard deviation of $1/4^\circ$ since the relative order of the different tower pairs in M_Ω as a function of different weighted averages of risks does not seem to depend on changes in the standard deviation of the bearing error distribution over the range $1/4^\circ$ to $3/4^\circ$. This corresponds to the results obtained for the present CCS which are graphically presented in Figures 7.3.3 and 7.3.4.

Since variation in σ^2 does not need to be considered, the application of the decision theoretic procedures of Chapter 5 reduces to a search through M_Ω for that tower allocation which minimizes the value of a weighted average $\underline{w}' \underline{R}^*$ for some given set of weights $\underline{w}$. The program in Appendix 3

was used to generate the vector $\hat{R}^*$ for each member of M_Ω and store the results on tape. The results on tape were then searched given $\underline{w}$. Due to the manner in which $\hat{R}^*$ is computed, the estimator $\hat{R}^*$ should be considered a random vector and so the m_s of M_Ω with the minimum calculated value of $\underline{w} \hat{R}^*$ is not necessarily the m_s with the minimum $\underline{w} \hat{R}^*$. Because of this tower allocations with computed values close to the minimum computed value over M_Ω must be considered as potential candidates. The question now is - how close is close enough? An answer is available with Tukey's w-procedure or the honestly significant difference procedure described by Steel and Torrie(1960) on page 109. Although this procedure may be conservative, it is easy to apply since it requires only a single value for judging the significance of all differences of the form $\underline{w} \hat{R}_1^* - \underline{w} \hat{R}_2^*$. Following the results of Appendix 2, should $\underline{w} (\hat{R}_1^* - \hat{R}_2^*) > 0.01$, the tower systems 1 and 2 can be judged to have significantly different performance with an experiment wise error rate of about $\alpha=0.05$. (The value 0.05 will tend to be conservative.)

Given the member of M_Ω with the minimum computed value of $\underline{w} \hat{R}^*$, $\underline{w}$ fixed, all other members of M_Ω with weighted averages within 0.01 of the computed minimum will be treated as a homogeneous set. In practice any tower system in this set will suffice as an optimal allocation for the weight set

$\underline{w}$.

As a starting point, the set M_{Ω} was searched for those members which are best for each individual species. For some species many tower site pairs were close to the computed minimum. Representative members of the optimality set for each species are plotted in Figures 7.4.2 to 7.4.11 along with a plot showing the probability distribution given by Dick Huempfer for that species. Visually speaking each set of tower allocations seems different from the others. The extent of this difference can be quantified. Let $M_{\Omega,j}$ denote the set of all tower site pairs deemed "best" and equivalent by the above procedure for species j . Averaging all the risk vectors $\underline{R}^*$ for members of this homogeneous set of tower allocations yields an average performance level for each other species at the optimum for species j . Doing this for each species yields Table 7.4.1. To make this table more interpretable the set M_{Ω} was subsequently searched for those systems which have the highest values of R_j^* for each species j . Using Tukey's procedure a set of systems with performance values within 0.01 of the maximum risk for each species was obtained. Averaging over these maximum risks for each species results in Table 7.4.2.

To compare Tables 7.4.1 and 7.4.2 the concept of regret will be useful (Lindgren (1968) page 196). The maximum regret for a species in this thesis is the difference of maximum risk and minimum risk both given in Table 7.4.2. Computed values

Figure 7.4.2—Results for Grouse

Optimal Tower Allocations For Grouse

Figure 7.4.3 - Results for Squirrel

Optimal Tower Allocations For Squirrel

Figure 7.4.4 - Results for Hawk

Optimal Tower Allocations For Hawk

Figure 7.4.5 - Results for Badger

Optimal Tower Allocations for Badger

Figure 7.4.6— Results for Raccoon

Optimal Tower Allocation for Raccoon

Figure 7.4.7 - Results for Skunk

Optimal Tower Allocation for Skunk

Figure 7.4.8— Results for Deer

Optimal Tower Allocations for Deer

Figure 7.4.9 - Results for Muskrat

Ratings: 4-5 2-3 0-1

Present CCS Tower Sites ⊕

Optimal Tower Allocations for Muskrat

Figure 7.4.10- Results for Hare

Ratings: 4-5 2-3 0-1
Present CCS Tower Sites

Optimal Tower Allocations for Hare

Figure 7.4.11 - Results for Red Fox

Ratings: 4-5 2-3 0-1
Present CCS Tower Sites $\oplus$

Optimal Tower Allocations for Red Fox

Table 7.4.1 - Average values of risks over the set of optimum tower allocations for each species.

Systems Best For-	Value of Risk	Grouse	Sqrl	Hawk	Badger	Raccoon	Skunk	Deer	Muskrat	Red Fox	Hare
Grouse	.564		.905	.590	.693	.704	.548	.539	.841	.749	.871
Squirrel	.873	.676		.648	.746	.737	.795	.655	.856	.762	.867
Hawk	.589	.572	.909		.692	.699	.800	.524	.860	.751	.878
Badger	.684	.592	.904	.589		.700	.793	.534	.855	.744	.876
Raccoon	.687	.578	.896	.589	.699		.787	.539	.830	.749	.882
Skunk	.782	.597	.885	.607	.716	.704		.575	.822	.748	.854
Deer	.517	.578	.912	.589	.694	.700	.800		.849	.757	.882
Muskrat	.788	.601	.899	.623	.743	.721	.791	.597		.763	.841
Red Fox	.739	.580	.890	.595	.699	.702	.785	.560	.865		.863
Hare	.837	.621	.884	.638	.744	.728	.788	.618	.809	.777	

Table 7.4.2 - Highest and lowest average risks for each species compared with risks for the current CCS with $\sigma = 1/4^\circ$.

