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COMMITTEE REPORT:
LLL RE-EVALUATION OF THE SIMPLE MIRROR
FOR A FUSION REACTOR

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MASTER

COMMITTEE REPORT:
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ABSTRACT

In response to potential cost savings, a re-evaluation is made of the prospects for using the simple mirror configuration in a fusion reactor. Because of flute instability problems, this configuration was largely dropped in early mirror research in favor of the magnetic-quadrupole-well configuration, a more complex and costly arrangement but one not bothered by flute-mode instability. Here, both line-tying and feedback stabilization of simple-mirror flute instability are examined.

Based on both theoretical considerations and experimental evidence, line tying was found to require so much exterior cold plasma that fusion energy release was greatly reduced due to a large loss of energy from the hot ions to the cold electrons. Line tying showed promise, however, for the two-component mode of simple-mirror operation. Feedback stabilization, although not yet defined in enough detail to assess its practicality, may work and deserves further investigation.

INTRODUCTION

Historically, magnetic-mirror plasma-confinement research started with a magnetic field produced by two circular coils, known as the simple mirror configuration. A serious problem in this early mirror research was that the plasma was unstable to the interchange mode, and all or spokes of the plasma would move quickly towards the walls. Researchers noticed, however, that cold plasma between the mirrors and end walls at least retarded growth of the instability. This cold-plasma stabilizing effect is thought to be the result of tying plasma potential to wall potential via electrical conduction through the plasma on the flux tube passing from the plasma to the wall. This stabilization technique is called line tying. In the early 1960s, however, Ioffe demonstrated the stabilizing effect of a radial magnetic well and showed one way of making such a well by adding axial current-carrying conductors (Ioffe Bars) to a simple mirror. Thereafter, most mirror research abandoned interest in line tying in favor of the magnetic-well configuration.

New ways of making magnetic wells were found (e.g., baseball coils, Yin Yang, and Furth-Andreoletti axisymmetric configuration). Although these configurations were more complex and costly than the simple mirror, they were widely adopted by plasma-confinement researchers because they eliminated the interchange instability thereby permitting concentration on loss-cone instabilities and classical processes. Today, we look forward to a culmination of the magnetic-well approach in the 2XIIB,

Baseball II-T, and MX experiments. Moreover, we are already studying magnetic-well applications to large mirror devices, such as the Fusion Engineering Research Facility (FERF).

Our studies indicate that FERF and mirror fusion reactors can certainly be designed around the minimum-B configuration. On the other hand, there is no doubt that magnet development and construction costs could be reduced considerably if it were possible to return to simple circular coils. Therefore, Lawrence Livermore Laboratory initiated a re-examination of the simple mirror. This report addresses the question: "Has progress in plasma technology during the last ten years changed the prospects for utilizing the simple mirror with some stabilizing devices?" Specifically, what are the prospects for line-tying and feedback stabilization of the simple mirror?

PLASMA PARAMETERS FOR A FERF AND A POWER-PRODUCING REACTOR

G. A. Carlson

Plasma parameters for a FERF and a power-producing reactor are presented here to provide specific examples for the line-tying and feedback discussions to follow in later sections. In addition, a cost analysis is made of the savings that could be realized by using a simple mirror arrangement rather than the Yin-Yang configuration in the FERF.

The FERF and power-reactor parameters listed in Table 1 are based on a scaling analysis of simple-mirror machines. Figures 1 and 2 show the respective geometries of the two machines analyzed and the magnetic-field bundles passing through them.

A detailed description of the scaling analysis is given in Appendix A of Ref. 1. The analysis begins by specifying the desired fusion power, injection energy, central vacuum field, vacuum-mirror ratio, blanket-shield thickness, superconducting-coil-current density, and an estimate of allowable plasma β . Also specified is the ratio of trapped injection current to be provided to the classical trapped injection current. Consistent with other studies, this ratio is set at unity for the reactor and at 3 for FERF. The length-to-diameter aspect ratio of the simple-mirror machine is chosen so that the vacuum-mirror ratio is produced by a pair of short solenoids, separated by the length of the machine. This design provides maximum access along the machine for injection. With this information, principal machine dimensions can be determined. Then, plasma density, fusion-power density, first-wall neutron load, injection requirements, and machine Q can be calculated.

The field-line bundles in Figs. 1 and 2 are calculated simply by maintaining constant the product of the centerline magnetic-field strength and the cross-section area of the field-line bundle. Expander guide coils probably are required to maintain this field-line expansion outside the mirrors. Expansion of the field-line bundle to a fairly large area is required to decrease the power density of the leakage flux before

Fig. 1. Simple-mirror reactor.

Fig. 2. Simple-mirror FERF.

Table 1. Simple mirror designs.

	Reactor	FERF
Input		
Injection energy, E_o - keV	200	65
Fusion power, P_{FUS} - MW	1000	5
$B_{O, VAC}$ - kG	50	35
R_{VAC}	3	2
β	0.9	0.65
$R = R_{VAC}/\sqrt{1 - \beta}$	9.5	3.4
Blanket-shield thickness, t_B - m	1.5	1.0
Overall coil current density - kA/cm ²	5	5
Injection current/classical injection current	1	3
Results		
Coil aspect ratio, $L/2r_c$ (so simple coil pair gives R_{VAC})	1.50	1.20
Plasma radius at midplane, r_p - m	2.68	0.31
L - m	12.5	3.14
Coil cross section - m	1.39×1.39	0.53×0.53
Peak ion density, $n_{i, o}$ - cm ⁻³	2.80×10^{14}	3.05×10^{14}
Peak fusion-power density - W/cm ³	44	52
Peak neutron-wall load - MW/m ²	21	2.92
- cm ⁻² s ⁻¹	9.4×10^{14}	1.3×10^{14}
Injection current (trapped) - A	3800	645
Injection power (trapped) - MW	770	42
$Q_{classical}$	1.30	0.36

it is collected (by either direct conversion equipment or simple cooled targets). Calculated leakage-flux power density is indicated on the figures.

Figures 3 and 4 show approximate axial density profiles, both inside and outside the mirror, for the simple-mirror power reactor and the simple-mirror FERG. Inside the mirror, density is calculated as described in Ref. 2. Outside the mirror, density is calculated as leakage current density (which decreases with axial distance because of the expanding magnetic-field-line bundle) divided by $v_{||}$.

For equal plasma volumes and magnetic-field strengths, a simple mirror coil and its structure will be considerably less expensive than a corresponding minimum-B Yin-Yang coil and structure. A simple mirror produces plasma volume more efficiently (i. e., with less ampere-meters of conductor), therefore, superconductor costs are lower. Also, the structure of the simple mirror coil is a simple hoop in almost pure tension, compared to Yin-Yang coil structures, which must resist heavy bending loads. In an earlier analysis,² the cost ratio of a Yin-Yang versus a simple mirror coil and structure was found to be 9.6. Using this ratio as a first estimate, the

Fig. 3. Axial density profile - reactor.

