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SUBROUTINE LAPINV FOR NUMERICAL INVERSION
OF THE LAPLACE TRANSFORM

R. D. May

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SUBROUTINE LAPINV FOR NUMERICAL INVERSION
OF THE LAPLACE TRANSFORM

R. D. May

SEPTEMBER 1967

OAK RIDGE NATIONAL LABORATORY
Oak Ridge, Tennessee
operated by
UNION CARBIDE CORPORATION
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SUBROUTINE LAPINV FOR NUMERICAL INVERSION
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R. D. May

ABSTRACT

The subroutine LAPINV computes the inverse Laplace transform of a function whose values are known on a set of equally spaced real points. The method of Legendre polynomial approximation employed and the use of this subroutine are described in this memorandum. A few general remarks on the problem of numerical inversion of the Laplace transform are also included.

I. Introduction:

The subroutine LAPINV for the IBM 360/75 is a program for finding the inverse Laplace transform of a function known on a set of equally spaced points on the real axis. The method of inversion assumes an expansion of the inverse transform in terms of Legendre polynomials (after a change of variable). Other systems of orthogonal polynomials would work equally well [1], [2]. Details of the method are set forth in section II.

The subroutine uses the FORTRAN/360 programming language†. Section III contains directions for the use of the subroutine. Various comments on the problem of numerical inversion of the Laplace transform and on the effective use of subroutine LAPINV may be found in section IV. In section V are the results of a few test cases. Appendices provide FORTRAN listings of the subroutines LAPINV and BREAD and the values of the constants to be stored in the array BTABLE.

For routine use of subroutine LAPINV, sections III, IV-C, and V-A should provide sufficient instructions.

†FORTRAN/360 is a FORTRAN dialect in use at the Oak Ridge National Laboratory.

II. Method of Inversion:

We wish to solve the integral equation $F(s) = \int_0^\infty e^{-st} u(t) dt$ for u , given only the values $F(s_i)$, $i = 1, \dots, n$, of F on a set of n equally spaced points $s_1, \dots, s_n$ on the real axis in the s -plane. The substitutions $x = 2e^{-bt} - 1$, $g(x) = e^{at} u(t)$ reduce the equation to the form

$$2bF(s) = \int_{-1}^{+1} \left(\frac{x+1}{2}\right)^{(s+a-b)/b} g(x) dx \quad (\text{where } b > 0).$$

We assume an expansion $g(x) = \sum_{\ell=0}^{\infty} C_\ell P_\ell(x)$ of g in terms of the

Legendre polynomials P_ℓ . ($P_0(x) = 1$, $P_1(x) = x$, $(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$, $n = 1, 2, \dots$.) Choosing b and a so that $s_i = bi - a$, $i = 1, \dots, n$ (i.e., so that $b = \text{increment in } s_i \text{ values}$ and $a = b - s_1$), we have

$$2^i b F(s_i) = \int_{-1}^{+1} (x+1)^{i-1} \sum_{\ell=0}^{\infty} C_\ell P_\ell(x) dx, \quad i = 1, \dots, n.$$

We assume the series for g is uniformly convergent on $[-1, +1]$ (or at least permits term by term integration). Then expanding

$(x+1)^{i-1}$ and using the result $\int_{-1}^{+1} x^k P_\ell(x) dx = 0$ for $k < \ell$,

we find

$$2^i bF(s_i) = \sum_{\ell=0}^{\infty} C_{\ell} \left[\sum_{k=0}^{i-1} \binom{i-1}{k} \int_{-1}^{+1} x^k P_{\ell}(x) dx \right]$$

$$= \sum_{\ell=0}^{i-1} C_{\ell} B(\ell, i-1), \quad i = 1, \dots, n,$$

where $B(\ell, i) = \sum_{k=\ell}^i \binom{i}{k} \int_{-1}^{+1} x^k P_{\ell}(x) dx$, $\binom{i}{k} = i! / k! (i-k)!$.