Species	Average Risk Over Set of Best Tower Sites Per Species	Average Risk Over Worst Tower Sites Per Species	Risks for the Current CCS ($\sigma = 1/4^\circ$)
Grouse	.564	.747	.618
Squirrel	.873	.923	.902
Hawk	.589	.741	.669
Badger	.684	.833	.769
Raccoon	.687	.798	.738
Skunk	.782	.835	.797
Deer	.577	.753	.618
Muskrat	.788	.904	.791
Red Fox	.739	.833	.786
Hare	.837	.920	.852

Table 7.4.3 - Increases in risk.

Systems Best For-	Minimum Risk	Maximum Regret	Grouse	Sqrl	Hawk	Badger	Raccoon	Skunk	Deer	Muskrat	Red Fox	Hare
Grouse	.564	.183		.032	.001	.009	.017	.029	.022	.053	.010	.034
Squirrel	.873	.050	.112		.059	.062	.050	.013	.138	.068	.023	.030
Hawk	.589	.152	.008	.036		.008	.012	.018	.007	.072	.012	.041
Badger	.684	.149	.028	.031	.000		.013	.011	.017	.067	.005	.039
Raccoon	.687	.111	.014	.023	.000	.015		.005	.022	.042	.010	.047
Skunk	.782	.053	.033	.012	.018	.032	.017		.058	.034	.009	.017
Deer	.517	.236	.014	.039	.000	.010	.013	.018		.061	.018	.045
Muskrat	.788	.116	.037	.026	.034	.059	.034	.009	.080		.024	.004
Red Fox	.739	.094	.016	.017	.006	.015	.015	.003	.043	.077		.026
Hare	.837	.083	.057	.007	.049	.060	.041	.006	.101	.021	.038	

for this regret are given in Table 7.4.3. Note that the sum of the minimum risk and maximum regret equals the maximum risk. The remaining values (or regrets) in Table 7.4.3 are obtained from Table 7.4.1 by subtracting the minimum risks for each species from the appropriate values of Table 7.4.1. Thus Table 7.4.3 contains the increases in risk obtained on the average for those species for which the system is not optimal. Comparing these values against the maximum regrets obtained per species over M_{Ω} , it is clear that systems optimal for grouse, hawk, badger, raccoon, deer and red fox also result in the greatest departures from optimality on the average for squirrel, muskrat and hare. Skunk seem to be intermediate between these two groups. One possible explanation for this is that the geographic probability distribution specified for skunk places moderately high probability for skunk in regions where other species also have high geographic probability values. Consequently, tower allocations appropriate for skunk would also tend to work moderately well for all other species.

These results tend to indicate that there is a natural dichotomy of the species studied on the Cedar Creek Natural History Area into at least two groups such that tower systems of M_{Ω} which function well for one group do not function well for the other group and vice versa. This dichotomy cannot be solely attributed to the probability distributions specified

for the species. Grouse and hare are in different groups but have much the same habitat requirements as shown by the similarities between their distributions. Figures 7.3.1 and 7.3.2 give one interpretation in terms of the probability of receiving signals at both towers from hare, muskrat and squirrel. These three species have the shortest ranges. The next in line are skunk and then come all the others. A second reason for the dichotomy may be found in the fact that the radii of the disks used in defining the loss function for hare, muskrat and squirrel was 0.0125 as opposed to 0.025 for the remainder. So whatever the cause, the dichotomy does seem to exist. Pictorially speaking, the optimal allocations of Figures 7.4.3, 7.4.9 and 7.4.10 do seem to depart from the other solutions. One can draw consolation from the fact that the magnitude of the maximum possible departure (or maximum regret) for squirrel, muskrat, hare and skunk tends to be small in comparison to the maximum departures (or maximum regrets) for the other species. Consequently, the systems of M_{Ω} tend to be rather similar in their risk values for squirrel, muskrat, hare and skunk. These species all have high risks for the stated research mission with the present transmitters. Altering either of these two factors could change the performance of the systems of M_{Ω} significantly in relation to these species. It is also interesting to note that the present CCS gives quite reasonable performance for the four species: squirrel,

muskrat, hare and skunk relative to the best available in M_{Ω} . Figures 7.3.3 and 7.3.4 in conjunction with the above analysis indicate that the best way to improve the performance of the CCS for these species is to improve their transmitters, or seek tower allocations vastly different from the ones investigated here. The question still exists as to whether or not data should be collected on these species for projects necessitating a disk of radius 0.0125. Only the scientists can answer this question. To minimize the impact of the large risks, the scientist should choose study subjects as close to the towers as possible - however off the baseline as indicated in Section 4.2. Such issues for these species can be researched further using the computer program in Appendix 3.

The dichotomy between species necessitates the search for a compromise tower allocation which is "best" in some overall sense. One way to arrive at a solution is to quantify the research mission on the natural history area in a set of weights so that weighted averages of risks can be investigated. Such an examination will most likely result in a set of solutions not "best" for either group of species mentioned above by themselves.

To illustrate this point four sets of weights will be investigated reflecting different research missions. These weights are given in Table 7.4.4 along with the minimum calculated weighted average over M_{Ω} for each weight set. Table 7.4.5 gives the calculated values of the weighted averages

Table 7.4.4 - Four different sets of weights and minimum values of corresponding averages.

