REACTOR KINETICS
QUARTERLY PROGRESS REPORT
JANUARY - MARCH 1957

EDITED BY
M. E. REMLEY

ATOMICS INTERNATIONAL

A DIVISION OF NORTH AMERICAN AVIATION, INC.
P.O. BOX 309 CANOGA PARK, CALIFORNIA

CONTRACT: AT(11-1)-GEN-8
ISSUED: JANUARY 1, 1959

DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency Thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

DISCLAIMER

Portions of this document may be illegible in electronic image products. Images are produced from the best available original document.

DISTRIBUTION

This report has been distributed according to the category "Progress Reports" as given in "Standard Distribution Lists for Unclassified Scientific and Technical Reports" TID-4500 (14th Ed.), October 1, 1958. A total of 620 copies was printed.

ABSTRACT

Investigation of the energy coefficient of radiolytic gas production for the KEWB I reactor as a function of core temperature shows the coefficient to be independent of temperature above about 30°C. The temperature coefficient of reactivity is found to increase markedly with temperature and reach a value of 0.031 %/°C at 90°C with a core pressure of 71 cm Hg. A void coefficient of reactivity has been determined in the central exposure tube of the reactor, and is related to the void coefficient derived from the temperature coefficient measurements. Investigations of the transient power oscillations show a decrease in the period of oscillation and an increase in the damping of the oscillation with increasing reactivity input.

An evaluation of the "boron-bubble" experiment and its extension to more complex geometries has been made in connection with the determination of "absolute reactivity." This study has indicated a possible solution to the difficulties involved in the experimental determination of the absolute reactivity as contrasted with the reactivity in dollars observable in kinetic experiments.

A method has been devised for determining the transfer function of a reactor from power fluctuations during steady operation by treating the fluctuations as the response of the system to an inherent random or stochastic driving force.

A power series solution of the one-group reactor kinetics equations has been obtained for the reactivity as a prescribed cubic function of time.

TABLE OF CONTENTS

	Page No.
I. The KEWB I Program	5
A. Introduction	5
B. Energy Coefficient of Gas Production	6
C. Temperature Coefficient of Reactivity	6
D. Effect of Voids Upon Reactivity	8
E. Transient Power Oscillations	15
F. References	21
II. Theoretical Reactor Kinetics Studies	22
A. The Theory of a Modified Boron Bubble Experiment	22
B. The Determination of Reactor Transfer Functions from Measurements at Steady Operation.	33
C. Power Series Solutions of Reactor Kinetics Equations	35

LIST OF FIGURES

1. Energy Coefficient of Gas Production as a Function of Core Temperature	7
2. Reactivity Compensation by Core Temperature Rise (Referred to 25°C).	9
3. Temperature Coefficient of Reactivity as a Function of Core Temperature	10
4. Reactivity Compensated by Calibrated Voids in the Center of the Central Exposure Facility	12
5. Void Coefficient of Reactivity as a Function of Void Volume in the Central Exposure Facility.	13
6. Reactivity Compensation by a Fixed Void Volume as a Function of Position in the Central Exposure Tube.	16
7. Variation of Oscillatory Parameters with Reactivity Input at Initial Core Pressure of 60 Cm Hg and Initial Core Temperature of 25°C	19
8. Variation of Oscillatory Parameters with Reactivity Input at Initial Core Pressure of 60 Cm Hg and Initial Core Temperature of 35-40°C.	20

I. THE KEWB I PROGRAM

J. W. Flora	D. R. Muller
D. P. Gamble	G. C. Murphy
E. L. Gardner	M. E. Remley
E. B. Hecker	G. Schumann
D. L. Hetrick	R. E. Wimmer

A. INTRODUCTION

The KEWB (Kinetic Experiments on Water Boilers) program is devoted to the experimental study of the dynamic behavior of homogeneous reactors, with particular emphasis on resolving and understanding the parameters which affect the inherent shutdown mechanisms and the magnitude of the energy release following relatively large reactivity releases under various operating conditions. More detailed descriptions and discussions of the KEWB objectives together with previous results have been reported earlier.^(1,2,3,4)

The current effort resulted in the completion of the investigation of the energy coefficient of radiolytic gas production. Additional information was obtained on the temperature coefficient of reactivity, so that a more precise determination of the coefficient over the temperature range from 25 to 95°C is presented below. The void coefficient of reactivity was studied in the central exposure facility of the core. The neutron flux distribution through the central exposure tube was also determined for the present reactivity loading of 4 percent. A study of the power coefficient of reactivity was initiated, and the preliminary results are now under analysis. A group of 15 transients with reactivity releases ranging from 0.4 to 0.7 percent was undertaken to study the oscillatory behavior of the reactor power following the transient power peaks.

The remainder of the quarter was scheduled for overhaul and repair of core instrumentation and for location and replacement of a leaking valve in the plumbing for the gas system. A failure occurred in one of the pressure transducers which was directly exposed to the fuel solution. This transducer, fabricated of Type 416 stainless steel, had been originally gold plated and installed at the bottom of the reactor core vessel. The high radioactivity of this defective transducer has so far prevented careful examination of it, so the exact cause of

failure is presently unknown. Replacement was made with a transducer having the same characteristics but specially fabricated of Type 316 stainless steel with a one-piece body.

A discussion of the experimental results is given below.

B. ENERGY COEFFICIENT OF GAS PRODUCTION

To conclude the investigation of the energy coefficient of gas production, the coefficient was examined for temperature dependence. Care was taken to avoid history-dependent effects resulting from variation in the saturation of dissolved gases in the core solution, as this had previously been demonstrated to be a source of error.⁽⁴⁾ A rigorous schedule of saturation preparation of the fuel solution was followed for each experiment.

In the range of from 25 to 90°C, the coefficient has a value of $14.0 \pm 0.3 \text{ cm}^3$ (at STP) of H_2 and O_2 per watt-hour, independent of reactor fuel temperature. Each of the experimental runs which contributed to these data resulted in a final core pressure of 68 cm Hg absolute. The dependence of the coefficient on the core pressure has been reported earlier. Below 25°C, the coefficient decreases with decreasing temperature. At 17.5°C, a value of $12.9 \pm 0.6 \text{ cm}^3/\text{watt-hour}$ was measured. Investigations of behavior at lower temperatures have been prohibited by cooling system limitations. The complete results of the investigation are shown in Fig. 1, where 95 percent confidence limits are shown on the experimental points.