The first n of the coefficients C_i can be computed by

$$C_0 = 2bF(s_1)/B(0, 0)$$

$$C_m = [2^{m+1} b F(s_{m+1}) - \sum_{\ell=0}^{m-1} C_{\ell} B(\ell, m)]/B(m, m), \quad m=1, 2, \dots, n-1.$$

The value of u at any point t may now be approximated by

$$u(t) = e^{-at} g(2e^{-bt}-1) \doteq e^{-at} \sum_{\ell=0}^{n-1} C_{\ell} P_{\ell}(2e^{-bt}-1), \quad \text{where the values}$$

of the P_{ℓ} are easily computed from the standard recursion relation given above. The constants $B(i, j)$ required are also easily obtained from the relations

$$B(0,0) = 2, B(0, 1) = 2, B(1, 1) = 2/3$$

$$B(0, n+1) = B(0, n) + B(1, n)$$

$$B(\ell, n+1) = B(\ell, n) + \{(\ell+1) B(\ell+1, n) + \ell B(\ell-1, n)\} / (2\ell+1)$$

$$B(n, n+1) = B(n, n) + n B(n-1, n) / (2n+1)$$

$$B(n+1, n+1) = (n+1)B(n, n) / (2n+3)$$

where $n = 1, 2, \dots$ and $\ell = 1, 2, \dots, n-1$. Values of the $B(i, j)$ for $j = 0, 1, \dots, 19$ and $i = 0, 1, \dots, j$ have been computed and are read from data cards by a separate subroutine before the first call on subroutine LAPINV.

A checking option is programmed into the subroutine to allow up to 5 independent estimates of each desired value $u(t)$. This procedure takes values $\{F(s_i)\}$ of F at $N \cdot M$ equally spaced points $s_1, \dots, s_{N \cdot M}$ and breaks the set of s_i into M sets, each containing N points and having a spacing of M times the spacing of the original set. (So for $k = 1, \dots, M$, the k^{th} set contains the points $s_k, s_{k+M}, \dots, s_{k+(N-1)M}$.) The above inversion technique is then applied to each of these M sets to give M independent estimates of u .

III. Use of Subroutine LAPINV:

Before calling LAPINV:

Location BTABLE (I, J) must contain the value $B(I-1, J-1)$ when $1 \leq J \leq 20$, $1 \leq I \leq J$. (Other entries of BTABLE may be arbitrary.) This storage of the constants $B(i, j)$ may be accomplished by calling subroutine BREAD before the first call on LAPINV. BREAD reads the $B(i, j)$ into BTABLE from data cards provided with subroutine LAPINV. [Calling statement for BREAD is CALL BREAD (BTABLE), where BTABLE is a double precision array, dimensioned (20, 20).]

Calling Statement:

CALL LAPINV (F, N, M, SINIT, SINC, ARG, U, UAVE, NUMARG, BTABLE)

Dimensioned Parameters:

F(100), ARG (NUMARG), U(5, NUMARG), UAVE(NUMARG), BTABLE (20, 20).

Types of Parameters:

Double precision F, BTABLE; Real SINIT, SINC, ARG, U, UAVE; Integer N, M, NUMARG.

Entry:

N = number of terms in Legendre polynomial approximation

M = number of checks (independent estimates of inverse transform)

SINIT = initial value of s at which value $F(s)$ is given

SINC = increment in values s at which values $F(s)$ are given

$F(I)$ = value of given function F at point $S_I = SINIT + (I-1) SINC$,

$I = 1, 2, \dots, N \cdot M$. (For $I > N \cdot M$, $F(I)$ may be arbitrary.)

NUMARG = number of arguments at which inverse transform is to be evaluated

$ARG(I)$ = value of I^{th} argument at which inverse transform is to be evaluated, $I = 1, 2, \dots, NUMARG$

$BTABLE$ must contain the values read in by $BREAD$.

U , $UAVE$ may contain any values.

Cautions:

Parameters N , M and $SINC$ must satisfy $2 \leq N \leq 20$, $1 \leq M \leq 5$, $SINC > 0$. The choice of N , $SINC$, and $SINIT$ greatly affects the performance of $LAPINV$. See sections IV-C and V-A for suggestions of reasonable values for these parameters.

Exit:

No programmed abnormal exits. Exponential overflow or underflow cancels execution of program. For normal exit:

$U(I, J) = I^{th}$ estimate of value of inverse transform at J^{th} argument, $I = 1, \dots, M$; $J = 1, \dots, NUMARG$

$UAVE(J) = \text{average of } U(I, J), U(2, J), \dots, U(M, J); J = 1, \dots, NUMARG.$

No other parameters are redefined or altered during execution of $LAPINV$.