Species	Weight Set 1	Weight Set 2	Weight Set 3	Weight Set 4
Grouse	.2000	.0385	.3077	.0000
Squirrel	.2000	.0385	.0000	.5714
Hawk	.1000	.0769	.1538	.0000
Badger	.1000	.0769	.1538	.0000
Raccoon	.1000	.0769	.1538	.0000
Skunk	.0500	.1538	.0000	.1429
Deer	.1000	.0769	.1538	.0000
Muskrat	.0500	.1538	.0000	.1429
Fox	.0500	.1538	.0769	.0000
Hare	.0500	.1538	.0000	.1429
Minimum Calculated Average	.670	.732	.610	.849

Table 7.4.5 - Performance of current CCS with respect to the weights above.

Weight Set 1	Weight Set 2	Weight Set 3	Weight Set 4
.745	.770	.681	.863

Figure 7.4.12 – Weight Set 1 Tower Allocations

Figure 7.4.13 – Weight Set 2 Tower Allocations

⊕ = Present CCS Tower Site

Figure 7.4.14—Weight Set 3 Tower Allocations

Figure 7.4.15 — Weight Set 4 Tower Allocations

⊕ = Present CCS Tower Site

for the present CCS. Weight set 1 reflects the current research emphasis on the natural history area, weight set 2 reflects an inversion of this emphasis. The ratios of the nonzero weights in sets 3 and 4 reflect the emphasis of weight set 1. The zero entries in these two sets were chosen to reflect the dichotomy evident in the data given in Table 7.4.3. Figures 7.4.12, 7.4.13, 7.4.14 and 7.4.15 give representative members of the sets of tower allocations which were judged best for the given set of weights and which were used in calculating the minimums listed in Table 7.4.4. The present CCS is denoted by the two circled dots. Judging from the values given in Tables 7.4.4 and 7.4.5 and the last four figures, it would seem the present CCS is both numerically and geographically closer to those systems which are optimum for research missions weighted in favor of squirrel, skunk, muskrat and hare. In any case, the actual difference in performance is slight between the current CCS and those systems optimal for the weights given in Table 7.4.4.

These weighted averages can be given a monetary interpretation. Suppose a group of scientists are willing to pay a fixed amount of money $= a_i$ to obtain a fix on species i within the disk of radius c_i . (The amount a_i depends on the species and c_i but suppose it does not depend on the likelihood of obtaining such a fix.) Thus, hypothetically the measurement process can be viewed as a financial transaction between the

scientists and the measurement system for each fix with the research monies of the scientists being spent for information. If a fix is inside the disk the transaction is deemed fair and if a fix is outside the disk or not obtained at all the transaction is not fair and the a_i dollars spent for the fix are lost. Let N_i = the total number of fixes which can be gathered on species i before all the money is spent for both good and bad fixes. Then the total assets of the

scientists collecting data is $\sum_{j=1}^{10} a_j N_j$. Define

$$w_i = \frac{a_i N_i}{\sum_{j=1}^{10} a_j N_j} \quad . \quad \text{Then} \quad \sum_{i=1}^{10} w_i R_i \quad \text{is the expected proportion}$$

of the research budget which is lost on poor fixes or no fixes at all. From Table 7.4.4 and 7.4.5 one sees less money is lost on the average at the CCS under research missions 2 and 4 as opposed to 1 and 3. The average loss for 1 and 3 combined at the CCS over the minimum loss is about \$0.07 or 10% of the total loss, if one thinks in terms of the research dollar being split between the two categories. Using this analysis, it is possible to balance the cost of moving a tower against the cost of not moving the tower and losing money on poor fixes. More general analyses are possible by making a_i a function of the likelihood of obtaining a fix.

The natural dichotomy in the tower allocation requirements of the species studied at the Cedar Creek Natural History Area suggests the introduction of a third tower. The present system in use is closest to optimum over M_2 for a research mission weighted in favor of hare, muskrat and skunk. The figures for the other species on pages 170, 172, 173, 174, 176 and 179 suggest placing a third tower east of the present two. Such an arrangement would be triangular with three effective two tower systems - each of which is close to optimal for a different group of species. Such an arrangement has the potential to reduce the risks involved in tracking markedly.

As yet a formal evaluation of the risks for such a system is not possible due to the work remaining on the three tower estimation problem. Once this estimation theory is computerized a program similar to that in Appendix 3 could be written. Once written, the best location for a third tower on the natural history area could be sought. The results of this thesis provides the motivation for continuing work on this problem.

Various extensions of the preceding work are possible. The stochastic model for radio wave propagation can be adapted for use over more irregular terrain with fixed barriers such as mountains. Such a development was not needed for the Cedar Creek Natural History Area since the terrain is level. To adapt this stochastic model to uneven terrain the value of $P[E_i(\underline{x})]$ calculated in Section 5.3 must be adapted according to the terrain's departure from evenness over lines of signal propagation from the transmitter to the towers. This can be accomplished by making $(F_j F_s)^2$ in (5.3.1) a function of the signal path - as it must be over irregular terrain.