C. TEMPERATURE COEFFICIENT OF REACTIVITY

Final measurements of the temperature coefficient of reactivity were made after completing the loading of the reactor to 4 percent excess. Two techniques for measurement were used. First, critical rod positions at various temperatures were noted and reactivity changes with temperature were then interpolated from the rod calibration curves. Second, period measurements were used to determine directly the reactivity compensated by a core temperature variation between known pairs of temperatures. The detailed procedure for the transient technique was as follows: A critical rod position at temperature T_1 was determined. The reactor was shut down and the core cooled to a lower temperature T_2 . The control rods were then withdrawn to the position corresponding to

Figure 1. Energy Coefficient of Gas Production as a Function of Core Temperature

criticality at temperature T_1 . The reactivity made available by this temperature difference between T_1 and T_2 were then computed from the observed stable reactor period. This technique was followed for a series of temperature steps, and the results thus obtained were in good agreement with the results from the first technique.

The measurements were made over the temperature range from 25 to 95°C, with the core pressure adjusted at 71 cm of Hg, so that the boiling point of the solution was 98°C. The reactor power varied between 4 watts and 5 kilowatts. No power dependence was noted in the results. The composite results of approximately 130 measurements of the reactivity compensation by core temperature rise are shown in Fig. 2. The deviations on the plotted points show the 95 percent confidence limits on the composite results.

From the curve in Fig. 2, the temperature coefficient of reactivity was determined as a function of temperature. These results are shown in Fig. 3, which is the first derivative of the curve in Fig. 2. The general shape of the temperature coefficient curve is as would be expected, with the coefficient increasing rapidly in magnitude as the core temperature approaches the boiling point. It should be noted here that the measured temperature is a mean core temperature, and that localized boiling can occur during operation at mean temperatures near boiling. This latter effect will contribute markedly to the increase in the magnitude of the temperature coefficient.

The magnitude of the coefficient is smaller than those observed for other similar reactors, e.g., SUPO⁽⁵⁾ at Los Alamos and the Livermore Research Reactor.⁽⁶⁾

D. EFFECT OF VOIDS UPON REACTIVITY

The introduction of a void into the core region causes a decrease in reactivity of the core. The magnitude of this effect is dependent upon both the size and the position of the void. Such voids arise during reactor operation, and are caused by radiolytic gases and steam. Experiments to determine the effectiveness of this process have been performed recently.

Since direct access to the fuel solution is impractical (the core is a welded stainless-steel vessel), the experiments were performed in the central exposure

Figure 2. Reactivity Compensation by Core Temperature Rise (Referred to 25°C)

Figure 3. Temperature Coefficient of Reactivity as a Function of Core Temperature

(AI)

facility, a tube 1.84 inches I.D., located on a horizontal diameter of the spherical core. A series of lucite cylinders, containing calibrated voids, were inserted into the core region of this tube, and graphite plugs were inserted into the reflector regions. For our particular geometry and fuel composition,⁽²⁾ which has a hydrogen-to-U²³⁵ ratio of 152, small additions of water or lucite have about the same reactivity effect as fuel solution. Neutron streaming from the core, which is appreciable for this size void, is considerably reduced by the plugs. Thus, we approximate an axially symmetric homogeneous core and reflector. It should be held in mind, however, that production and absorption of neutrons in the region under study is not identical to the core solution.

Progressively larger sections of the lucite were removed, leaving a right circular cylindrical void at the center of the core, and the loss of reactivity for each step was noted. A plot of reactivity lost as a function of void volume is shown in Fig. 4, where the deviations indicate the 95 percent confidence limits on the experimental data. All the determinations were made at a reactor power of 4 watts. Graphical differentiation of the curve in Fig. 4 yields a void coefficient of reactivity, which is shown in Fig. 5 as a function of the void volume. This function is dependent on the magnitude of the void for small void volumes and appears to reach a constant value of -0.0026 percent reactivity per cm³ of void for volumes greater than 60 cm³.

If one assumes that the temperature coefficient and the void coefficient are the results of basically the same phenomenon, i.e., solely from the effects of fuel solution density changes, one coefficient may be derived from the other.

Consider the following as determined for the fuel solution in the KEWB reactor:

$$\delta\rho/\delta T = \text{temperature coefficient of reactivity} = -0.016 \text{ \%}/^\circ\text{C at } 30^\circ\text{C},$$

$$D = \text{density of fuel solution} = 1.235 \text{ gm/cm}^3 \text{ at } 30^\circ\text{C},$$

$$\begin{aligned} \delta D/\delta T &= \text{temperature coefficient of density for fuel solution} \\ &= -4.4 \times 10^{-4} \text{ gm/cm}^3 \text{ } ^\circ\text{C};^* \end{aligned}$$

*This value is derived from data given in Table 4.3.15 of Volume 2 of the Reactor Handbook. Note that this differs significantly from the equivalent constant for water.

Figure 4. Reactivity Compensated by Calibrated Voids in the Center of the Central Exposure Facility

whence:

$$\frac{\delta \rho}{\delta D} = \frac{\frac{\delta \rho}{\delta T}}{\frac{\delta D}{\delta T}} = 36.4\% \cdot \text{cm}^3 / \text{gm} .$$

The density of the fuel solution, after introduction of a small void volume, v , is given by:

$$D = \frac{M}{V + v} ,$$

where

M = mass of solution,

V = original volume of solution.

Then

$$\begin{aligned} \frac{\delta D}{\delta v} &= - \frac{M}{V} \left(\frac{1}{V + 2v} \right) \text{ for } v \ll V , \\ &= - D_0 \left(\frac{1}{V + 2v} \right) . \end{aligned}$$

Thus, for the KEWB fuel solution,

$$\frac{\delta D}{\delta v} = - 1.073 \times 10^{-4} \text{ gm} / \text{cm}^3 \text{ per cm}^3 \text{ of void} .$$

Then, finally,

$$\frac{\delta \rho}{\delta v} = \text{void coefficient of reactivity} = \left(\frac{\delta \rho}{\delta D} \right) \left(\frac{\delta D}{\delta v} \right) = - 0.0039 \% / \text{cm}^3 .$$

This value for the void coefficient is in excellent agreement with the theoretical value of $-0.0038 \text{ \%}/\text{cm}^3$ derived in Reference (1). However, both the referenced value and the derived value were found to be 1.5 times larger than the maximum observed experimental value. Some discrepancy is certainly to be expected because of the non-distributed nature of the experimental void geometry, as well as the previously mentioned differences in the neutron source and sink effects in the region immediately surrounding the void. In addition, the voids were shielded by the stainless-steel tube which is welded in the sphere to form the central exposure facility through the reactor core.