IV. Comments and Cautions:

A. The perverse nature of the inverse Laplace transform

The process of taking the inverse Laplace transform of a function given on the real axis is inherently unstable (an "unbounded operator"), tending to magnify errors by an indeterminate factor. Therefore, no numerical inversion technique is entirely satisfactory. A sufficiently well-behaved function u is determined by the values of its Laplace transform $F = \int_0^{\infty} e^{-st} u(t) dt$ on an infinite set of equally spaced real points $s_1, s_2, \dots$. (This fact can be seen from the results of section II if "sufficiently well-behaved" is taken to mean that the Legendre polynomial expansion given there is well defined, converges to g on $[-1, +1]$, and permits term by term integration.) Three types of difficulties arise in making practical use of this fact: 1) the function u in question may not be "well-behaved", 2) we can use only a finite number of points $s_1, s_2, \dots, s_N$, and 3) the values of F at these points are known only within certain limits.

Difficulty 1) may often be ignored. One troublesome case arises when the function $e^{at} u(t)$ does not have a well-defined, finite limit as $t \rightarrow 0$ or $t \rightarrow \infty$ for the value of a chosen[†]. The Legendre polynomial expansion in section II of $g(x) = (\frac{x+1}{2})^{-a/b} u(-\frac{1}{b} \ln(\frac{x+1}{2}))$ will then be of questionable validity. Choice of a negative a may "damp out" bad behavior of u as $t \rightarrow \infty$. If $u(t)$ tends to ∞ as $t \rightarrow \infty$, LAPINV may still give accurate values $u(t)$ for small t .

[†] In terms of the input parameters of LAPINV, have $a = K \cdot \text{SINC} - \text{SINIT}$ for $K = 1, 2, \dots, M$, and $b = M \cdot \text{SINC}$.

For large t , the computed values for $u(t)$ tend to level off at a constant value rather than increase as the true $u(t)$. See section V-C for an example.) Some simple jump discontinuities in u will also allow a good enough polynomial approximation for LAPINV to produce a general idea of the behavior of u . (See section V-D.)

Difficulty 2) should not in itself prove too serious, since 20 terms of the Legendre polynomial expansion should give a reasonable approximation to most functions occurring. (If more than 20 terms are wanted, LAPINV is easily modified by adjusting appropriate dimension statements and do-loop parameters, and the additional constants $B(i, j)$ needed can be computed from the relations given in section II. However, it is unlikely that more than 20 terms can be reliably computed because of the cancellation errors mentioned below.)

The major difficulty in computing the inverse transform arises from the lack of precision with which the values of the Laplace transform F are known. This lack of precision may be greatly magnified by the unstable character of the inverse transform, which in subroutine LAPINV takes the form of cancellation errors in the computation of the coefficients C_i . These errors compound rapidly as the order i of the coefficients C_i increases, and they generally result in meaningless values for higher order coefficients (and also for u in which these coefficients are used). The number of coefficients that can be reliably computed depends on the accuracy with which F is known. For example, when F is known with single precision accuracy (7 significant digits), about 10 terms of the Legendre

expansion seems to be the limit. Section V-A gives more guidelines for the choice of N based on results of several test cases. Of course the use of a smaller value of N (i.e., a smaller number of terms) to avoid cancellation errors may result in a poor polynomial approximation. If possible, several values of N should be used in a search for an optimum value. [Even this optimum value may well be none too good, especially if f is known to say 3 or 4 figures only. In such cases one might, if desperate, try a successive approximations approach, as in Bellman, Kalaba, and Lockett, Numerical Inversion of the Laplace Transform, Chapter five. A FORTRAN-IV program for this procedure may be found in Appendix One (p. 187) of that book. Subroutine LAPINV may be used to obtain the initial approximation needed for this approach, and it has the advantage that this initial estimate of $u(t)$, unlike that used by Bellman, can be found at any number of arguments t even for low order N . This may hasten convergence of the process, but has not been tested.]

The double precision arithmetic used in LAPINV is intended to reduce cancellation errors (and any round-off errors) enough to permit use of large N when F is known to many significant digits. However, the inverse transform u of F so obtained is still at best a smooth representative of an infinite class of widely varying functions having Laplace transforms within the error bounds of the given values of F . For example, sharp spikes or rapid oscillations in u cannot be resolved.