Besides the above change, it is also possible to incorporate uncertainty about transmitter performance and the influence of terrain into the decision model. This development would be analogous to the approach taken for σ^2 in this thesis and could be combined with the steps taken to handle uncertainty in σ^2 . Some of the results of this thesis (Figures 7.3.3 and 7.3.4 and the attendant discussion) tend to indicate variation in σ^2 might be of less importance in the tower allocation problem than variation in the parameters controlling the radio transmitter's performance. Indications of the effect of variation in these parameters can be seen in the optimal tower allocations for hare, grouse and deer. These three species have similar probability distributions and different transmitter performances as indicated by Figures

7.3.1 and 7.3.2. A basic directional orientation of southwest to northeast is present in each but the solutions are shifted and the towers are placed differing distances from each other. These differences can be interpreted as a result of transmitter differences and serve to provide justification for future changes in the decision model. Such changes were not necessary in Chapter 7 since the theoretical performance curves given in Figures 7.3.1, 7.3.2, 7.3.3 and 7.3.4 agree with the experience of those using the CCS and so the conclusions which follow should be valid.

The general approach of this thesis is also adaptable to problems of designing a system for monitoring the physiological processes of free roaming animals. Here the problem is one of locating a fixed number of towers (due to economic factors) at sites over some study region (the sites chosen for engineering feasibility) in such a way that the probability of continuous reception of transmitted physiological information of acceptable quality from free roaming animals is maximized. Although the loss function for this problem would be different from that in this thesis, the use of the concept of risk, the stochastic radio wave propagation model of Section 5.3 and the implementation of the mathematical model with a computer program using simulation techniques all seem fruitful and useful approaches to the problem.

The results of Section 4.4 imply the need for future work.

From the point of view of optimal tower allocation the fact that the expected mean squared error or expected distance of an estimate $\hat{x}$ from the true but unknown point x do not exist is no impediment to pursuing the subject using a probabilistic criterion. However, from a data analytic point of view the lack of such moments for certain induced distributions of location estimates produces problems which cannot always be resolved by optimal design. For example, one conceivable reason for tracking animals is to obtain information concerning the path the animal used and then examine the path. Since the distributions of location estimates change radically as the animal's unknown location changes and lack certain moments it is not possible to carry out an analysis of the path using regression techniques. Thus further work is indicated on this and other data analytic problems.

This discussion hints at a basic fact about two tower systems. Considering the spectrum of instrumental systems devised by men for investigating some phenomenon, some systems are so "good" that the data collected by the system more than fills the needs of the scientists using the systems and so only simple analysis techniques are needed in examining the data. Other measurement systems produce data which cannot be accepted on face value. This would seem to be the case for a two-tower triangulation system due to the variation in accuracy of position estimation over the xy plane. In such cases two things can be done - the system can be redesigned

4

to produce superior data, or more sophisticated data analysis procedures can be devised. Both possibilities can be realized with two tower triangulation systems - the latter being suggested in the preceding paragraph for path analysis. The former could be accomplished by redesigning the measurement system to provide concomitant information about the transmitter location in addition to the two measured bearings. The introduction of a third tower could provide such information as well as the introduction of devices to be carried by the animal which respond to signals sent from the towers and so, as with radar, yield estimates of the range from tower to transmitter in addition to the bearings measured by both towers. Due to environmental attenuation of the signal from animal to towers, the use of a third tower may be the more reliable of these two possibilities.

In addition to the data collected by Kuechle (1964) presented on page 36 of this thesis, there is another independent set of data which offers further information about the dynamic nature of the bearing error distribution at the CCS. These data were collected by Sargeant, Forbes and Warner (1965) on free roaming red fox and hence were subject to the full range of influences - instrumental, environmental, subject and reader- which affect triangulation data collected at the CCS. Information obtained from these data relating to the variance of the bearing error distribution combined with information provided by the data collected by Kuechle should help to give a practical value for σ_0^2 and range for σ_e^2 in the components of variance model discussed in Chapter 3 section 4.

Not all of the data collect by Sargeant, Forbes and Warner (1965) is relevant to the purpose of this appendix. The portion which will be studied herein was collected during the winter of 1964-65 by mapping trails of radio-tagged fox in fresh snow and comparing these maps against the data recorded on the same fox with the CCS. In total, 9.18 miles of fox tracks incorporating 45 hours and 25 minutes of fox time were collected in seven separate units. Along these seven trail units a total of 32 locations were found and verified in the field where fox had spent time resting, digging or some other activity in a confined region. The CCS recorded a total of 495 transformable bearing pairs on these 32 locations which corresponds to

approximately 6.2 hours being spent by single fox at these locations. These fixes (or transformable pairs of bearings) were extracted from the film record by two observers and included many bearings read from extremely poor quality film records. In other words, every possible bearing was measured and care was exercised so that minimal reader discrimination entered. The bearings were rounded to the nearest degree prior to recording and an effort was made to avoid the bias introduced by the reader's awareness on previous bearings on the same location. The readers were unaware in advance of the precise locations of the 32 trail sites.

Once the data had been extracted as above it was condensed and studied. A somewhat different procedure was followed in studying the bearings on the 32 trail sites from what is normally done with bearings at the CCS. The film quality on each bearing was evaluated and classified as good or bad with regard to clarity and definition of the signal image's shape. Then the data collected on good images was analyzed to determine estimates of the true rounded bearings closest to the unrounded bearings of the 32 sites. To do this, the data collected on good film images was split into 32 groups - one group for each trail site - and then dicotomized as to which tower yielded each bearing. Rather than consider consecutive bearing pairs as individual units, each of the 32 groups of bearings were studied tower by tower. Since the bearings were rounded to the nearest degree, all bearings collected by a given tower on a

single trail were ranked and the sample mode selected as the rounded bearing for that tower closest to the true unrounded bearing of the site. Then all the bearings, both those taken on good film images and poor film images, were compared against the sample modes determined with good quality film images on each tower-trail site set of data; and, the absolute value of the observed bearing's deviation from the sample mode was recorded in degrees. Since each fix is calculated from a pair of bearings and therefore each pair of bearings has two absolute deviations, these deviations were totaled for each fix and a histogram prepared of the totals so calculated for the 495 fixes. These data are given in Table I page 200 taken from Sargeant, Forbes and Warner (1965).