A traverse of the core through the central exposure tube with one experimental void was made. The reactivity compensation by the void as a function of position of void is plotted in Fig. 6. The 60 cm^3 void was found to be 1.4 times as effective at the center of the core as at the core boundary. The depression in the curve is probably attributable to the shadowing by the control rods located in vertical reentrant thimbles in the core.

Inasmuch as these experiments are an approximate simulation and have dealt with single large voids rather than homogeneously distributed voids, the results are not directly applicable to the problem of bubble-void equilibria in power and transient operation. The general magnitudes have been determined, however, and positional dependence has been established. Further studies of this problem are required.

E. TRANSIENT POWER OSCILLATIONS

In some of the reactor transients studied previously,^(2,3) damped power oscillations were observed superimposed upon the exponential decay after the transient power peak. It was found desirable in subsequent transients to initiate a manual scram immediately after the first peak, thus incidentally concealing any oscillatory effects. During this report period, a series of transients (the D-series) were conducted as a set of preliminary experiments specifically to produce and study the oscillatory effects.

Parameters studied in the analysis of the oscillatory behavior include: (1) the ratio of the second peak power to the first peak power, P_2/P_1 ; (2) the period of the oscillation, t , i. e., the time between the two power peaks; and (3) the "fractional resurgence," R/P_2 , a term developed for describing the resolution of the

Figure 6. Reactivity Compensation by a Fixed Void Volume as a Function of Position of the Central Exposure Facility

second power peak. This latter term is defined as the ratio of the difference in power between the second peak and the minimum after the first peak (defined as the "resurgence," R) to the power at the second maximum. Values of these parameters along with other pertinent information for each of the transients in this series are listed in Table I.

Fifteen transients giving damped oscillations after the first power peak have been divided into 4 groups having common characteristics. Group I (transients No. 1-D and 2-D) was run at a low core pressure, 15.3 cm Hg. Initial temperature was maintained at 25°C, and the reactivity input was 0.45 percent. No oscillations were observed. It should be noted that reactivity inputs as large as 0.68 percent failed to produce oscillations at this core pressure under the previous reactor loading of one percent, whereas reactivity inputs as small as 0.33 percent produced pronounced convergent oscillations for that loading at pressures near atmospheric. Low core pressures are apparently not conducive to oscillatory transients. The state of saturation of gases in the core solution also appears to have little effect here.

Group 2 (transients No. 4-D, 5-D, and 9-D) consists of three transients with an initial core temperature of 25°C, and with the fuel solution unsaturated with respect to dissolved gases. Consequently, the oscillatory behavior observed in this group cannot be correlated with the other data, since the degree of saturation of the solution is unknown. However, the data are presented for comparison with others in this series of transients.

Group 3 (transients No. 3-D, 6-D, 7-D, 10-D, 14-D, 15-D) consists of transients run at the same initial temperature and pressure as Group 2, but with the core solution saturated with radiolytic gases. The variation with reactivity input of the three parameters under consideration is shown in Fig. 7. The ratio of the second power peak to the first, P_2/P_1 , is shown to decrease with increasing reactivity input along with a decrease in the period of oscillation. The fractional resurgence, R/P_2 , increases with reactivity release.

The fourth group of transients (Nos. 8-D, 11-D, 12-D, and 13-D) were induced with the same initial conditions as Group 3, except for an increase in initial core temperature to 35 to 40°C. The results for this group are shown in Fig. 8, where the parameter trends are seen to be very similar to those

TABLE I

D - SERIES TRANSIENTS TO INVESTIGATE
TRANSIENT POWER OSCILLATIONS

Transient Number	Initial Pressure (cm Hg)	Reactivity Input (%)	Initial Temperature (°C)	First Power Peak, P_1 (kw)	Second Power Peak, P_2 (kw)	First Power Minimum, P_m (kw)	$\frac{P_2 - P_m}{P_2} = \frac{R}{P_2}$	$\frac{P_2}{P_1}$	Period of Oscillation (sec)	Condition of Core*
1	15.3	0.446	25	25	-	-	-	-	-	U
2	15.3	0.449	25	25	-	-	-	-	-	S
3	57.5	0.436	25	24	14	14	0	0.583	15.4	S
4	57.5	0.477	25	45	22.2	22.2	0	0.493	11.6	U
5	63.0	0.636	25	137.5	43	35	0.186	0.312	12.4	U
6	57.5	0.656	25	145	45	37	0.189	0.310	10.9	S
7	57.6	0.637	25	130	40	35	0.125	0.308	11.2	S
8	57.5	0.653	40	157	48	39.5	0.177	0.306	10.1	S
9	57.5	0.632	25	148	45	42	0.067	0.304	10.5	-
10	57.5	0.458	24	31.5	15.5	15.0	0.032	0.492	15.0	S
11	56.6	0.517	35	52.5	25	22.0	0.112	0.476	13.9	S
12	57.8	0.683	34	150	43	30.5	0.291	0.287	10.9	S
13	57.5	0.425	34	28	16.2	16.2	0	0.581	15.0	S
14	57.6	0.548	25	60	26.5	25.8	0.026	0.441	12.0	S
15	57.8	0.578	29	61	26.5	23.2	0.124	0.435	12.0	S

*S and U are used to indicate that the core solution is either saturated or unsaturated, respectively, with radiolytic gases.

(A)

Figure 7. Variation of Oscillatory Parameters with Reactivity Input at Initial Core Pressure of 60 Cm Hg and Initial Core Temperature of 25°C

Figure 8. Variation of Oscillatory Parameters with Reactivity Input at Initial Core Pressure of 60 Cm Hg and Initial Core Temperature of 35-40°C

obtained with the third group. It is noted that the ratio, R/P_2 , appears to increase more rapidly with reactivity input. This result, which indicates an increased damping of the oscillation, is the only effect attributable to the increased initial temperature.

These preliminary studies of the oscillatory behavior have not shown any system for predicting or controlling the number and amplitude of the oscillations. However, the results do show a decrease in the period of oscillation, a decrease in the ratio of the second to the first power peak, P_2/P_1 , and an increase in the fractional resurgence, R/P_2 , with increasing reactivity input. Both of the latter conclusions indicate a decrease in the oscillatory behavior of the reactor with increasing reactivity release. In none of the transients was any indication of divergent oscillation observed.