(More precisely: The Laplace transform of a rectangular pulse of height h , width w , beginning at $t = a$, is $h \cdot (e^{-as}/s) \cdot (1 - e^{-ws}) \doteq hwe^{-as}$ for sw small. That of a sine curve of amplitude A and frequency $b/2\pi$ is $b A/(s^2 + b^2) \doteq A/b$ for b large. Let E measure the estimated error in the values of f . Then when $w \ll (E/h) e^{as}$ for the s values used, or when $b \gg A/E$, the corresponding pulse or oscillation will be unresolvable.)

To summarize, subroutine LAPINV may not always work.

B. Checking procedure

There is nothing particularly subtle about the checking option of subroutine LAPINV. If other methods of inverting Laplace transforms are available, comparison of their results with those of LAPINV may be preferable. Or, using LAPINV alone, one can simply call on LAPINV several times using various sets of equally spaced points $\{s_i\}$ and check the results for consistency. This process has the advantage of allowing one to alter the parameters N , $SINC$, and $SINIT$ between trials. It may also be necessary if values of F are given for only a limited number of points. (Say one wants a 10-term expansion, $N = 10$, and 3 checks, $M = 3$. This requires 30 values of f as LAPINV is programmed. If F is known at only 12 equally spaced points, one can call LAPINV 3 times with $N = 10$, $M = 1$ and using the first 10, middle 10, and last 10 points.)

Some type of checking procedure is recommended, since it is easy to get absurd results (usually in the form of wild oscillations) by an unfortunate choice of parameters N , $SINC$, and $SINIT$. Checking the computed value of the inverse transform u by calculating its

Laplace transform and comparing with the original F values is not recommended, for just as the inversion process tends to magnify errors, the process of taking direct Laplace transforms tends to decrease them. A false sense of security regarding the accuracy of the computed inverse transform u may therefore result from comparing the computed and given values of F .

C. Choice of SINC and SINIT

The behavior of the inverse transform u for large t is determined by that of F for small s ; the behavior of u for small t is determined by that of F for large s (see any text on Laplace transforms). Using this criterion, the points s where F is evaluated should be chosen according to the range of t where $u(t)$ is to be computed. Because of the e^{-st} term appearing in the Laplace transform, "large" may mean " > 5 " and "small" may mean " < 0.1 " in this discussion.

The expressions e^{-bt} and e^{-at} occur in LAPINV, where $b = M \cdot \text{SINC}$ and $a = K \cdot \text{SINC} - \text{SINIT}$, $K = 1, 2, \dots, M$, and t is any argument for which $u(t)$ is to be computed (M = number of checks). One must have $|at| < 174$, $bt < 174$ (on the 360/75) for all t used to avoid exponential overflow and underflow. In practice, however, these values should be kept much smaller, say < 3 . As a rough guide, try keeping $|at| < 0.5$ and $.05 < bt < 2$ for the range of t being used as arguments for u .

[Because of the distortion caused by the substitution $x = 2e^{-bt} - 1$, the argument x at which the Legendre polynomials of section II are evaluated is treated as -1 (resp., $+1$) when bt becomes too large (resp., too small).

The values of u obtained for such t will be approximations to $\lim_{t \rightarrow \infty} u(t) = \lim_{x \rightarrow -1} \left(\frac{1+x}{2}\right)^{a/b} g(x)$ (resp., $\lim_{t \rightarrow 0} u(t)$) rather than the desired $u(t)$.

If at becomes large, $g(x) = e^{at} u(t)$ may tend too rapidly to infinity as $x \rightarrow -1$ ($t \rightarrow \infty$) to allow a good polynomial approximation.]

V. Test Cases:

Recall that in our notation $F(s) =$ Laplace transform of u , $u(t) =$ inverse transform of F , $x = 2e^{-bt} - 1$, $t = \frac{-1}{b} \ln \left(\frac{1+x}{2}\right)$, $g(x) = e^{at} u(t)$, $b = M \cdot \text{SINC}$, $a = K \cdot \text{SINC} - \text{SINIT}$ for $K = 1, \dots, M$, and $M =$ number of checks, $N =$ number of terms in the Legendre polynomial expansion of g .