Before these data can be analyzed in an attempt to obtain information directly about the variances of the bearing error distribution for the high and poor quality film images, it is necessary to devise a model relating the variance to these data. Let $\underline{x}'_k$ denote the xy coordinates of the k^{th} site $k=1, \dots, 32$. Let $\underline{\theta}_k$ denote the bearings of $\underline{x}'_k$ relative to the tower locations at the CCS. Since the data is in terms of degrees, all quantities of angular measure in this appendix will use this unit. Hence the vector $\underline{\theta}_k$ has components θ_{1k} and θ_{2k} , $\underline{\theta}'_k = (\theta_{1k}, \theta_{2k})$ and these components are in the interval $[0, 360)$. Each component is closest to one of the integers $0, 1, \dots, 360$ unless one of the components is equal to $I - \frac{1}{2}$, $I=0, \dots, 359$. In rounding the components

θ_{ik} ($i=1,2$), ($k=1,\dots,32$) the closest integer will be taken. In the case of ties the largest of the two will be taken. If 360 is a rounded value, it will be changed to 0 so that all rounded bearings remain in $[0,360)$. Let θ_{ik}^R be the rounded counterpart of θ_{ik} .

The bearings θ_k are unknown parameters and consequently $\theta_k^R = (\theta_{1k}^R, \theta_{2k}^R)$ is also unknown. Sargeant, Forbes and Warner estimated θ_{1k}^R and θ_{2k}^R with sample modes. Since sample modes need not correspond with population modes, the question arises as to the degree of confidence which can be placed in such estimates. Let $\eta_1^R, \dots, \eta_n^R$ be a sample of n rounded bearings taken from a single tower on a single fox trail site and rounded to the nearest degree as described in the preceding paragraph. Assume the sample points are independent random variables. Let η denote the true but unknown unrounded bearing from the tower to the fox trail site and let

η^R be its rounded counterpart. It is necessary to know the conditional probability $P[\eta^R = \text{mode of } \eta_1^R, \dots, \eta_n^R | n]$.

A lower bound for this probability can be calculated quickly.

Let $P[\eta_i^R = \eta^R] = \alpha$ for each $i=1,\dots,n$. Then

$$P[\eta^R = \text{mode of } \eta_1^R, \dots, \eta_n^R | n] \geq$$

$$\sum_{k=[n/2]+1}^n \binom{n}{k} \alpha^k (1-\alpha)^{n-k}. \quad (A1-1)$$

Before this lower bound can be calculated it is necessary to determine the range of α for the data of Table I page 200.

Table I - Observed percentages of absolute deviation sums calculated from pairs of bearings.

Sum of Absolute Deviations From Sample Mode (One sum per pair of bearings)	Fixes Based on High Quality Film Images		Fixes Based on Poor Quality Film Images	
	Number of Fixes	Percent of Total	Number of Fixes	Percent of Total
0	254	64.1	36	36.4
1	115	29.1	49	49.5
2	27	6.8	12	12.1
≥3	0	0.0	2	2.0
Total Number of Fixes	396		99	

Assuming the truncated normal distribution is an acceptable distribution for the observed bearings about η , an evaluation of α is possible.

For mathematical convenience a measure of distance is introduced on the interval $[0, 360)$ which places points close to 360 also close to 0. Refer to section 4.3 of this thesis and define $m(\alpha, \beta) = 180/\pi \cdot D(\alpha\pi/180, \beta\pi/180)$ the distance from α to β when α and β are measured in degrees. Then $m(.,.)$ is a metric on $[0, 360)$.

In terms of $m(.,.)$, it must be recognized that $\eta_i^R = \eta^R$ if and only if $-\frac{1}{2} \leq m(\eta^R, \eta_i) < \frac{1}{2}$. Since $m(\eta^R, \eta_i) = m(\eta, \eta_i) + m(\eta^R, \eta)$ then it follows that $\eta_i^R = \eta^R$ if and only if $-\frac{1}{2} - m(\eta^R, \eta) \leq m(\eta, \eta_i) < \frac{1}{2} - m(\eta^R, \eta)$. If σ is the standard deviation from the random variable η_i about $\eta = E(\eta_i)$, then $m(\eta, \eta_i)/\sigma$ will be assumed to have a truncated standard normal distribution as defined in section 4.3. For $L \geq 4$ the untruncated standard normal distribution is an adequate approximation so the probability that η_i^R equals η^R is,

$$P \left[-\frac{1}{2} - m(\eta^R, \eta) \leq m(\eta, \eta_i) < \frac{1}{2} - m(\eta^R, \eta) \right] = \frac{N \left(\frac{\frac{1}{2} - m(\eta^R, \eta)}{\sigma} \right) - N \left(\frac{-\frac{1}{2} - m(\eta^R, \eta)}{\sigma} \right)}{(A1-2)}$$

Values of $P[\eta_i^R = \eta^R]$ calculated from expression (A1-2) are given in Table II page 202. A table of lower bounds calculated using expression (A1-1) is given in Table III