F. REFERENCES

1. M. E. Remley et al., "Program Review of the Water Boiler Reactor Kinetic Experiments," NAA-SR-1525, March 15, 1956.
2. The KEWB Staff, "Quarterly Progress Report, The KEWB Program, July - September, 1956," NAA-SR-1811, March 1, 1957.
3. D. L. Hetrick et al., "Preliminary Results on the Kinetic Behavior of Water Boiler Reactors," NAA-SR-1896, April 15, 1957.
4. M. E. Remley, "Reactor Kinetics Quarterly Progress Report October - December, 1956," NAA-SR-2134, January 15, 1958.
5. L. D. P. King, "Design and Description of Water Boiler Reactors," Proceedings of the International Conference on the Peaceful Uses of Atomic Energy (Geneva, August, 1955) (New York: United Nations, 1956), II, P/488.
6. J. W. Flora, J. W. Shortall, and W. E. Drummond, "Temperature Effect on Reactivity of the C. R. & D. Water Boiler," LRL-148, June, 1954.

II. THEORETICAL REACTOR KINETICS STUDIES

A. THE THEORY OF A MODIFIED "BORON-BUBBLE" EXPERIMENT (R. E. Skinner)

1. Introduction

The necessity of an experiment separate from any observation of the kinetic behavior of a reactor to determine the mass coefficient of multiplication, $\mu = \partial k_{eff} / \partial m$, and thus make possible the measurement of reactivity in "absolute"* units, has been discussed by de Hoffman[†] and by the author.[§] It follows from the cited discussions that any device used to measure reactivity (e. g., a control rod) cannot be calibrated in absolute units without an explicit determination of μ or its equivalent. For example, if the reactivity measuring device is calibrated in grams of fissionable material added to the core, then it is necessary to know μ in order to obtain the calibration in absolute units. If the device is calibrated in dollars, it is necessary to know β_{eff} (the effective fraction of all fission neutrons which are delayed) in order to obtain the calibration in absolute units. Thence, if the device is calibrated in both grams of fissionable material and in dollars, then a knowledge of μ will yield simultaneously the calibration in absolute units and β_{eff} , and similarly a knowledge of β_{eff} will yield the calibration in absolute units and the quantity μ .

Further, it was pointed out[§] that, from an analysis of kinetic experiments based only on the space-independent kinetic equations, it is not possible to determine β_{eff} and l (the effective neutron generation time) separately, but only their ratio, $l^* = l / \beta_{eff}$. Likewise, such an analysis yields the reactivity coefficients only in units of dollars. It follows, therefore that if it is desired to measure β_{eff} and l separately and the reactivity coefficients in absolute units, a knowledge of the conversion factor between reactivity in dollars and in absolute units, and thus a knowledge of μ , is required.

*Defined by $\rho = (k_{eff} - 1) / k_{eff}$, where k_{eff} is the effective multiplication. Throughout this discussion, ρ is used to denote the reactivity in "absolute" units.

†F. de Hoffman, "Criticality of the Water Boiler and Effective Number of Delayed Neutrons," AECD-3051 (1944).

§R. E. Skinner and D. L. Hetrick, "The Transfer Function of a Water Boiler Reactor," NAA-SR-1948 (1957).

For clarity, it is useful to distinguish carefully between the basic idea which underlies all the experiments of the "Boron-Bubble" class and specific theories for various types of "Boron-Bubble" experiments. The specific theories involve considerations of particular problems that have to be overcome in applying the basic concept which underlies the experiments to an actual reactor. It is further useful to distinguish between the problems encountered in attempting to evolve the basic concept into the theory of a practical experiment, assuming that one has available a pseudo U^{235} atom, and the problem generated by the lack of a physically realizable pseudo U^{235} atom. In the discussion that follows, the problem generated by the lack of a physically realizable pseudo U^{235} atom is not treated, and further, the discussion is restricted to homogeneous reactors of the solution type.

2. Basic Concept

For simplicity, it is assumed that there is only one fissionable isotope present. The extension to the case of a number of fissionable isotopes is simple. For explicitness, this fissionable isotope is taken to be U^{235} , or simply "25."

The concept of pseudo atom of a fissionable isotope is introduced by means of the definition: a pseudo-atom of a fissionable isotope is defined to be an atom which has every physical property possessed by an atom of the fissionable isotope except that for a pseudo atom $\nu_{pseudo}(v) = 0$, where $\nu_{pseudo}(v)$ is the number of neutrons per fission of a pseudo atom by a neutron with speed v .

Now assume that the original density of 25 at a point r in the reactor core is $N(r)$ atoms per unit volume, and that a fraction a (where $a \leq 1$) of the 25 atoms are replaced, in proportion to the original density distribution of 25, by pseudo-25 so that the densities of 25 and pseudo-25 after the replacement are $N^{25}(r) = (1 - a)N(r)$ and $N_{ps}^{25}(r) = aN(r)$, respectively. Then clearly the effect of the replacement on the physical parameters of the reactor can be represented by replacing $\nu(v)$ (the number of neutrons per fission of 25 induced by a neutron with speed v) by a $\nu_{eff}(v)$ given by

$$\nu_{eff}(v) = (1 - a)\nu(v) . \quad \dots(1)$$

Now it has been shown by Ussachoff,* among others, from the time dependent neutron transport equation, that the effective multiplication constant, k_{eff} , is an eigenvalue of the equation

$$0 = -\hat{\Omega} \cdot \nabla F(r, v) - \Sigma_t(r, v) F(r, v) + \int \frac{\nu(v')}{k_{eff}} [\Sigma_f(r, v') g(v, v', r) F(r, v')] d^3 v' + \int w(v, v', r) F(r, v') d^3 v' , \quad \dots(2)$$

subject to the usual boundary conditions, where the eigenfunction $F(r, v)$ is the neutron flux at point r with speed v in the direction $\hat{\Omega}$ per unit velocity and per unit solid angle. The important aspect of this result is that $\nu(v)$ and k_{eff} appear in only one term of eq. (2), and these are the only quantities appearing in eq. (2) that are affected by the replacement of 25 by pseudo-25.