Fig. 4. Axial density profile - FERF.

potential cost savings in using a simple mirror coil in FERF were determined (Table 2). It must be emphasized that this cost comparison is only a first estimate; a detailed design and cost analysis of a complete simple-mirror FERF would be required to verify these figures.

Table 2. Cost comparison: Yin-Yang versus simple-mirror FERF.

	Cost (M\$)	
	Yin-Yang FERF ^a	Simple-mirror FERF
Coil and structure	75.5	7.9
Remainder of facility	150.9	150.9
Total cost	226.4	158.8

^aRef. 3.

LINE-TYING STABILIZATION OF THE SIMPLE MIRROR

Line tying is generally understood as the electrical conduction from the hot, confined plasma to the wall through a cold plasma, thereby tying plasma potential to wall potential. Also, cold plasma generally has been found (experimentally) to have a stabilizing effect on interchange modes. There is a lack of general agreement, however, on how line tying depends on such important parameters as $N_{\text{cold}}/N_{\text{hot}}$ and connection lengths. Moreover, there is neither theoretical nor experimental agreement on a model for line tying. In spite of this dearth of understanding, some light can be shed on the subject.

We have found that sheaths play an important role. They tend to thermally insulate the plasma from the wall by drastically reducing electron conduction (streaming) to the wall. The sheath is good in that it helps maintain a favorable energy balance ($Q \gtrsim 1$, where Q = fusion power/trapped beam power). On the other hand, a sheath strongly tends to prevent the electrical conduction needed for line tying. There are three sheaths of interest in a mirror machine (Fig. 3 and 4): one at the mirror edge, a distributed sheath throughout the magnetic expander, and one at the end wall. It is argued (Appendix A) that these sheaths cause so much electrical impedance that $N_{\text{cold}}/N_{\text{hot}} > 1$ is required for interchange stability. Another analysis (Appendix B) indicates that Q is degraded unless $N_{\text{cold}}/N_{\text{hot}} \ll 1$. In summary, then, it appears that line tying will not stabilize a simple mirror in a reactor regime, simply because getting good electrical end conduction automatically gives too much heat conduction. On the other hand, line tying has good prospects for stabilizing interchange modes for the two-component mirror, in that this mirror requires $N_{\text{cold}}/N_{\text{hot}} > 1$. For line-tying or feedback stabilization to be effective, however, localized finite- β ballooning must be avoided. This problem is discussed in the next section.

Beta Limit of Stability due to Ballooning

W. A. Newcomb and L. D. Pearlstein

In this report, two methods of stabilizing a simple axially-symmetric mirror are considered: line tying and axial feedback. Both methods rely on good electrical communication along the magnetic-field line in the region of the trap. In this section, we consider the stability limitations imposed by ballooning, a condition that impedes electrical communication.

Since it takes energy to bend field lines, there is competition between this energy sink and the energy reservoir due to the bad curvature localized in the interior of the magnetic trap. Putting it another way, "Can the wave grow before it propagates to the end of the trap?" This competition produces the rough stability criterion

$$\frac{V_\alpha^2}{L^2} \gtrsim \gamma_{\text{HYD}}^2, \quad (1)$$

where V_α is Alfvén velocity, $(\sqrt{B^2/4\pi\rho})$, L is trap length, and γ_{HYD} is the hydromagnetic growth rate $(v/\sqrt{R_c R_p})$, where R_c is the radius of curvature and R_p is the density-gradient distance). Incorporating these definitions into Eq. (1), maximum beta becomes

$$\beta < \frac{L^2}{R_c R_p} \sim 0(1). \quad (2)$$

To determine the precise value of the $0(1)$ term, we resort to the energy principle. This procedure along with detailed results using a normal mode distribution (Holdren distribution) and a distribution that describes injection at the loss-cone angle, are presented in the following paragraphs. The two distributions should bracket the response. In fact, stability criteria are somewhat insensitive to the choice of distribution.

In this model, we assume the trap to be long and thin (i. e., $R_p \ll L$), in which case,

$$p_\perp + \frac{1}{2} B^2 = \frac{1}{2} B_{\text{vac}}^2(z), \quad (3)$$

where B is the magnetic field normalized to its maximum value and $p_\perp$ is perpendicular pressure. Note that, in this limit, vacuum-field strength is only a function of z , the distance along the magnetic-field line. For simplicity, we assume for the pressure profile at $z = 0$

$$p_\perp(\psi, 0) = \frac{\beta}{2} (1 - \psi) B_{\text{vac}}^2(0), \quad (4)$$

where $2\pi\psi = \int \tilde{B} \cdot d\tilde{A}$ normalized so that $0 \leq \psi \leq 1$. Now, with the above normalization, we have

$$p_{\perp}(\psi, z) = \frac{\beta}{2} (1 - \psi) B_{\text{vac}}^2(o) \frac{Q(B, R)}{Q(1/R, R)}, \quad (5)$$

where $R = 1/B(o)$ and, for Case I,⁴ the Holdren normal-mode distribution,

$$Q(B, R) = (2R - 3) \left(\tanh^{-1} x - x + \frac{x^3}{3} \right) + \frac{B+4}{10B} x^3, \quad x = \sqrt{1 - B} \quad (6a)$$

and Case II,

$$Q = 1 - B^2,$$

for injection at the loss-cone boundary.

Equilibrium demands (mirror mode stability)

$$B^2 + B \left. \frac{dp_{\perp}}{dB} \right|_{\psi} > 0.$$

This condition is trivially satisfied for Case II, provided $R_{\text{vac}} > 1$. For Case I, we obtain the relationship between β and R_{vac} shown as the dashed curve in Fig. 5 (see also Ref. 4).

To determine stability to ballooning, we use the energy principle, which, in the limit $R_p \ll L$, the energy integral is represented

$$w = \frac{1}{2} \int_{-L}^L (f\xi^2 + g\xi^2) \frac{dz}{L} > 0 \quad (\text{stability}), \quad (7)$$

where

$$f = \frac{\psi(B^2 - p_{\parallel} + \rho_{\perp})}{r^2 B^3} \quad (8)$$

and

$$g = \frac{\psi k}{r B^2} \left. \frac{\partial(p_{\perp} + p_{\parallel})}{\partial \psi} \right|_z, \quad (9)$$

where k is the inverse of the radius of curvature, r is the radial coordinate, $p_{\parallel}$ is the parallel pressure, and ξ is the radial displacement of a flux line. A detailed derivation of this expression along with results is being prepared for submittal to The Physics of Fluids.