A. Choice of N

The following table is a rough guide for choosing N based on the results given below and other tests. The first number listed is a suggested N . A smaller N than listed will give better results if $g(x)$ is well approximated by a few terms of its Legendre polynomial expansion. A larger N than listed will give better results if some high order terms in the expansion of g are large. The second number listed is the maximum N for which meaningful results are likely to be obtained.

<u>Number of significant figures to which F is given</u>	<u>Suggested N</u>	<u>Maximum N</u>
16 (Double precision)	12-15	18-20
7 (Single precision)	8-9	10-11
5	5-6	7-8
3	3-4	4-5

The function $F(s) = 1/(s + 1/2)$ has the inverse Laplace transform $u(t) = e^{-t/2}$. Values of F were computed to several different degrees of accuracy. Using LAPINV and various choices of N , the corresponding estimates of u were found for 20 arguments in the interval $[0, 4]$. Results are given in the form of the maximum percentage error in the computed $u(t)$ for t in $[0, 4]$. (Since the computed u tends to oscillate about the true u values, the average error will be somewhat less.)

Double Precision Accuracy for F

<u>N</u>	<u>% error (maximum for $0 \leq t \leq 4$) in computed $u(t)$</u>
12	$6 \times 10^{-6} \%$
15	$2 \times 10^{-3} \%$
18	.4%

Note: For the above case, $a = 0.0$, $b = 0.5$, and $g(x) = \frac{x+1}{2}$. Since g has an exact Legendre expansion of two terms, all errors must be cancellation errors. For the remaining cases below, $a = 0.0$, $b = 1.0$, and $g(x) = (\frac{x+1}{2})^{1/2}$.

Single Precision Accuracy for F

<u>N</u>	<u>% error in computed $u(t)$ (maximum for $0 \leq t \leq 4$)</u>
8	8% ($< 1\%$ over range $0 \leq t \leq 2$)
10	50% ($< 12\%$ over range $0 \leq t \leq 1.6$)
12	large oscillations (see graph of figure 1)

F Given to Five Significant Figures

<u>N</u>	<u>% error in computed $u(t)$ (maximum for $0 \leq t \leq 4$)</u>	
3	50%	(< 4.5% over range $0 \leq t \leq 1.6$)
5	6%	(< 1.5% over range $0 \leq t \leq 2$)
7	50%	(< 8% over range $0 \leq t \leq 1.6$)

F Given to Three Significant Figures

<u>N</u>	<u>% error in computed $u(t)$ (maximum for $0 \leq t \leq 4$)</u>	
3	50%	(< 6% over range $0 \leq t \leq 2$)
4	5%	(< 4% over range $0 \leq t \leq 2$)
5	large oscillations (see graph of figure 1)	

When N is too large for the given data, the computed values of u tend to oscillate wildly about the true u values as illustrated in figure 1.

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Figure 1

B. Choice of SINC and SINIT

The guidelines given in section IV-C appear reasonable from the results listed below and other tests.

The function $F(s) = 1/(s+1)^2+1$ has the inverse Laplace transform $u(t) = e^{-t} \sin(t)$. Values of F were given with single precision accuracy (7 significant figures) and LAPINV was used to compute u with $N = 8$. The best results, obtained with $a = 0.0$, $b = 0.3$, were as follows:

<u>Argument</u>	<u>Exact $u(t)$ to 4 places</u>	<u>Computed $u(t)$ to 4 places</u>
0.2	0.1627	0.1634
0.4	0.2610	0.2611
0.6	0.3099	0.3094
0.8	0.3223	0.3219
1.0	0.3096	0.3094
1.2	0.2807	0.2809
1.4	0.2430	0.2434
1.6	0.2018	0.2022
1.8	0.1610	0.1613
2.0	0.1231	0.1232
2.2	0.0896	0.0896
2.4	0.0613	0.0611
2.6	0.0383	0.0381
2.8	0.0204	0.0201
3.0	0.0070	0.0067
3.2	0.0024	0.0027
3.4	0.0085	0.0089
3.6	0.0121	0.0124
3.8	0.0137	0.0140
4.0	0.0139	0.0141

Results of similar accuracy were obtained for $b = 0.2$ and $0 \leq a \leq 0.2$. For $b = 0.1$ or $b = 0.4, 0.5, 1.0$ the results were slightly less satisfactory (about two-place accuracy).