Table II - Values of $P[\eta_i^R = \eta^R]$ for Different
Standard Deviations and Separations

Value of 5	Values of $m(\eta^R, \eta)$				
	$-1/2^\circ$	$-1/4^\circ$	0°	$1/4^\circ$	$1/2^\circ$
$1/4^\circ$	.500	.840	.955	.840	.500
$1/3^\circ$	.499	.761	.866	.761	.499
$1/2^\circ$	.477	.625	.683	.625	.477

Table III - Values of Lower Bound for Different
Sample Sizes and Bernoulli Probabilities

Sample Size n	$P[\eta_i^R = \eta^R]$				
	.5	.6	.7	.8	.9
8	.363	.594	.806	.944	.955
12	.387	.664	.882	.981	.999
16	.402	.716	.926	.993	1.000
20	.412	.755	.952	.997	1.000

page 202. The range of cases considered in Table III reflects the current set of field data and the results from Table II.

Since 396 high quality bearing pairs were available for estimating sample modes (Table I page 200), on the average approximately 11 observations were used per mode. It would seem that for fox trail sites with bearings such that $-\frac{1}{4}^{\circ} \leq m(\eta^R, \eta) \leq \frac{1}{4}^{\circ}$ and $\frac{1}{4}^{\circ} \leq \sigma \leq \frac{1}{2}^{\circ}$ and a sample size 8 or larger that the probability of a correct mode classification is at least 0.6. If the true but unknown bearings are scattered randomly among the integers 0,1,...,359, one would anticipate half the total numbers of tower-fox trail combinations to have relatively high to very high probabilities of correct mode classification. Should the sample size be low the probability of misclassification is of importance. Unfortunately not anything is known directly about the sizes involved. It would be surprising if a few sites had most of the total number of high quality bearings. If the true but unknown bearing η is close to $I - \frac{1}{2}$ for some $I=0, 1, \dots, 359$ the probability of a misclassification increases. However in this case, an error would most likely result in $I+1$ being used in place of I or vice versa and such a change - although increasing the number of instances an absolute deviation of zero degrees from the sample mode is recorded and perhaps decreasing the sample variance - would not greatly alter conclusions arrived at from the pooled data of Table I. In summary given the probabilities

for correct classification and the effects of misclassification in various cases it seems the data of Table I warrants further analysis. Since misclassification tends to increase the number of entries in the zero class of deviations from the sample mode it could happen that the sample variances are reduced slightly. So, at worst results from Table I might be optimistic. Nevertheless, it seems the data should be analyzed further and to do this additional modeling must be done.

The development which follows depends on the following set of assumptions-

- (1) The observed bearings are all independent.
- (2) The bearing error distribution is fixed for a given film quality and is the same for each tower.
- (3) The truncated standard normal distribution is an appropriate distribution for the errors in (2) and is adequately approximated by the untruncated standard normal distribution.
- (4) All sample modes were also the true but unknown rounded bearings for each tower-fox trail site combination.

A fifth assumption will be added at the needed time. Let a tower-fox trail site be given at random and let η be the true but unknown bearing for this combination. It was shown in the preceeding paragraphs that an observed bearing η_i is rounded to η_i^R and equals η^R with $P \left[\eta_i^R = \eta^R \right]$ given in expression (A1 -2). Using the metric $m(.,.)$ this is equivalent to

$m(\eta^R, \eta_i^R) = 0$. Analogous to the development leading to (A1-2) it can be shown that $P[|m(\eta^R, \eta_i^R)| \leq 1]$ approximately equals

$$N\left(\frac{3/2 - m(\eta^R, \eta)}{\sigma}\right) - N\left(\frac{-3/2 - m(\eta^R, \eta)}{\sigma}\right). \quad (\text{A1-3})$$

Then it is possible to calculate $P[|m(\eta^R, \eta_i^R)| = 1]$ and $P[|m(\eta^R, \eta_i^R)| \geq 2]$. However all these probabilities are based on a difference specified by the tower-fox trail site. Since the trail sites were selected by fox with no knowledge of the bearings it seems reasonable to treat $m(\eta^R, \eta)$ as a random variable distributed uniformly over the interval $[-\frac{1}{2}, \frac{1}{2}]$. That is, in selecting a site two fractional bearing adjustments are selected independently and at random from the uniform distribution over $[-\frac{1}{2}, \frac{1}{2}]$. Then an additional randomization selects two integer bearings from the set given above. The fractional increments determine the precise location about these integer bearings. This constitutes the fifth assumption mentioned above in this appendix. Treating $m(\eta^R, \eta)$ as a uniform random variable over $[-\frac{1}{2}, \frac{1}{2}]$ and taking the expectation of (A1-2) and (A1-3) with respect to this distribution yields the average probabilities of a rounded observed bearing having an absolute deviation of zero degrees and strictly less than two degrees respectively. Let the former be denoted as p_0 and the latter q_2 . Then (A1-4) and (A1-5) given p_0 and q_2 respectively are-

$$p_0 = N(1/\sigma) - N(-1/\sigma) + \frac{2\sigma}{\sqrt{2\pi}} \left(\exp(-\frac{1}{2}(1/\sigma)^2) - 1 \right) \quad (\text{A1-4})$$

$$q_2 = N(1/\sigma) - N(-1/\sigma) + 4(N(2/\sigma) - N(1/\sigma)) \\ + \frac{2\sigma}{\sqrt{2\pi}} (\exp(-2/\sigma^2) - \exp(-1/2\sigma^2)) \quad (A1-5)$$