Stability is determined by numerical solution of the Euler-Lagrange equation

$$\frac{\partial}{\partial z} f \frac{\partial \xi}{\partial z} - g\xi = 0,$$

subject to appropriate boundary conditions. We assume the vacuum field to be of the form:

Fig. 5. Holdren distribution (index = n).

$$B_{\text{vac}} = \frac{1}{2} \left(1 + \frac{1}{R_{\text{vac}}} \right) - \left(1 - \frac{1}{R_{\text{vac}}} \right) \cos \left(\frac{z}{L} \right)^n \quad n = 1, 2. \quad (10)$$

Results are given in Fig. 5 for both Cases I and II. The fact that $n = 2$ leads to an improved beta limit is easily understood in terms of the behavior of the bad radius of curvature, which is infinite at $z = 0$ and finite away from the midplane but larger than the minimum value for $n = 1$. Thus, the drive for instability is diminished.

In conclusion, for a vacuum-mirror ratio of less than four and for a vacuum-field shape of $n = 2$, the ballooning mode does not lead to beta limits severely less than that obtained for optimum injection ($\beta = 0.85$).⁵

Simple-Mirror Experiments and Experimental Evidence for Line Tying

T. C. Simonen

There have been a number of experiments investigating plasma stability in simple-mirror geometry. In some studies, minimum-B stabilizing fields could be applied independently. Here, we discuss only a few experiments,⁶⁻¹² all of which have hot ions ($T_i \gg T_e$) and most of which have moderately high plasma beta. (Experimental parameters are summarized in Table 3.) It should be emphasized that line-tying and minimum-B stabilization have also been studied in hot-electron plasmas and in low-density hot-ion plasmas. For example, line tying of hot-electron plasmas has been studied at Oak Ridge National Laboratory, Lawrence Berkeley Laboratory, and LLL. In addition, controlled line-tying experiments of low-density hot-ion plasmas were performed in DCX-1.5.¹³

To compare results from different simple-mirror experiments, it is necessary to estimate the flute-instability plasma-flow velocity across magnetic-field lines. Rosenbluth and Longmire¹⁴ show that this velocity is less than the thermal ion velocity ($v_{\theta i}$) by a factor $(a_i/R_B)^{1/3}$, which depends on the ion-gyro radius (a_i) and field-line radius of curvature (R_B). For experiments considered here, this weakly varying factor is typically ~ 0.2 . Consequently, we estimate plasma lifetime, due to flute instability, as

$$\tau_{\text{flute}} = \frac{5 R_p}{v_{\theta i}}. \quad (11)$$

Plasmas with much longer lifetimes are said to be MHD stable.

A similar scaling law has been derived by Kadomtsev.¹⁵ His model considers a conducting wall (with radius R_w) that retards plasma flow such that, at high density ($\omega_{pi}^2 \gg \omega_{ci}^2$),

$$\tau = \frac{C R_w}{v/\theta} \left(\frac{R_B}{a_i} \right)^{1/2}. \quad (12)$$

Table 3. High-density simple-mirror experiments.

Experiment	n_e (cm ⁻³)	τ (μ s)	$n\tau$ (cm ⁻³ s)	w_i (keV)	w_e (eV)	B (kG)	β (%)	a_i (cm)	L (cm)	R_p (cm)	τ_{flute} (μ sec)	$\frac{\tau}{\tau_{flute}}$
Linear Syllac ^a	1.5×10^{16}	19	3×10^{11}	2.5	600	60	40	0.17	500	1	0.13	120
Livermore ^b	5×10^{13}	100	5×10^9	4	150	20	1	0.63	66	5	0.37	270
General Atomic ^c	1×10^{13}	~100	1×10^9	0.07	—	1-3	—	~1	60	<8.6	<4	>25
OGRA-II ^g	10^{13}	200-400	3×10^9	1	—	2-10	10-0.4	2-0.4	200	6	0.7	300-600
Modified ^d PR2	without min B ~ 10^9 with min B	100 500		~1	—	4		1	~100	~5	0.6	150
MTSE II ^e	without min B 5×10^{12} with min B	<100 250	$<0.5 \times 10^9$ 1.25×10^9	1.5	22	1.8	10	3	150	8	0.54	<180
DECA ^f	without min B 1×10^{11} with min B	15 70	1.5×10^6 7.0×10^6		—	10						

^aRef. 6.^bRef. 7.^cRef. 8.^dRef. 10.^eRef. 11.^fRef. 12.^gRef. 9.

In Kadomtsev's analysis, the constant C is taken as 20. Thus, plasma lifetimes estimated by Kadomtsev are approximately two orders of magnitude longer than those estimated by Rosenbluth and Longmire. Nevertheless, the energy scaling of both models is the same,

$$\tau \sim E^{-3/4}. \quad (13)$$

From the viewpoint of $n\tau$, the most impressive experiment is the linear Syllac experiment⁶ ($n\tau = 3 \times 10^{11}$ cm⁻³ sec). Less impressive, in terms of mirror confinement, is the fact that mean ion lifetime is only one bounce time. This experiment does indicate, however, that stability can be achieved at relatively high beta ($\beta \sim 0.4$).

Longer confinement times (~ 100 μ sec) were reported in the Toy Top experiment.⁷ In this case, stability was observed for ~ 100 ion bounces, and external plasma density was determined to exceed 0.1%. Subsequent 2XII experiments, with extensive measurements of external plasma density, suggest that external density in the Toy Top test probably exceeded several percent.

In another experiment, Scott, Jensen and Wharton⁸ demonstrated that flute stabilization depended on the conductivity of end plates. An external plasma density exceeding 1% was reported.

Studies using the OGRA-IIT device⁹ indicate a $\tau \sim B^{1/2}$ scaling law between 2 to 10 kG, as predicted by Kadomtsev's theory. Energy scaling has not been determined for these experiments.

Experiments in Russia,¹⁰ England,¹¹ and France¹² have reported factor-of-5 improvements in plasma lifetimes with the addition of minimum-B fields. Presumably, these experiments were line tied without Ioffe bars and were limited by other processes with minimum-B fields.

Plasma lifetimes (τ) for the experiments in Table 3 are typically $\tau \sim 100 \tau_{\text{flute}}$. In every case, these long plasma lifetimes in simple-mirror geometry were attributed to line tying. In several cases, stability improved with increasing external plasma density. In no case, were accurate measurements of external-plasma parameters reported or direct evidence identifying specific line-tying processes documented. Nonetheless, the following general conclusions can be made from these experiments:

- In no case, was plasma motion observed at the flute-instability flow velocity.
- This stability was observed even at high beta.
- Most experiments attribute stability to line tying.
- When measured, external densities exceeded central densities by 1%.
- Increased external plasma improved stability.
- No direct experimental evidence specifically identified line tying as the stabilizing mechanism.
- No favorable scaling laws toward higher $n\tau$ were indicated.
- Improved stability was observed with minimum-B geometry.

In summary, plasma confinement approximately one hundred times greater than expected for unimpeded flow across field lines has been experimentally observed in simple-mirror geometry with external plasma connecting the internal plasma to the vacuum-chamber walls. How these results scale with higher energies is unknown at present, but very unfavorable scaling has been predicted by Kadomtsev.¹⁵ Results from the OGRA-II experiments support Kadomtsev's position. On the other hand, experiments in minimum-B geometry show that interchange-stability problems can be eliminated.