C. Bad behavior of $u(t)$ as $t \rightarrow \infty$

The function $F(s) = 1/s^2$ has the inverse transform $u(t) = t$. Thus the function $g(x) = t e^{at}$, $t = \frac{-1}{b} \ln \left(\frac{1+x}{2} \right)$, which is to be expanded in Legendre polynomials is not well-defined at $x = -1$ ($t = \infty$) unless $a < 0$. Nevertheless, LAPINV gives reasonable results for $a = 0$ and good results for $a = -0.25$. (For $a = +0.25$ results were rather poor as might be expected.) The following data was obtained for $F(s) = 1/s^2$ given to seven significant figures and with $N = 8$, $b = 0.75$:

<u>Argument</u>	<u>Exact $u(t)$ (to 3 places)</u>	<u>Calculated $u(t)$</u>		
		<u>$a = -.25$</u>	<u>$a = 0.0$</u>	<u>$a = +.25$</u>
0.4	0.400	0.400	0.398	0.33
0.8	0.800	0.801	0.794	0.77
1.2	1.200	1.209	1.152	0.51
1.6	1.600	1.584	1.676	2.46
2.0	2.000	1.986	2.061	2.75
2.4	2.400	2.419	2.330	1.84
2.8	2.800	2.845	2.645	1.41
3.2	3.200	3.238	3.080	2.04
3.6	3.600	3.595	3.612	2.39
4.0	4.000	3.928	4.179	4.93

Similar results were obtained for $N = 10$.

When $u(t) \rightarrow \infty$ as above, there is a tendency for the calculated values of $u(t)$ to level off at a constant value (especially for large b) as illustrated in the following results for $a = 0$, $b = 1.5$, $N = 9$. (See also the section in reference [1] where this same example is treated.)

<u>Argument</u>	<u>Exact u(t)</u>	<u>Computed u(t)</u>	
0.5	0.50	0.51	} fairly good approximation for $t < 4$
1.0	1.00	1.04	
1.5	1.50	1.43	
2.0	2.00	2.02	
2.5	2.50	2.69	
3.0	3.00	3.16	} leveling off at about $u(t) = 3.6$ for $t \geq 4$.
3.5	3.50	3.43	
4.0	4.00	3.57	
4.5	4.50	3.63	
5.0	5.00	3.66	

D. Discontinuities

Even when u is discontinuous, the function $g(x) = e^{at} u(t)$, $t = \frac{-1}{b} \ln \left(\frac{1+x}{2} \right)$, can sometimes be approximated well enough on $[-1, +1]$ by polynomials for LAPINV to give rough results.

The function $F(s) = e^{-2s}/s$ is the Laplace transform of $u(t) = \begin{cases} 0, & t \leq 2 \\ 1, & t > 2 \end{cases}$. Figure 2 shows the results of applying LAPINV with given to 7 significant figures, $N = 10$, $b = 0.1$, and $a = 0.0, 0.5$ (which is not an optimal choice of parameters).

E. Time

Tests indicate that a single call on subroutine LAPINV with $N = 20$, $M = 1$, $NUMARG = 10$ takes less than 34 milliseconds. This time should increase at most linearly with $M \cdot NUMARG$ and should be considerably less for smaller values of N .

Figure 2

References

- [1] M. K. Miller and W. T. Guy, Jr., Numerical Inversion of the Laplace Transform by use of Jacobi Polynomials, SIAM Journal of Num. Anal. 3, 624 (1966).
- [2] A. Papoulis, A New Method of Inversion of the Laplace Transform, Quart. of Applied Math. 14, 405-414 (1957).
- [3] R. Bellman, R. Kalaba, J. Lockett, Numerical Inversion of the Laplace Transform, American Elsevier Publishing Co., New York, 1966.