Let p_1 = the probability of an absolute degree deviation of magnitude one degree and $p_2 = 1 - q_2$ the probability of an absolute degree deviation of magnitude at least two degrees.

$$p_1 = q_2 - p_0 = 4(N(2/\sigma) - N(1/\sigma)) \\ + \frac{2\sigma}{\sqrt{2\pi}} (\exp(-2/\sigma^2) - 2\exp(-1/2\sigma^2) - 1) \quad (A1-6)$$

The probabilities calculated above are not exactly what is needed since they apply only to one tower and the data given in Table I is based on both towers. Presuming outcomes at each tower to be independent and letting Y = the sum of the absolute degree deviations of the two bearings per fix, it follows that $P[Y=0] = p_0^2$, $P[Y=1] = 2p_0p_1$, and $P[Y \geq 2] = 1 - p_0^2 - 2p_0p_1$. Table IV gives these probabilities for different values of σ and also the percentages of Table I with simultaneous confidence intervals calculated using the procedure given by Miller (1966) page 216. Values of $\sigma = \frac{1}{2}^\circ$ and $\frac{1}{4}^\circ$ show acceptable agreement with the good and bad data respectively. These values seem to correspond well with the data collected by Kuechle on page 36. Recalling the components of variance model of Section 3.4, it seems a practical value of σ_0 is $\frac{1}{4}^\circ$ and a practical range for $(\sigma_0^2 + \sigma_e^2)^{\frac{1}{2}}$ is from $\frac{1}{4}^\circ$ to $3/4^\circ$. It also seems the truncated normal distribution can be used with the CCS.

Table IV - Probabilities derived by the model of Appendix I compared with percentages given by Sargeant, Forbes and Warner (1965).

Sum of Absolute Deviations	Values of the Standard Deviation (σ)					Fixes on High Quality Film Images	Fixes on Poor Quality Film Images
	$1/8^\circ$	$1/4^\circ$	$1/3^\circ$	$1/2^\circ$	$3/4^\circ$	$\eta = 3.96$ $\alpha = .05$	$\eta = 99$ $\alpha = .05$
0	.81	.64	.53	.37	.22	.64 $\pm$.059	.36 $\pm$.118
1	.18	.32	.40	.45	.42	.29 $\pm$.056	.50 $\pm$.123
≥ 2	.01	.04	.07	.18	.36	.07 $\pm$.031	.14 $\pm$.085

In applying the computational procedure of Section 7.2 to the Cedar Creek Natural History Area, ultimately estimates of weighted averages are calculated with (7.2.9). Since these estimates are calculated using pseudo-random number sequences, it becomes important to know the approximate magnitude of their variance as well as the dependence of the estimated values on the square size used in partitioning the study region.

Theorem 7.2.1 on page 151 gives an upper bound for the variance of these estimates. Table A2-1 on page 209 gives an evaluation of this bound for the prior probabilities given by D. Huempfer and for three sets of weights.

Table A2-2 on page 210 gives the results of a computational experiment in which the four mile by four mile square containing the Cedar Creek Natural History Area was partitioned into 100 equal subsquares and 144 equal subsquares. The prior distributions for the different species were taken as uniform for this experiment and different pseudorandom number sequences were used for the two trials. Two different weight sets were studied (weight sets 1 and 2 given in Table A2-1) and two different tower site pairs were used. However, the only difference in computer runs between the two partitionings were the partitions themselves and the random number sequences used. Consequently, for a given weight set the corresponding cell values estimate the same value although this value changes from cell to cell.

Table A 2 - 1 - Evaluation of Upper Bound on
Variance of Three Weighted Averages

Species (j 1, . . , 10)	Sum of Squared Probabilities $\sum_{i=1}^8 \sum_{k=1}^8 P_{jik}^2$	Weights		
		1	2	3
Grouse	.0190	.2	.0385	.1
Squirrel	.0182	.2	.0385	.1
Hawk	.0167	.1	.0769	.1
Badger	.0196	.1	.0769	.1
Raccoon	.0180	.1	.0769	.1
Skunk	.0174	.05	.1538	.1
Deer	.0185	.1	.0769	.1
Muskrat	.0405	.05	.1538	.1
Fox	.0172	.05	.1538	.1
Hare	.0220	.05	.1538	.1
Sum of Squared Weights $\sum_{j=1}^{10} w_j^2$ (j=1)		.13	.12	.1
(n _A = 2)				
Upper Bounds Calculated for Variance of (7.2.9) Using Expression (7.2.10)		.0034	.0031	.0026

Table A2-2 - Comparison of two independent partitions of a hypothetical natural history region or study area.

Partition I - Four mile by four mile square into 100 square subdivisions

	Weight Set 1	Weight Set 2
Tower	.6521	.7286
Pair 1	.6449	.7191
Tower	.6974	.7580
Pair 2	.6909	.7531

Partition II - Four mile by four mile square into 144 square subdivisions

	Weight Set 1	Weight Set 2
Tower	.6450	.7187
Pair 1	.6436	.7186
Tower	.7029	.7657
Pair 2	.6906	.7525

The difference of corresponding cell means should have an expectation of zero. A t-test of the hypothesis that the average of the cell mean differences is zero is not significant with $\alpha=0.05$. A pooled variance estimate of the combined tables is $.305 \times 10^{-4}$ computed with 12 degrees of freedom. These results tend to indicate that minor changes in the partitions of the study area will not result in major changes in the conclusions. However, uniform prior probability distributions were used in this experiment; consequently, a second experiment was conducted with different tower sites and the probability distributions given by D. Huempfner using the same weights as in Table A2-2 (namely those of Table A2-1). The computer program of Appendix 3 was executed twice with different pseudo-random number sequences. The same twelve tower site pairs were used between programs. The results and variance estimates are given in Table A2-3.