Hot-to-Cold-Plasma Energy Drain

B. G. Logan

As noted earlier, Q is reduced by the extra cooling on the cold electron of the cold end plasma. Here, only results are given, the calculational methods and assumptions used in obtaining these results are described in Appendix B.

Figure 6 represents the effect on Q of an external plasma with a distributed sheath and plasma pressure (βB_o^2), geometry (effective plasma cross-section area, A_{eff} , and effective plasma length, L_{eff}), and injection energy (E_o) held constant. For cold-plasma currents (J_c) not exceeding the hot injected current (J_h), temperature and Q are not changed greatly because the additional electrons from J_c are lost with an energy $q\phi_{\text{tot}}$. Unfortunately, ϵ ($\epsilon = n_{\text{cold}}/n_{\text{hot}}$) is small ($\sim 2 \times 10^{-4}$) for this magnitude of J_c , due to the short containment time of the cold plasma; probably too small to provide much line-tying effect. For $J_c > 10 J_h$, temperature is sufficiently low that

Fig. 6. kT_e and Q as a function of ϵ for a simple-mirror reactor. $E_0 = 200 \text{ keV}$, $R = 10$, $\beta = \text{constant}$.

$n\tau$ is dominated by electron drag and $Q \sim n\tau \sim kT_e^{3/2} \sim 1/\sqrt{\epsilon}$. At this point, ($\epsilon \sim 10^{-2}$), Q is degraded enough below unity that the reactor must have an efficient direct converter to produce net power, resulting in high recirculating power. Unfortunately, a line-tying theory by Lieberman (Appendix A) suggests that ϵ would have to be as high as unity to stabilize the lowest-order flute mode with a plasma sheath. At $\epsilon = 1$, Q is reduced to a value ($A = 0.2$), too low for net power unless a power-amplifying uranium blanket is used in a fission-fusion hybrid mode. A Q of 0.2 is at least acceptable for FERF, a neutron-producing test reactor, provided the parameters $E_0 = 200$ keV, $R = 10$, and $\beta = 0.9$ are attainable. Finally, the low Q values and electron temperatures at $\epsilon = 1$ are not invariably obtained; the values in Fig. 6 depend on choice of parameters made in Table 3. Inspection of the power balance in the two-component limit ($\epsilon \rightarrow 1$, Eq. B.31, Appendix B), shows that kT_e increases as $(L_{\text{eff}} \beta B_0^2)^{1/3}$ so that, by making the plasma longer and focusing more neutral beam power into a higher-density plasma (higher βB_0^2), one can force kT_e back up to high values (≈ 10 keV). Then, because only one species need be injected, an even higher Q can be obtained in the two-component mode,¹⁶ as indicated in Fig. 6. Thus, the two-component mode may be best for line tying ($\epsilon = 1$) in either a reactor or a FERF (Appendix B, Ref. B15).

Conclusion and Recommendations for Future Experimental and Theoretical Line-Tying Research

The prospects for stabilizing the simple mirror via line tying, under reactor conditions, appear poor because electron cooling is too great and Q is too low for the required amount of end plasma. This conclusion is quite strong in that the density required is several orders of magnitude higher for stabilization than electron cooling will allow. However, within an order of magnitude or so of density predictions, there is not good agreement between theoretical analyses and actual line-tying experiments. Further theoretical and experimental research is needed to better understand the role that plasmas in contact with walls play in the confinement of hot plasma. Nonetheless, our opinion is that the reactor prospects of line tying are poor.

Although the two-component mode of a simple-mirror reactor may not be entirely stabilized by line tying, flute growth rates certainly would be greatly reduced (according to the calculations in Appendices A and B) and would facilitate feedback stabilization.

FEEDBACK STABILIZATION OF THE SIMPLE MIRROR

To stabilize interchange modes with a much lower cold-plasma density than required for line tying leads to the idea of having an active (rather than passive) end wall that can sense a mode developing and change end field-line conditions (e.g., potential or electron emission) to counteract the growth of the mode. An analysis of this axial electrostatic feedback method (see Appendix A) indicates that the cold plasma at the ends, needed to carry stabilizing currents, can be several orders of magnitude lower

than that required for line tying. In fact, this density could be low enough to avoid excessive cooling of the hot plasma on the cold electrons. A second analysis (see Appendix D) also supports the possibility of axial feedback, in this case by electron injection. The required current density is derived. Finally, in a third analysis, based on recent discussions with Chuyanov at Kurchatov Institute in Moscow, several criteria are given for feedback. This analysis, discussed in Appendix E, treats an example based on mirror FERF parameters.

In addition to axial electrostatic feedback, work is underway on OGRA-III examining radial electrostatic feedback, which is interesting but seems applicable only to low-density plasmas.

The other type of feedback is magnetic, as is employed on SYLLAC. (See Appendix C for a discussion of axial-current feedback stabilization.) This stabilization method will work on a very thin plasma, but is not at this time considered applicable to our spherical plasmas. However, the two-component mirror may be able to use this type of feedback.

Conclusions and Recommendations on Feedback Stabilization

We think it possible, in principle, to stabilize the simple mirror by axial feedback. However, this technique is effective only on lower-order modes; higher-order modes must be stabilized by other methods (e.g., finite-orbit stabilization). Such practical matters as electrode interference with the direct converter, power levels, and cost have yet to be considered.

In contrast with microinstabilities, the study of MHD phenomena in mirror machines does not require a very hot plasma or a very large experiment. Consequently, the study of feedback of simple mirrors would be an ideal area for small experiments and could prove valuable to future mirror CTR work.

MODIFIED CONFIGURATIONS HAVING BETTER MHD STABILITY THAN SIMPLE MIRROR BUT MUCH LESS COMPLICATED THAN BASEBALL OR YIN-YANG CONFIGURATIONS

R. F. Post

The main advantage of the simple mirror is an economic one, derived from the simplicity and high magnetic efficiency of circular coils. The main disadvantage, of course, is the problem of suppressing flute instability. There are "modified simple mirror" configurations, however, which retain the advantages of circular coils and yet have improved stability against the flute mode.

One such configuration is a mirror-cusp combination, in which a simple long solenoid is capped on either end by a bucking coil shaped like a short circular coil. These cap coils are spaced away from the solenoid so as to leave a gap, thereby forming a ring mirror. Looking from the axis outward, field curvature will be favorable

in all directions, precluding radial outward fluting. However, there is no mirroring effect for field lines lying near the magnetic axis (these lines are not diverted through the ring mirror, but escape axially). Thus, the mirror-confined plasma will be hollow (i. e., the shape of a napkin ring), leaving the possibility of fluting on the inner surface of the plasma. It can be shown that only modes $m > 1$ are unstable in this case, so finite-orbit stabilization could be sufficient to stabilize the plasma. It also appears that contained-plasma profiles may be found that are stable against all fluting, even in the absence of finite-orbit effects.¹⁷

The mirror-cusp configuration, similar to that tested in the ELMO experiment at ORNL,¹⁸ illustrates that there may be configurations that retain most of the advantages of simple-mirror geometry yet are less prone to fluting.