APPENDIX 1

```

SUBROUTINE LAPINV(F,N,M,SINIT,SINC,ARG,U,UAVE,NUMARG,BTABLE)
  DIMENSION F(100),ARG(NUMARG),U(5,NUMARG),UAVE(NUMARG),P(20)
  DOUBLE PRECISION BTABLE(20,20),C(5,20),F,SINC2,P,SUM,DBLI
  REAL M=M
  SINCM=SINC*REALM
  SINC2=DBLE(2.0*SINCM)
  DO 5 NUMCHK=1,M
5 C(NUMCHK,1)=SINC2*F(NUMCHK)/BTABLE(1,1)
  DO 10 J=2,N
    SINC2=2.0*SINC2
    DO 10 NUMCHK=1,M
      SUM=0.0
      JM1=J-1
      DO 20 I=1,JM1
20 SUM=SUM+C(NUMCHK,I)*BTABLE(I,J)
      INDEX=NUMCHK+JM1*M
10 C(NUMCHK,J)=(SINC2*F(INDEX)-SUM)/BTABLE(J,J)
    P(1)=1.0
    DO 100 J=1,NUMARG
      P(2)=DBLE(2.0*EXP(-SINCM*ARG(J))-1.0)
      DO 50 I=3,N
        DBLI=I
50 P(I)=((2.0*DBLI-3.0)*P(2)*P(I-1)-(DBLI-2.0)*P(I-2))/(DBLI-1.0)
      DO 100 NUMCHK=1,M
        SUM=C(NUMCHK,1)
      DO 60 I=2,N
60 SUM=SUM+C(NUMCHK,I)*P(I)
100 U(NUMCHK,J)=SNGL(SUM)
    DO 300 NUMCHK=1,M
      CHKNUM=NUMCHK
      ALPHA=SINCM-SINIT-(CHKNUM-1.0)*SINC
      IF(ALPHA.EQ.0.0) 300,200
200 DO 210 J=1,NUMARG
210 U(NUMCHK,J)=EXP(-ALPHA*ARG(J))*U(NUMCHK,J)
300 CONTINUE
    DO 320 J=1,NUMARG
      ADD=0.0
      DO 310 NUMCHK=1,M
310 ADD=ADD+U(NUMCHK,J)
320 UAVE(J)=ADD/REALM
  RETURN
END

```

APPENDIX 2

```
SUBROUTINE BREAD(BTABLE)
DOUBLE PRECISION BTABLE(20,20)
DO 5 J=1,20
  5 READ 10.(BTABLE(I,J),I=1,J)
 10 FORMAT(3D25.16)
  RETURN
  END
```

APPENDIX 3

Table of the Constants $B(l, n) = \sum_{k=1}^n \binom{n}{k} \int_{-1}^{+1} x^k P_l(x) dx$ for $n = 0, 1, \dots, 19$ and $l = 0, 1, \dots, n$.

Note: The constant $B(I-1, J-1)$ is to be stored in location BTABLE (I, J) for $1 \leq J \leq 20, 1 \leq I \leq J$.