Therefore, if $F(r, v)$ is a solution of eq. (2), and the effective multiplication k_{eff} is the corresponding eigenvalue before the replacement of 25 by pseudo-25, then after the replacement, $F(r, v)$ is still a solution of eq. (2) provided that

$$\frac{\nu_{eff}(v)}{k'_{eff}} = \frac{\nu(v)}{k_{eff}} , \quad \dots(3)$$

where k'_{eff} the effective multiplication after the replacement. Hence,

$$k'_{eff} = (1 - a) k_{eff} ; \quad \dots(4)$$

that is, the replacement of a fraction a of the 25 atoms present by pseudo 25 atoms reduces the effective multiplication by the factor $(1 - a)$. Equations (1) and (4) embody the basic concept behind all experiments of the "Boron-Bubble" class.

*L. N. Ussachoff, "Equation for the Importance of Neutrons and Reactor Kinetics," *Proceedings of the International Conference on the Peaceful Uses of Atomic Energy* (Geneva, August 1955) (New York: United Nations, 1956), P/656.

3. Dependence on Method of Establishing Reactivity Worth

All experiments of the "Boron-Bubble" class are dependent on an analysis of the general procedure for establishing the reactivity effect of a perturbation. In a solution type reactor, for example, it is possible to determine the reactivity effect of a perturbation as being equivalent to the uniform addition of a given amount of uranium (of a given enrichment in the uranyl salt solution) to the core. This is in contrast to a heterogeneous reactor, where it is only possible to establish the reactivity effect in grams of uranium of a given enrichment added to the core in the form of a fuel rod at a given location, with a given number of rods in a given configuration already in place, or as some appropriately weighted average of a number of such additions.

In particular, the theories of the original* and modified Boron-Bubble experiments assume that it is possible to establish the reactivity effect of a perturbation as being equivalent to the uniform addition of a given quantity of a given enrichment of U^{235} to the core in the form of $[U^{235} + X]$.† The symbol X is used to denote the material added along with the U^{235} . The composition of this material is assumed to be fixed. Further, it follows that if β_{eff} is to be determined, the conversion factor between the reactivity in dollars and the reactivity in grams of U^{235} in the form of $[U^{235} + X]$ added uniformly to the core must be known.

4. Description of Modified Experiment

A certain amount of real fuel solution could be replaced in a homogeneous reactor by a "mock solution" (a solution which imitates the properties of pseudo-25 in solution by means of a neutron absorber, such as boron) and the basic concept applied to determine μ directly. However:

- 1) It is desirable to avoid contamination of the fuel solution.
- 2) Experience indicates that it is difficult to reproduce the original amount of solution in the core in the process of draining and refilling the core (due to small, variable amounts of solution being held up in the solution transfer system).

*F. de Hoffman, *op. cit.*

†This can be accomplished, for example, by means of a properly calibrated control rod. For the sake of generality in what follows, the means of establishing the reactivity worth is not stated explicitly.

The original Boron-Bubble experiment was designed primarily to overcome the first of these difficulties. The second difficulty was not encountered in the case of LOPO because of the accessibility of the core and associated equipment. Further, the original experiment involves a simple averaging process which, as described by de Hoffman,* requires complete spherical symmetry and complete accessibility of the core. The requirement of complete spherical symmetry can be relaxed by appropriate modifications of the averaging process described by de Hoffman to the simpler requirement of axial or plane symmetry with an associated increase in the difficulty of performing the experiment. The experimental requirement of complete accessibility of the core for performance of the original experiment cannot be removed, however.

Except for LOPO, the insides of the cores of all solution-type reactors constructed to date have been inaccessible. However, these water-boiler-type reactors have been provided with a central exposure tube. In the case of HYPO and SUPO, the central exposure tube is off-center. The presence of either the central exposure tube, or of truncation due to an incompletely full sphere, is sufficient to remove the spherical symmetry which results in the simple averaging process described by de Hoffman.

The theory of the modified "Boron-Bubble" experiment was evolved as an attempt to avoid both difficulties 1 and 2 above, to avoid the difficulties inherent in the averaging process when there is no spherical symmetry, and to minimize the number of experimental measurements required. The concept consists of the establishment of a relationship between an actual experiment performed outside the core proper and a *gedanken* experiment performed by making uniform additions of 25 and pseudo-25 to the core. The procedure will be to describe the two experiments individually and then to relate them.

The description of the actual experiment is an idealization of the operations that are performed in actually carrying out the modified Boron-Bubble experiment. The subsequent description of the *gedanken* experiment is a description of the basic concepts as applied to an experiment performed with uniform additions to the core.

*F. de Hoffman, *op. cit.*

a. Actual or External Experiment

Consider three small containers ("bubbles") of the same volume,

- 1) "solvent bubble" containing solution without any $[U^{235} + X]$ in the core,
- 2) "25 bubble" containing the same amount of solvent as the "solvent bubble" and M grams of 25 in the form of $[U^{235} + X]$,
- 3) "pseudo-25 bubble" containing the same amount of solvent as the "solvent bubble" and M grams of pseudo-25 in the form of $[U^{235} + X]$.

Initially, the "solvent bubble" is located at point r . The reactor is assumed always to be critical ($k_{eff} = 1$) for this configuration. Furthermore, the critical mass (in units of grams of 25 in the form of $[U^{235} + X]$ uniformly distributed in the core) is denoted by M_c . Since criticality is dependent on the location, r , of the solvent bubble, M_c is a function of r .

The solvent bubble is then replaced by the 25 bubble and the excess worth of the 25 bubble as compared to the solvent bubble is established to be equivalent to δm grams of 25 added uniformly to the core in the form of $[U^{235} + X]$. Second, the solvent bubble is replaced by the pseudo-25 bubble, and the effect of the replacement is established to be equivalent to the addition of $\delta m'$ grams of 25 in the form of $[U^{235} + X]$ uniformly to the core.

Let δk_M be the change in multiplication due to the replacement of the solvent bubble by the 25 bubble and let δk_M^* be the change in multiplication due to the replacement of the solvent bubble by the pseudo-25 bubble. The quantities Δk_{actual} and Δm_{actual} are defined by means of the relationships*

$$\Delta k_{actual} = \delta k_M^* - \delta k_M \quad \dots(5.1)$$

and

$$\Delta m_{actual} = \delta m' - \delta m \quad \dots(5.2)$$

Note that by definition δk_M , δk_M^ , δm , and $\delta m'$ are functions of r , and hence Δk_{actual} and Δm_{actual} are also functions of r .