Finally, it should be recalled that Furth and Andrioletti have shown that there is a class of axially-symmetric mirror fields that are also magnetic-well fields.¹⁹ These fields are at one end of a continuum of axially-symmetric fields, starting with the simple mirror. Between these limits are fields, such as those described above, that are either stable by the $\int dl/B$ criterion or, at least, are capable of suppressing low-order fluting modes. In considering mirror systems, attention should be paid to these possibilities.

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APPENDIX A

EXTERNAL-PLASMA STABILIZATION OF FLUTE MODE IN A SIMPLE MIRROR REACTOR

M. A. Lieberman

INTRODUCTION

Here, stabilization of the flute mode ($k_{\parallel} = 0$) in a collisionless, mirror-confined plasma (density n_0) is considered. The stabilization arises from the presence of external plasma between the mirror throat and an external conducting wall. The instability is described using the usual low-frequency slab model. The external-plasma region (density n_{ext}) appears as a source term in the electron-continuity equation. A general treatment is given of the admittance between the mirror-confined plasma and the conducting wall, including the effects of external-plasma impedance (Babykin *et al.*^{A1}), external-wall-sheath impedance (Kunkel and Guillory^{A2}), and sheath impedance at the mirror throat. These sheaths strongly limit the shorting of instability potential at the wall, and under the most favorable circumstances, stabilization of the $p = 1$ flute mode requires $n_{\text{ext}} \geq n_0$. The required n_{ext} can be reduced several orders of magnitude, using feedback currents at the external wall.

DISPERSION EQUATION

Ion and electron densities and ambipolar potential are shown in Fig. A-1.

The instability is examined using the slab model shown in Fig. A-2. In this model, effective acceleration, $g = v_{\text{th}}^2/R_c$ (where $R_c = R_e^2/R_p$), is the field line curvature and ℓ_B is the mirror scale length.

The wave is taken to vary as $\exp j(\omega t - ky)$.

The guiding center approximation is used for the particle motion in the mirror region:

$$\bar{v} = \frac{m}{q} \frac{\bar{g} \times \bar{B}}{B^2} + \frac{\bar{E} \times \bar{B}}{B^2} + \frac{m}{q} \frac{1}{B^2} \frac{d\bar{E}}{dt}. \quad (\text{A.1})$$

The continuity equation for each species in the mirror is written

$$\partial n / \partial t + \nabla \cdot (n \bar{v}) = S, \quad (\text{A.2})$$

where the source term S gives the current flow into the mirror region along field lines in response to the perturbed potential ϕ . The potential satisfies

A1. M. V. Babykin, P. P. Gavrin, E. K. Zavoiskii, L. I. Rudakov and V. A. Skoryupin, *Sov. Phys. JETP* 20, 1096 (1965).

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Fig. A-1. (a) Ion and electron densities near axis of mirror-confined plasma with cold low-density plasma background and sheath at nonemitting end plates. (b) Corresponding potential distribution (schematic). (Source: Kunkel and Guillory.^{A1})

Fig. A-2. Coordinates for slab model.

$$\nabla^2 \phi = - \sum_{\alpha} \frac{q_{\alpha} n_{\alpha}}{\epsilon_0}. \quad (\text{A.3})$$

The effect of the external plasma appears only in the source term S . Letting $y(\omega)$ be the admittance/unit area of the external density region and integrating Eq. (A.2) over the mirror volume, we find

$$S = - \frac{y(\omega)}{q \ell_p} \phi, \quad (\text{A.4})$$

where ℓ_p is the confined plasma length. Linearizing Eqs. (A.1) through (A.4), we obtain the dispersion equation

$$1 + \sum_{\alpha} \frac{\omega_{p\alpha}^2}{\Omega_{\alpha}^2} = \sum_{\alpha} \frac{1}{\omega - \omega_{\alpha}} \left(- \frac{1}{k R_p} \frac{\omega_{p\alpha}^2}{\Omega_{\alpha}^2} + j \frac{y_{\alpha}}{\epsilon_0 k^2 \ell_p} \right), \quad (\text{A.5})$$

where $\omega_{p\alpha}^2 = q_{\alpha}^2 n_{\alpha} / (\epsilon_0 m_{\alpha})$, $\Omega_{\alpha} = q_{\alpha} B / m_{\alpha}$, $\omega_{\alpha} = k g_{\alpha} / \Omega_{\alpha}$, and $R_p^{-1} = n_{\alpha} dn_{\alpha} / dx$ is the plasma radius. The usual flute instability for an electron-ion plasma with $T_i \gg T_e$ is obtained by putting $y_{\alpha} = 0$, $\omega_e \ll \omega_i$, and $1 + \omega_{pe}^2 / \Omega_e^2 \ll \omega_{pi}^2 / \Omega_i^2$ in Eq. (A.5):

$$1 = - \frac{\Omega_i^2}{k R_p} \left(\frac{1}{\omega - \omega_i} - \frac{1}{\omega} \right), \quad (\text{A.6})$$

which has the solution

$$\omega = \omega_i \left[\frac{1}{2} \pm \left(\frac{1}{4} - \frac{\Omega_i^2}{k R_p \omega_i} \right)^{1/2} \right].$$

For small k , we find

$$\omega = \pm j \left(\frac{g_i}{R_p} \right)^{1/2},$$

which is the usual result. Putting $K = 1 + \omega_{pi}^2 / \Omega_i^2 \gg 1$, $\epsilon_i = a_i^2 / \ell_B^2$, $\epsilon_e = \epsilon_i T_e / T_i$, $p = k R_p$, $\Omega = \omega / \Omega_i$, and the normalized sheath admittance

$$A(\omega) = \frac{y_e(\omega) \Omega_i R_p^2}{\epsilon_0 \omega_{pi}^2 \ell_p}, \quad (\text{A.7})$$

we obtain the dispersion equation

$$\Omega^2 - \Omega [p(\epsilon_i - \epsilon_e) + j A / p^2] + \epsilon_i + \epsilon_e + \epsilon_i A / p + p^2 \epsilon_i \epsilon_e = 0. \quad (\text{A.8})$$

EXTERNAL-PLASMA ADMITTANCE

The ac voltage drop across the external plasma is the sum of:

- Sheath impedance at the mirror
- Sheath impedance at the external wall
- Impedance of the bulk of the external plasma.