n										
0	C.200000000000000000 01									
1	C.200000000000000000 01	0.606666666666666650 00								
2	C.206666666666666666 01	0.133333333333333310 01	0.266666666666666660 00							
3	C.399999999999999995 01	0.235999999999999980 01	0.799999999999999940 00	0.11428571428571410 00						
4	C.639999999999999970 01	0.426666666666666610 01	0.18285714285714240 01	0.45714285714285680 00	0.50793650793650750-01					
5	C.106666666666666665 02 C.23088023088023050-01	0.76190476190476120 01	0.38095238095238050 01	0.12698412698412630 01	0.25396825396825370 00					
6	C.18285714285714210 02 C.13852813852813830 00	0.13714285714285700 02 C.10656010656010630-01	0.76190476190476070 01	0.30476190476190430 01	0.83116883116883010 00					
7	0.319999999999999940 02 0.52214452214452130 00	0.246666666666666670 02 0.74592074592074500-01	0.14933333333333310 02 0.49728049728049630-02	0.67878787878787760 01	0.22626262626262570 01					
8	0.568488888888888790 02 0.15912975912975950 01	0.4551111111111110990 02 0.31825951825951790 00	0.28961616161616050 02 0.39782439782439710-01	0.14480808080808050 02 C.23401435166141010-02	0.55695415695415600 01					
9	0.102399999999999980 03 0.42965034965034900 01	0.83781818181817980 02 0.10741258741258690 01	0.55854545454545360 02 0.18955162484574200 00	0.30075524475524370 02 0.21061291649526900-01	0.12889510489510460 02 0.11084890341856240-02					
10	0.18618181818181790 03 C.10741258741258710 02 C.52765192104077440-03	0.15515151515151480 03 0.31591937474290340 01	0.10741258741258710 03 0.70204305498423000 00	0.61378621378621220 02 0.11084890341856270 00	0.28643356643356580 02 0.11084890341856240-01					
11	0.341333333333333190 03 0.2594841628392927500 02 C.58063711314485180-02	0.28882051282051170 03 0.85947209653091860 01 0.25245691875863100-03	0.20630036630036570 03 0.22354228566076760 01	0.12378021978021930 03 0.54709057712153610 00	0.61890109890109770 02 0.63870082445533700-01					
12	0.63015384615384470 03 0.5824451519061500 02 C.36352932301262900-01	0.54013146813186610 03 0.21460347701833690 02 0.30294110451035740-02	0.39604670329670170 03 0.64381043105501190 01 0.12117644106414290-03	0.24756043956043890 03 0.15328819787024050 01	0.13106140917905580 03 0.27870581430952890 00					
13	0.11702857142857100 04 C.12952852720035340 03 C.17065642106083460 00	0.10142476190476140 04 0.51811410880141400 02 0.2047881852970150-01	0.76068571428571240 03 0.17270470293380470 02 0.15752937330538570-02	0.49220840336134270 03 0.47101282618310370 01 0.5834421233528110-04	0.27344911297852370 03 0.10239409264850050 01					
14	C.21845333333333270 04 C.24208434812521400 03 C.66897473863647180 00	0.19114666666666610 04 0.12084327205366310 03 0.10291919055951860 00	0.14617098039215650 04 0.43961197110423010 02 0.11435465617724300-01	0.97447320261437650 03 0.13379494772737430 02 0.81681897269459360-03	0.56416869625042800 03 0.33448736931843560 01 0.28166171472227350-04					
15	0.409999999999999980 04 0.624466666666666650 03 C.23156817875371700 01 C.1362879264781930-04	0.36141176470588140 04 0.27475748194014320 03 C.42882496066466130 00	0.28100803921568540 04 0.19751379278092570 03 0.61261422952094400-01	0.19233023735810060 04 0.35837932426975260 02 0.63373885812511550-02	0.11539814241486020 04 0.10034621079553060 02 0.42249257208341010-03					
16	0.77101176476587940 04 C.12748711388066500 04 C.7318697995435530 01 C.2100660236503100-03	0.68534379084967070 04 0.61183604675371090 02 0.15682924273736160 01 0.66078494656251890-03	0.54106088751289770 04 0.25484751948071290 03 0.270394524615338250 00	0.37874262125902880 04 0.91745107013056710 02 0.3605269484451000-01	0.23445971792225600 04 0.28229263696325100 02 0.34889709178500980-02					
17	0.14563555555555510 05 0.25782157501711200 04 0.21982691027064520 02 C.19096824566780-02	0.14036549707602790 05 C.1399107875089650 04 0.52096150755031690 01 0.11233425091562820-03	0.10424439766081830 05 0.59920746603940790 03 0.10419230151006320 01 0.32095511690179460-03	0.74460284043441630 04 0.22661825578439720 03 0.16805209920977960 00	0.47383817118551840 04 0.7553941854979960 02 0.21006512401222440-01					
18	C.2759410526157800 05 C.55262488996225800 04 0.60282688730827360 02 C.12132103418387820-01	0.24814694736842030 05 0.28924736101934590 04 0.16075383661552610 02 0.10398454787614130-02	0.20104276691729250 05 0.13349875431062160 04 0.36299253429312390 01 0.57771421942323030-04	0.14621292139439450 05 0.54394381388253040 03 0.68061100179960740 00 0.15614032714141340-05	0.95356253083300820 04 0.19424421924376100 03 0.10312287906054650 00					
19	C.524287999999999930 05 C.11574532787990510 05 C.16120037505056950 03 C.82566994488435700-01	0.4743588052180800 05 C.61785461933718150 04 C.46820108885649180 02 0.6951847812042670-02	0.38810929879124750 05 C.29748766486605090 04 0.11700027221417740 02 0.56366688048050100-03	0.2868633949226340 05 C.12740446209545020 04 0.74818239560571500 01 0.29666662156888570-04	0.19174226312817540 05 0.88360112515170830 02 0.43768893342185080 00 0.76068364504791240-06					

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