It follows, then, that Δk_{actual} is the change in multiplication due to the replacement of the 25 bubble by the pseudo-25 bubble, and that Δm_{actual} is the corresponding reactivity worth of this replacement in grams of 25 in the form of $[U^{235} + X]$ added uniformly to the core. Thus, from the definition of $\mu(m)$, it follows that

$$\mu(M_c) = \left. \frac{\partial k_{eff}}{\partial m} \right|_{m=M_c} \cong \frac{\Delta k_{actual}}{\Delta m_{actual}}, \quad \dots(6)$$

where M_c is the mass of 25 in the form of $[U^{235} + X]$ in the core with the "solvent bubble" in place and the reactor critical ($k_{eff} = 1$).

b. Gedanken Experiment (Uniform Additions)

Now consider an experiment performed by making uniform additions of 25 and pseudo-25 in the form of $[U^{235} + X]$ to the core. As before, the reactor is assumed always to be critical, $k_{eff} = 1$, with the solvent bubble in place, and M_c is taken to be the mass of 25, in the form of $[U^{235} + X]$ uniformly distributed in the core, for this (critical) configuration.

Let δk_m be the change in multiplication due to the uniform addition of δm grams of 25 in the form of $[U^{235} + X]$ to the core, and let δk_m^* be the change in multiplication due to the uniform addition of δm grams of pseudo-25 in the form of $[pseudo-25 + X]$ to the core, the reactor being in the critical configuration before the two types of additions. Then, if δm grams of 25 in the form of $[U^{235} + X]$ are added uniformly to the core, there are now $(M_c + \delta m)$ grams of 25 in the core, and the effective multiplication is $k_{eff} = 1 + \delta k_m$; and if δm grams of pseudo-25 in the form of $[pseudo-25 + X]$ are added uniformly to the core, there are M_c grams of 25 in the core, and the effective multiplication is $k'_{eff} = 1 + \delta k_m^*$. If there are $(M_c + \delta m)$ grams of 25 in the core, then in replacing δm grams of the $(M_c + \delta m)$ grams of 25 in the core by δm grams of pseudo-25, the density of 25 is reduced by the factor

$$(1 - a) = \frac{\delta m}{M_c + \delta m}.$$

Hence, from eq. (4) and the above, one has the result that

$$\frac{1 + \delta k_m^*}{1 + \delta k_m} = \frac{M_c}{M_c + \delta m} \quad \dots(7)$$

Now Δk_{ideal} is defined by the relation

$$\Delta k_{ideal} = \delta k_m^* - \delta k_m \quad \dots(8)$$

It follows that Δk_{ideal} is the change in multiplication due to the uniform replacement of δm grams of the $[M_c + \delta m]$ grams of 25 in the core by δm grams of pseudo-25. With this definition, eq. (7) can be written in the form

$$\Delta k_{ideal} = \frac{1 + \delta k_m}{1 + \left(\frac{M_c}{\delta m}\right)} \quad \dots(9)$$

In any actual experiment, $|\delta k_m|$ will generally be equal to or less than 10^{-3} , in which case δk_m may be neglected with respect to one, with the result that

$$\Delta k_{ideal} = \frac{\delta m}{M_c + \delta m} \quad \dots(10)$$

to better than 0.1 percent.

c. Relationship of the Two Experiments

It remains, now, only to relate the two separate experiments described in the preceding paragraphs. Letting δm be the same for both experiments, it follows from the definitions of δk_M , δm , and δk_m that

$$\delta k_M = \delta k_m \quad \dots(11)$$

Now δk_M , δk_m^* , δm , $\delta m'$, Δk_{actual} , and Δm_{actual} are by definition functions of r (the point at which the bubble experiment is performed). Thus, the assumption of the equivalence of δm for the two experiments reflected in eq. (11) implies that δk_m , δk_m^* , and, therefore, Δk_{ideal} are also functions of r .

The establishment of the relationship is completed by defining a function $h(r)$ such that

$$\Delta k_{actual} = h(r) \Delta k_{ideal} , \quad \dots(12)$$

where r is the point in the reactor at which the actual (or bubble) experiment is performed. Combining eqs. (6) and (12), the result is obtained that

$$\mu(M_c) = \frac{h(r)}{\Delta m_{actual}} \Delta k_{ideal} , \quad \dots(13)$$

where Δk_{ideal} is given by eq. (10).

5. Expressions for $h(r)$

a. First Order Perturbation Expressions

To obtain an explicit expression for $h(r)$, an expression for δk_{eff} obtained from the neutron transport theory expression for the reactivity as given by Ussachoff* has been used. Writing δk_M , δk_m^* , δk_m , and δk_m^* in terms of this expression, and assuming that the perturbed and unperturbed fluxes are equal (first-order perturbation theory), it is found from eqs. (5.1) and (8) that

$$\Delta k_{actual} = \frac{-N_M \Delta I_M}{N_M \Delta G_M + I} , \quad \dots(14.1)$$

and that

$$\Delta k_{ideal} = \frac{-N_m \Delta I_m}{N_m \Delta G_m + I} , \quad \dots(14.2)$$

*L. N. Ussachoff, *op. cit.*

where for the perturbation j (here $j = m$ or M)

$$\Delta I_j = \int_{R_j} dVol \int d^3v \int [F_c^+(r, v) \sigma_f(v') \nu(v') g(v, v', r) F_c(r, v')] d^3v', \quad \dots(15.1)$$

$$\begin{aligned} \Delta G_j = & \int_{R_j} dVol \int [F_c^+(r, v) \sigma_i(v) F_c(r, v)] d^3v \\ & - \int_{R_j} dVol \int d^3v \int \{F_c^+(r, v) \delta'[w(v, v', r)] F_c(r, v')\} d^3v', \end{aligned} \quad \dots(15.2)$$

$$I = \frac{M_c}{v} \int_{core} dVol \int d^3v \int [F_c^+(r, v) \sigma_f(r, v') \nu(v') g(v, v', r) F(r, v')] d^3v', \quad \dots(15.3)$$

$$\begin{aligned} \delta'[w(v, v', r)] = & \{S_u(v, v', r) \sigma_s^u(v') + f_u(v, v', r) \sigma_{in}^u(v') \\ & + \sum_i [S_i(v, v', r) c_i \sigma_s^i(v') + f_i(v, v', r) c_i \sigma_{in}^i(v')]\} \end{aligned} \quad \dots(15.4)$$