Mirror Sheath

In absence of an external plasma, sheath potential at the mirror is fixed by equating electron and ion currents

$$\Gamma_i = n_o \frac{\ell_p}{\tau_m} = \frac{n_o v_e}{4} e^{-\phi/T_e} = \Gamma_e, \quad (\text{A.9})$$

where

$$v_e = [8e T_e / (\pi m)]^{1/2}$$

and τ_m is mean ion-scattering loss time. The sheath conductance/unit area g_o , given by

$$g_o = -e \frac{\partial \Gamma_e}{\partial \phi} = \frac{en_o \ell_p}{T_e \tau_m}, \quad (\text{A.10})$$

is very small, leading to little stabilization effect in Eq. (A.8). To increase sheath admittance at the mirror, we introduce an external cold plasma, density n_{ext} . Then, assuming collisional electrons (Maxwellian distribution) with $T_{\text{ext}} = T_e$, Eq. (A.9) is modified to

$$\Gamma_e = \Gamma_i + \Gamma_{e \text{ ext}}, \quad (\text{A.11})$$

or

$$\Gamma_e = \frac{n_o v_e}{4} e^{-\phi/T_e} = \frac{n_o \ell_p}{\tau_m} + \frac{n_{\text{ext}} v_e}{4}. \quad (\text{A.12})$$

Thus, ambipolar potential is reduced and sheath conductance is increased to

$$g_m = \frac{e}{T_e} \left(\frac{n_{\text{ext}} v_e}{4} + \frac{n_o \ell_p}{\tau_m} \right), \quad (\text{A.13})$$

even though $n_{\text{ext}} \ll n_o$, $v_e \gg \ell_p / \tau_m$, such that the first term in Eq. (A.13) dominates.

In addition to the above resistive voltage drop, a reactive voltage drop may appear across a sheath. For the above sheath, which separates confined density (n_o) from external density (n_{ext}), the reactive element of sheath impedance is inductive and can be lumped into the bulk external-plasma contribution (C) as a negligible correction.

Wall Sheath

Assuming a nonemitting wall,

$$\Gamma_e = \frac{n_{\text{ext}} v_e}{4} e^{-\phi/T_e} = \frac{n_{\text{ext}} v_i}{4} = \Gamma_i, \quad (\text{A.14})$$

which leads to

$$g_{\text{ow}} = \frac{e}{T_e} \frac{n_{\text{ext}} v_i}{4} \quad (\text{A.15})$$

at the wall. This admittance is $(m/M)^{1/2}$ times smaller than that of Eq. (A.13) and, thus, will dominate. To increase wall-sheath admittance, we must make the wall an emitting surface. Assuming an electron emission current of $j_o < j_e - j_i$, where j_e and j_i are random electron and ion currents, we have

$$g_w = \frac{e}{T_e} \frac{n_{\text{ext}} v_i}{4} (1 + j_o/j_i) = \frac{\epsilon_o}{\lambda_{\text{Dext}}^2} \frac{v_i}{4} (1 + j_o/j_i), \quad (\text{A.16})$$

which tends to

$$g_w = \frac{\epsilon_o}{\lambda_{\text{Dext}}^2} \frac{v_e}{4}, \quad (\text{A.17})$$

as $j_o \rightarrow j_e - j_i$. For $j_o \geq j_e + j_i$, dc sheath potential is clamped at zero for $\Gamma_e < n_{\text{ext}} v_e/4$, the linearization leading to Eq. (A.16) fails, and the sheath disappears.

The reactive component of wall-sheath impedance depends on the details of the sheath's structure. Neglecting electron density in the sheath and using Child's law for the ions,^{A2}

$$j\omega c_\omega = \frac{j\omega \epsilon_o}{2\lambda_{\text{Dext}}} \left(\ln \frac{j_e}{j_o + j_i} \right)^{-1/2}, \quad (\text{A.18})$$

valid for $j_o < j_e - j_i$.

External-Plasma Impedance

Including the effects of an effective collision frequency ν and a finite electron inertia, electrons in the external plasma satisfy the parallel-force equation

$$m \frac{dv_e}{dt} = -q \nabla_{\parallel} \phi - m \nu v_e, \quad (\text{A.19})$$

which leads to

$$j_{\parallel} = -\frac{\epsilon_o \omega^2}{\nu + j\omega} \nabla_{\parallel} \phi \quad (\text{A.20})$$

or

$$y_p = \frac{\epsilon_o \omega_{pext}^2}{(\nu^2 + \omega^2) \ell_{ext}} (\nu - j\omega), \quad (A.21)$$

where ℓ_{ext} is external-plasma length. This shows a resistive voltage drop due to finite collision frequency and an inductive voltage drop due to finite electron inertia.

The above result is not valid for arbitrarily large ℓ_{ext} . Neglecting collisions, the actual dispersion equation in the external plasma is

$$k_{\parallel}^2 \left(1 - \frac{\omega_{pext}^2}{\omega^2} \right) + k_y^2 \left(1 + \frac{\omega_{piext}^2}{\Omega_i^2} \right) = 0, \quad (A.22)$$

where $\phi \propto e^{j(\omega t - k_y y)} \sin k_{\parallel} z$, and k_y is the flute wave number. For Eq. (A.21) to be valid, we require $k_{\parallel} \ell_{ext} \ll 1$, a condition normally met in practice.

To summarize, for a cross-sectional plasma area A_p , the total admittance/unit area y_e to be used in solving dispersion Eq. (A.8) is given by the circuit in Fig. A-3. In this circuit,

$$y_e = Y_e / A_p;$$

$$\frac{1}{R_m} = \frac{e}{T_e} \left(\frac{n_{ext} v_e}{4} + \frac{n_o \ell_p}{\tau_m} \right) A_p = \frac{\epsilon_o v_e A_p}{4 \lambda_{Dext}^2} \left(1 + 4 \frac{n_o}{n_{ext}} \left(\frac{m T_i}{M T_e} \right)^{1/2} \frac{1}{N} \right),$$

where N is the mean number of ion bounces in the mirror;

$$R_p = \frac{\nu \ell_{ext}}{\epsilon_o \omega_{pext}^2 A_p};$$

$$L_p = \frac{\ell_{ext}}{\epsilon_o \omega_{pext}^2 A_p};$$

$$G_w = \frac{\epsilon_o v_{iext}}{4 \lambda_{Dext}^2} (1 + j_o / j_i) A_p, \quad j_o < j_e - j_i;$$

and

$$C_w = \frac{\epsilon_o}{2 \lambda_{Dext}} \left(\ln \frac{j_e}{j_o + j_i} \right)^{1/2} A_p, \quad j_o < j_e - j_i.$$

Fig. A-3. Model of external-plasma admittance.