The results of this experiment are important to Chapter 7 since the same partitioning, prior probability distributions and weight sets were used in this experiment as will be used in Section 7.4. Consequently, the pooled variance estimate of Table A2-3 will be helpful in Section 7.4 when different weighted averages are to be compared.

Table A2-3 - Comparison of two computer runs using probabilities specified by Huempfer.

Tower Pair (i = 1, ..., 12)	Weight Set (j = 1, 2)	Run 1	Run 2	Residual r_{ij}
1	1	.6862	.6872	-.0010
	2	.7486	.7473	+.0013
2	1	.6893	.6921	-.0028
	2	.7537	.7551	-.0014
3	1	.6949	.6913	-.0036
	2	.7569	.7545	.0024
4	1	.6939	.6938	.0001
	2	.7553	.7553	.0000
5	1	.6792	.6794	-.0002
	2	.7483	.7473	.0010
6	1	.6788	.6785	.0003
	2	.7500	.7488	.0012
7	1	.6846	.6835	.0011
	2	.7558	.7550	.0008
8	1	.6866	.6861	.0005
	2	.7563	.7553	.0010
9	1	.6698	.6731	-.0033
	2	.7478	.7509	-.0031
10	1	.6704	.6719	-.0015
	2	.7523	.7529	-.0006
11	1	.6775	.6834	-.0059
	2	.7642	.7673	-.0031
12	1	.6699	.6714	-.0015
	2	.7574	.7574	.0000

$$1/4 \left\{ 1/12 \sum_{j=1}^{12} r_{1j}^2 + 1/12 \sum_{j=1}^{12} r_{2j}^2 \right\} = 1/4 \{ 5.31 + 2.75 \} \times 10^{-6} =$$

$$1.02 \times 10^{-6}$$

The results to the tower allocation problem on the Cedar Creek Natural History Area which were presented in Section 7.2 and 7.3 were obtained by computer simulation. The computational procedure used in this simulation was discussed in Section 7.2 along with various properties of the estimates produced. Appendix II also dealt with the same procedure and computer program in greater detail. This appendix will give an outline of the computer program mentioned above. No copy of the program will be presented in this appendix. Copies can be obtained from the author on request. Figure A3-1 on page 214 gives an abbreviated flowchart of the program. Each portion of this flowchart will be discussed in sequence.

The first part of the program involves considerable input and output. The partition of the study region Ω must be specified (page 144) along with those tower sites to be investigated by the program. The user can either have the program select those sites to be studied or read in fixed tower sites. In either case the program will only investigate those sites which are not too close or too far apart. This necessitates that maximum and minimum tower separation distances be given. The number of species and their names must be given in addition to the geographic prior distributions for each species (page 144 expression (7.2.3)). If it is so desired, the program will specify uniform prior distributions. Information regarding the radio transmitters on each species is

FIGURE A3-1 - OUTLINE OF COMPUTER PROGRAM

needed. The type of transmitter (whip or loop) and the range (page 104) must be set. The habitat attenuation factor (page 101) and the standard deviation in decibels (page 103) are also needed. Other parameters to be read in include the specific preference weights to be investigated, the different variances to be used for the bearing error distribution, the disk radii (pages 97 and 98) which denote the limits of acceptable accuracy for the species, a large positive prime number to seed the pseudo-random number generator, and the desired precision in the estimates of $\underline{w}_R^*$. After some bookkeeping all the above information is printed out by the program.

Next, given the above information regarding the properties of the radio transmitter for each species and the dimensions of the study area, a table of transmission probabilities is constructed using the independence assumption on page 102 and expressions (5.3.6) and (5.3.7). The entries in the table give the probability of receiving a readable signal from a given species at a given distance. In the remainder of the program the table is entered if a transmission probability at a certain distance for a given species is needed. If none of the entries for the species of interest correspond to the distance in question, the program interpolates between the two closest probability values. This computational approach saves considerable time in later parts of the program.

At this point the program counts the total number of possible tower site pairs and proceeds if this number is less than 1000. (This count depends on the minimum and maximum tower separation distances.) After this count, the program sequentially selects acceptable tower site pairs until each pair has been considered. Upon selection, a tower site pair is investigated by having $\hat{R}^*$ computed for each bearing error variance specified in the input. These computations were discussed on pages 145 and 146. Briefly, on each partition set of Ω , points are chosen at random. (Different points being chosen for different tower sites.) At each randomly chosen point, the performance of a given tower pair is simulated generating a scatter of fixes in the plane about the point. This simulated scatter offers information concerning the nature of the induced error distribution in F^2 for the given tower pair. On the basis of each simulated scatter the quantity $\hat{\ell}_j(x)$, $j=1, \dots, 10$ in expression (7.2.6) is calculated. Doing this on each partition square results in (7.2.8) on page 150. This expression is calculated for each bearing error variance and the results are printed out as well as stored on tape. When this is complete for one tower site pair, it is repeated for another acceptable tower site pair until all have been considered.

This completes the program. Further analysis can be run on the results stored on tape. The computational time for 294 tower pairs with 64 squares was about 2 hours on a CDC 3300.

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