and

$$\sigma_t(v) = \sigma_t^u(v) + \sum_i \sigma_t^i(v) c_i. \quad \dots(15.5)$$

Here R_M is the region occupied by the bubble, R_m is the core, V is the volume of the core, the subscript c denotes the values of the flux and adjoint flux (or importance function) for the critical configuration, and $\sigma_f(v)$ is the fission area per unit mass of U^{235} for neutrons of speed v . $\sigma_t^i(v)$ is the total interaction area per unit mass of the element i for neutrons of speed v , the subscript or superscript u refers to U^{235} , the subscript or superscript i to the i^{th} element in X , c_i is the concentration of the i^{th} element in X divided by the concentration of uranium, $g(v, v', r)$ is the fission kernel (whose exact form need not concern us here), $\sigma_s^i(v)$

is the elastic scattering area per unit mass for element i , $\sigma_{in}^i(v)$ is the inelastic scattering area per unit mass for element i , $S_i(v, v', r)$ is the elastic scattering kernel for the i^{th} element, $f_i(v, v', r)$ is the inelastic scattering kernel for the i^{th} element, $N_M = M/\Delta V$ is the density of 25 or pseudo-25 in the 25 or pseudo-25 bubbles, respectively, and $N_m = \delta m/V$.

From the definition of $h(r)$ (eq. 12) and from eqs. (14.1) and (14.2), it follows that

$$h(r) = \frac{N_M}{N_m} \frac{\Delta I_M}{\Delta I_m} \frac{1 + \frac{N_m \Delta G_m}{I}}{\frac{N_M \Delta G_M}{1 + \frac{I}{N_M \Delta G_M}}} \quad \dots(16)$$

This is one of the possible first-order perturbation expressions for $h(r)$.

From eq. (11) and the perturbation theory expressions for δk_M and δk_m , it is found that

$$\frac{\Delta I_m - \Delta G_m}{\Delta I_M - \Delta G_M} = \frac{N_M}{N_m} \frac{1 + \frac{N_m \Delta G_m}{I}}{\frac{N_M \Delta G_M}{1 + \frac{I}{N_M \Delta G_M}}} \quad \dots(17)$$

Combining eqs. (16) and (17), one obtains a first-order expression for $h(r)$ entirely in terms of the first-order perturbation theory integrals ΔI_m , ΔG_m , ΔI_M , and ΔG_M :

$$h(r) = \frac{1 - \frac{\Delta G_m}{\Delta I_m}}{1 - \frac{\Delta G_M}{\Delta I_M}} \quad \dots(18)$$

It is seen that $h(r)$ as given by eq. (18) depends only upon the flux, the adjoint flux, the scattering and fission kernels, $\nu(v)$, and the cross sections expressed in units of area per unit mass.

b. Consistent First-Order Expressions

In terms of consistent first-order perturbation theory, all terms of order higher than the first are neglected in contrast to first-order theory where only those higher terms due to perturbations in the flux are neglected. Applying this approximation to eq. (16) yields the result that

$$h(r) = \frac{N_M}{N_m} \frac{\Delta I_M}{\Delta I_m} \quad \dots(19)$$

Applying the consistent first-order approximation to eq. (17) and combining the result with eq. (19) yields eq. (18). Thus, eq. (18) is also a consistent first-order perturbation expression for $h(r)$.

B. THE DETERMINATION OF REACTOR TRANSFER FUNCTIONS FROM MEASUREMENTS AT STEADY OPERATION (*M. N. Moore*)

The time-dependent differential equations of reactor kinetics are generally non-linear. However, for small changes in power, they may be linearized, with the parameters slowly-varying functions of time. A problem of reactor kinetics has been to evaluate these parameters over ranges in the variables for which the parameters are sensibly constant. The experimental verification of these calculated values has so far been limited to the analysis of the power response of a reactor to controlled variations in one or several of the available parameters. An effort is made to introduce these input variations in such a way as to prevent the response from entering a range of values which are no longer governed by the constant coefficient linear kinetics equations, but a lower bound to the amplitude of the input signal is governed by the inevitable fluctuations in power which are present even when the reactor is operated steadily, i.e., the "reactor noise." More specifically, then, the smallest permissible input variation is that one which results in a response signal which has a certain signal-to-noise ratio. In order to attain a given output signal-to-noise ratio in a noisy reactor, it might be necessary to make power excursions which exceed the range of linearity of the kinetics equations, thereby defeating the purpose of the experiment.

One may look upon noise in a system as the response of that system to a random or stochastic driving force. Whatever the nature of that force is, one thing is clear: The response of the system is less random than the driving force itself, since the system introduces a causal relationship into the situation. With this distinction in mind, it is possible to investigate the transfer function from a study of system noise. Since the noise, from this point of view, is a response to an input representing the statistical nature of the underlying processes, it is clear that the noise is that response least likely to violate the linear approximations. Transfer functions obtained from noise analysis are least subject to non-linear distortions.

The differential equation for reactor noise may be generally written

$$z(D)y(t) = f(t) , \quad \dots(1)$$

where $y(t)$ is the power response, $f(t)$ is the stochastic "force," and $z(D)$ is a polynomial in the differential operator D , with constant coefficients, having a form such that $y = \text{constant}$ is a solution to

$$z(D)y(t) = \text{constant} .$$

It is possible to write the formal solution of (1) and form the product $y(t)y(t + \Delta t)$. Upon utilization of the higher randomness of $f(t)$, it is possible to evaluate the integral

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T y(t)y(t + \Delta t) dt ,$$

and hence the auto-correlation function

$$\rho_y(\Delta t) = \frac{\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T y(t)y(t + \Delta t) dt}{\int_{-\infty}^{\infty} y^2(t) dt} , \quad \dots(2)$$

which is related to the transfer function $T(\nu)$. It can be shown that the relationship between $\rho_y(\Delta t)$ and $T(\nu)$ is as follows:

$$|T(\nu)|^2 = \int_{-\infty}^{\infty} \rho_y(\Delta t) \cos 2\pi\nu \Delta t d\Delta t . \quad \dots(3)$$

The relationship (3) provides a means of experimentally determining the transfer function of a reactor from measurements at steady operation. The method is as follows: A power trace is obtained with sufficient amplification to include reactor noise, and from this trace $\rho_y(\Delta t)$ is calculated numerically from (2) together with the right hand side of (3). The functional form of $|T(\nu)|^2$ is known from the model, and the algebraic signs of the constants in $|T(\nu)|^2$ are known from the physics. These constants as well as the proportionality constants in (3) are then determined from a least-squares fit for various values of ν . Upon repeating the experiment at various temperatures and power levels, the temperature and power coefficients may be evaluated, and upon periodic repetition of the experiment the long-term changes may also be determined.