STABILIZATION CRITERIA

Effect of External Plasma Without Sheaths

If the sheaths are neglected ($R_m \rightarrow 0$, $G_w \rightarrow \infty$) and the external plasma is assumed collisionless ($w \gg \nu$), the treatment of Babykin et al. is recovered. The external plasma resists the shorting of the instability potential only by the effect of finite electron inertia (i. e., the inductor L_p in Fig. A-3). Using $\epsilon_e \ll \epsilon_i$ ($T_e \ll T_i$) and $y_e = -j \epsilon_o \omega_{pe}^2 / (\omega \ell_{ext})$ in Eq. (A.8) and expanding,

$$\Omega^3 - p\epsilon_i \Omega^2 + (\epsilon_i - G/p^2)\Omega + \epsilon_i G/p = 0, \quad (A.23)$$

where

$$G = \frac{M}{m} \frac{n_{ext}}{n_o} \frac{R_p^2}{\ell_{ext} \ell_p}.$$

In the usual case $\epsilon_i = a_i^2 / \ell_B^2 \ll 1$ and if mode number p is not too large, the condition for stability of Eq. (A.23) is $\epsilon_i < G/p^2$, or

$$\frac{n_{ext}}{n_o} > \frac{M}{m} \frac{\ell_p \ell_{ext}}{\ell_B^2} \frac{a_i^2}{R_p^2}, \quad (A.24)$$

the result obtained by Babykin et al.^{A1} This criterion is very favorable for stabilization, since a_i^2 / R_p^2 typically is $10^{-2} - 10^{-4}$. Thus, $n_{ext}/n_o \gtrsim 10^{-5} - 10^{-7}$ for stabilization, a condition easily met in a reactor, with very little degradation in Q . Although the experimental results of Babykin et al.^{A1} and Coensgen et al.^{A3} are supposed to be in agreement with Eq. (A.20), the external plasma densities said to be present in these experiments seem much too low, in view of the high neutral gas pressures and wall-bombardment rates for the experiments.

Sheath Effects

Kunkel and Guillory^{A2} treat the case in which a wall sheath is present in addition to bulk external-plasma impedance. They neglect the sheath at the mirror. In this case, sheath conductance at the wall G_w is much larger than sheath susceptance ωC_w , provided $\omega \ll \omega_{piext}$, a condition usually met in practice. If R_w is compared to $(v_i / \ell_B) L_p$, sheath resistance dominates provided

$$\frac{M}{m} \frac{1}{1 + j_o/j_i} \gg \left(\frac{\pi T_i}{8 T_{iext}} \right)^{1/2} \frac{\ell_{ext}}{\ell_B}, \quad (A.25)$$

A3. F. H. Coensgen, W. F. Cummins, W. E. Nexsen, Jr., and A. E. Sherman, Phys. Fluids 9, 187 (1966).

a condition generally met in practice, even if $j_o/j_i \sim (M/m)^{1/2}$. Except for very long connection lengths, sheath resistance is the dominant source of voltage drop between the confined plasma and the wall. Setting $y_e = \frac{1}{4}\epsilon_o v_{iext} (1 + j_o/j_i)/\lambda_{Dext}^2$ in Eq. (A.8),

$$A = (2\pi)^{-1/2} \frac{n_{ext}}{n_o} \left(\frac{T_i T_{iext}}{T_e^2} \right)^{1/2} \frac{R_p^2}{a_i \ell_p} (1 + j_o/j_i). \quad (A.26)$$

Putting $\Omega = u + 1/2 p (\epsilon_i - \epsilon_e)$ and dropping terms quadratic in ϵ ($p\epsilon \ll 1$) for the drift Eq. (A.1) to be valid yields the equation

$$u^2 - u j A / p^2 + \bar{\epsilon} (2 + j A / p) = 0, \text{ with } \bar{\epsilon} = 1/2 (\epsilon_i + \epsilon_e).$$

The unstable root is

$$u = 1/2 j A / p^2 - 1/2 j (A^2 / p^2 + j 4 \bar{\epsilon} A / p + 8 \bar{\epsilon})^{1/2}. \quad (A.27)$$

Since $\bar{\epsilon} \ll 1$, $4 \bar{\epsilon} A / p \ll A^2 / p^2 + 8 \bar{\epsilon}$ for $0 < A < \infty$. Expanding the square root in Eq. (A.27), yields the solution

$$\Omega = \frac{1}{2} p (\epsilon_i - \epsilon_e) + p \bar{\epsilon} A (A^2 + 8 \bar{\epsilon} p^4)^{-1/2} - j 1/2 p^{-2} [(A^2 + 8 \bar{\epsilon} p^4)^{1/2} - A]. \quad (A.28)$$

The growth rate for zero sheath admittance $A = 0$ is $\text{Im} \Omega = -\sqrt{2\bar{\epsilon}}$ [from Eq. (A.28)].

For a reduction in growth rate greater than a factor of F , Eq. (A.28) requires

$$A = \sqrt{2} (F - F^{-1}) p^2 \bar{\epsilon}^{-1/2}, \quad (A.29)$$

or

$$\frac{n_{ext}}{n_o} \gtrsim \sqrt{2\pi} (F - F^{-1}) p^2 \frac{a_i^2 \ell_p}{R_p^2 \ell_B} \left(\frac{T_e^2}{T_i T_{iext}} \right)^{1/2} \left(1 + \frac{T_e}{T_i} \right)^{1/2} (1 + j_o/j_i)^{-1}, \quad (A.30)$$

which is in basic agreement with the more qualitative expression of Kunkel and Guillory.^{A2} (There is an error in their Eq. (13).) This result is very discouraging, since F generally must be very large. For $j_o = 0$ (nonemitting wall) and reactor conditions, $F \gtrsim 10^5$ and the flute cannot be stabilized with $n_{ext} \gtrsim n_o$ unless a_i/R_p is 10^{-3} or less. For an emitting wall surface with $j_o = j_e$, the wall sheath disappears but the sheath at the mirror remains, giving

$$\frac{n_{ext}}{n_o} \gtrsim \sqrt{2\pi} (F - F^{-1}) \frac{m}{M}^{1/2} p^2 \frac{a_i^2 \ell_p}{R_p^2 \ell_B} \left(\frac{T_e^2}{T_{iext} T_i} \right)^{1/2} \left(1 + \frac{T_e}{T_i} \right)^{1/2}, \quad (A.31)$$

an improvement over Eq. (A.30), but not enough to be of interest in a mirror machine reactor (i.e., for $F \sim 10^5$, $a_i/R_p \sim 10^{-2}$, $n_{ext} \gtrsim n_o$ for stability). For a wetwood burner, $F \sim 10^2$ but $a_i/R_p \sim 10^{-1}$.

It appears that there is no difference, in principle, between the localized sheaths assumed here and a distributed sheath, which may exist in the external-plasma region due to magnetic field expansion. In both cases, the change in $j_{||}$ with ϕ is limited by the random ion current (for an emitting wall, the random electron current), whether or not the sheath is distributed.