This method should provide information which is most exact at the highest frequencies and becomes less exact at the very lowest frequencies. The low-frequency limit should fall within the range of pile oscillator experiments, thus providing a comparison of the consistency of the two methods. The accuracy of the present method can be estimated by determining the transfer function of the reactor on control and off, hence determining the transfer function of the control system. Since the control system is electromechanical, its transfer function can be determined by other, more exact methods. The accuracy with which the present method duplicates these other methods is a lower limit on the accuracy of the reactor transfer function determined from reactor noise.

C. POWER SERIES SOLUTIONS OF REACTOR KINETICS EQUATIONS (*Edwin Blue*)

The following analysis of the one-group kinetics equations was made in order to develop a numerical solution which might provide a partial check for codes involving more general kinetics equations.

The reactor kinetics equations with one group of delayed neutrons are given by

$$\dot{n} = \frac{\rho^* - 1}{l^*} n + \lambda c \quad \dots(1)$$

and

$$\dot{c} = \frac{1}{l^*} n - \lambda c , \quad \dots(2)$$

where n is the neutron density, c the density of delayed neutron precursors with decay constant λ , ρ^* the reactivity expressed in units of dollars, and l^* is the ratio of the average neutron lifetime to the effective fraction of delayed neutron precursors born per particle born in fission. That is, $\rho^* = \rho/\beta$, $l^* = l/\beta$, and the dot denotes differentiation with respect to time. If c is eliminated between eqs. (1) and (2), and if the reactivity is a linear function of time, then one obtains

$$\ddot{n} + (at + b)\dot{n} + (ct + d)n = 0 . \quad \dots(3)$$

Here*

$$\rho(t) = \rho_0 + Mt ,$$

$$a = -M/l ,$$

$$b = l^{*-1} + \lambda - \rho l^{-1} ,$$

$$c = -\lambda M l^{-1} ,$$

$$d = -M l^{-1} - \lambda \rho_0 l^{-1} .$$

If we make the transformation

$$\xi = \sqrt{|a|} \left(t + \frac{ab - 2c}{a^2} \right) , \quad \dots(4)$$

and let

$$\eta(\xi) = n(\xi) e^{(c/a)t(\xi)} , \quad \dots(5)$$

* In eq. (3) and most subsequent equations, the symbol c is used to represent a constant, and is not to be confused with the dependent variable in eqs. (2) or (8).

then η satisfies the equation

$$\eta'' + \epsilon(a) \xi \eta' + A\eta = 0, \quad \dots(6)$$

where $\epsilon(a) = a/|a|$, $A = a^{-3}(c^2 - abc + a^2d)$, and the prime denotes differentiation with respect to ξ . The above equation is the Weber equation, which has the series solution

$$\eta(\xi) = c_1 \left[1 + \sum_{\nu=1}^{\infty} \epsilon^2(a) F(A, \nu) \xi^{2\nu} \right] + c_2 \left[\xi + \sum_{\nu=1}^{\infty} \epsilon^2(a) G(A, \nu) \xi^{2\nu+1} \right],$$

$$F(A, \nu) = \frac{A(A+1) \dots (A+2\nu-2)}{(2\nu)!}, \quad G(A, \nu) = \frac{(A+1)(A+3) \dots (A+2\nu-1)}{(2\nu+1)!}. \quad \dots(7)$$

The above series solution may be evaluated numerically with the aid of a high-speed digital computer such as the IBM 704. It was deemed worthwhile to attempt to generalize the initial differential equations somewhat before programming the solution. The first step in this direction is to include a constant source term q . This change has the effect of modifying eq. (1) so that it now becomes

$$\dot{n} = \frac{\rho^* - 1}{l^*} n - \lambda c + q. \quad \dots(8)$$

If now one makes the change of variable (4) in (8), the result is no longer the Weber equation. In view of this, a power series solution of the original one-group kinetic equations (1), (2) was obtained, with the reactivity now taken to be a cubic function of time,

$$\rho = \rho_0 + At + Bt^2 + Ct^3, \quad \dots(9)$$

and with a constant source term q included.

The second order differential equation for the neutron density is now

$$\ddot{n} + (a + bt + ct^2 + dt^3) \dot{n} + (e + ft + gt^2 + ht^3) n - q = 0 , \quad \dots(10)$$

where

$$a = l^{-1} \beta + \lambda - l^{-1} \rho_0 ,$$

$$b = -l^{-1} A ,$$

$$c = -l^{-1} B ,$$

$$d = -l^{-1} C ,$$

$$e = -l^{-1} (\lambda \rho_0 + A) ,$$

$$f = -l^{-1} (\lambda A + 2B) ,$$

$$g = -l^{-1} (\lambda B + 3C) ,$$

$$h = -l^{-1} \lambda C .$$

If one assumes a power series solution for n of the form

$$n(t) = \sum_{j=0}^{\infty} \alpha_j t^j , \quad \dots(11)$$

then substituting eq. (11) in eq. (10) and equating coefficients of like powers of t to zero leads to the following recursion relation:

$$\alpha_n = -\frac{a}{n} \alpha_{n-1} - \frac{[(n-2)b + e]}{n(n-1)} \alpha_{n-2} - \frac{[f + (n-3)c]}{n(n-1)} \alpha_{n-3} \\ - \frac{[g + (n-4)d]}{n(n-1)} - \frac{h}{n(n-1)} \alpha_{n-5} . \quad \dots(12)$$

If we make the change of variable

$$\xi_n = \alpha_n t^n ,$$

then eq. (12) becomes

$$\begin{aligned} \xi_n = & -\frac{a}{n} \xi_{n-1} t - \frac{[e + (n-2)b]}{n(n-1)} \xi_{n-2} t^2 - \frac{[f + (n-3)c]}{n(n-1)} \xi_{n-3} t^3 \\ & - \frac{[g + (n-4)d]}{n(n-1)} \xi_{n-4} t^4 - \frac{h}{n(n-5)} \xi_{n-5} t^5 . \end{aligned} \quad \dots(13)$$

The series solution in the form

$$n(t) = \sum_{j=1}^{\infty} \xi_j(t) \quad \dots(14)$$

has been coded for numerical evaluation on the IBM 704.