FEEDBACK STABILIZATION

The possibility of stabilizing or greatly reducing the growth rate of the flute instability by applying feedback signals at the external wall is considered in this section. Letting $\omega_{pi}^2/\Omega_i^2 \gg 1$ and $y_i = 0$ in Eq. (A.5), the dispersion equation can be put in the form

$$j\omega C + Y_{NL} + 2G = 0, \quad (A.32)$$

where

$$C = \epsilon_0 k^2 \ell_p A_p \omega_{pi}^2 / \Omega_i^2, \quad (A.33)$$

$$Y_{NL} = j\epsilon_0 k^2 \ell_p \frac{\omega_{pi}^2}{\Omega_i^2} \left(\frac{\omega_e}{\Omega_i} + \frac{\omega_i + \omega_e}{kR_p(\omega - \omega_i)} \right), \quad (A.34)$$

$$G = y_e A_p = G_w, \quad (A.35)$$

and the factor of 2 in Eq. (A.32) is inserted to account for the sheath at both ends of the plasma. This suggests the circuit model shown in Fig. A-4.

Dispersion Equation with Feedback

Let us now consider using one end of the system to sense instability potential and the other end to apply a feedback signal, $\tilde{e}$. This new circuit is shown in Fig. A-5. The possibility of both sensing and feedback at the same end will be considered later. Also, note that, to control a mode with mode number $p = kR_p$, it is, in principle, necessary to sense the potentials on only two field lines spaced at an angle not commensurate with $2\pi/p$ around the circumference of the external end wall. We consider a simple feedback signal having components proportional to both $\phi(t)$ and $\dot{\phi}(t)$:

$$e(t) = -\beta\phi(t) - \delta\dot{\phi}(t), \quad (A.36)$$

where β and δ are real "amplification" factors. In the frequency domain,

$$\tilde{e} = -(\beta + j\omega\delta)\hat{\phi}, \quad (A.37)$$

and the new dispersion equation is

$$j\omega C\tilde{\phi} + Y_{NL}\tilde{\phi} + G(\tilde{\phi} - \tilde{e}) = 0, \quad (A.38)$$

Fig. A-4. Circuit model of flute instability, including line tying.

Fig. A-5. Circuit model of feedback stabilization of flute mode at the ends.

or

$$j\omega C + Y_{NL} + G(1 + \beta + j\omega\delta) = 0. \quad (A.39)$$

This is identical to Eq. (A.8), with A replaced by $A(T + \beta + j\omega\delta)$. The equation is still a quadratic, and, putting $A' = A(1 + \beta)$ and $H = A\Omega_i\delta/p^2$ in Eq. (A.8), we obtain

$$\Omega^2(1 + H) - \Omega(p(\epsilon_i - \epsilon_e) + jA'/p^2 + \epsilon_i pH) + \epsilon_i + \epsilon_e + \epsilon_i jA'/p + p^2 \epsilon_e \epsilon_i = 0. \quad (A.40)$$

Proceedings as in the "Sheath Effects" section, we put

$$\Omega = u + \frac{1}{2(1 + H)} [p(\epsilon_i - \epsilon_e) + \epsilon_i pH] \quad (A.41)$$

in Eq. (A.40) and drop terms of order ϵ^2 to obtain the equation

$$u^2(1 + h) - u jA'/p^2 + (\epsilon_i + \epsilon_e) + \frac{1}{2} [\epsilon_i + \epsilon_e(1 + H)^{-1}] jA'/p = 0. \quad (A.42)$$

The solution is

$$u = \frac{j}{2(1 + H)} \left\{ \frac{A'}{p} \pm \left[\frac{A'^2}{p^4} + 4(1 + H)(\epsilon_i + \epsilon_e) + 2(\epsilon_e + \epsilon_i + H\epsilon_i) jA'/p \right]^{1/2} \right\}. \quad (A.43)$$

Feedback Proportional to ϕ

Now, let us consider the case $H = 0$, corresponding to feedback proportional to ϕ alone and not to its derivative. In this case, Eq. (A.40) is identical to Eq. (A.8), with A' replacing A . Therefore, a reduction in growth rate by a factor greater than F requires

$$\frac{n_{ext}}{n_o} \geq \sqrt{2\pi} \frac{(F - F^{-1})}{1 + \beta} p^2 \frac{a_i^2}{R_p^2} \frac{\ell_p}{\ell_B} \left(\frac{T_e}{T_i T_{iext}} \right)^{1/2} \left(1 + \frac{T_e}{T_i} \right) (1 + j_o/j_i)^{-1}. \quad (A.44)$$

For large β , a significant reduction in the required external density over Eq. (A.30) is obtained. In essence, the feedback reduces the effective sheath impedance seen by the instability. However, there is an upper limit on β size for the above treatment to be valid. It is required that $|\tilde{e} - \tilde{\phi}| \lesssim \phi_o$, the dc ambipolar potential at the sheath. Thus, $1 + \beta \lesssim \phi_o/|\tilde{\phi}|$.

Feedback Proportional to ϕ and $\dot{\phi}$

To study the effect of a finite H (feedback of $\dot{\phi}$), we put $\epsilon_e = 0$ in Eq. (A.43), corresponding to the case $T_i \gg T_e$. Provided $H\epsilon_i \ll 1$ and following the discussion preceeding Eq. (A.28), the square root in Eq. (A.43) may be expanded to obtain the growth rate

$$\text{Im } \Omega = \frac{1}{2(1+H)p^2} \left[\left(A'^2 + 4(1+H)\epsilon_i p^4 \right)^{1/2} - A' \right]. \quad (\text{A.45})$$

For a factor-of-F reduction in growth rate, we now have

$$\frac{n_{\text{ext}}}{n_o} \geq \sqrt{2\pi} \frac{F - F^{-1}}{(1+\beta)(1+H)^{1/2}} p^2 \frac{a_i^2}{R_p^2} \frac{\ell_p}{\ell_B} \left(\frac{T_e^2}{T_i T_{\text{iext}}} \right)^{1/2} (1 + j_o/j_i)^{-1}. \quad (\text{A.46})$$

It appears that the feedback of the time derivative $\dot{\phi}$ is not as helpful as ϕ feedback. Again, there is only a reduction in growth rate, not complete stabilization.

Effect of Response-Time Delay

Let us now drop the $\dot{\phi}$ term in $e(t)$ and consider the effect of a response-time delay: $e(t) = -\beta\phi(t - \tau)$, where τ is the delay time. In this situation, $\tilde{e} = -\beta e^{-j\omega\tau}$ and A in Eq. (A.8) is replaced by $A' = A(1 + \beta e^{-j\omega\tau})$. For $\beta \gg 1$, the imaginary parts of two roots of the dispersion equation are

$$\text{Im } \Omega_1 = j \frac{A'}{p} \quad (\text{A.47})$$

$$\text{Im } \Omega_2 = -j \frac{\epsilon_i p^2}{A'}. \quad (\text{A.48})$$

It is clear that, for $\omega\tau \ll 1$, there is little change in Eq. (A.44); however, when $\omega\tau = \pi$, Ω_1 becomes unstable, unless $|\beta| < 1$ for these ω . This amplification- β requirement is easy to meet.

APPENDIX B

EFFECT OF EXTERNAL PLASMA ON SIMPLE-MIRROR-REACTOR Q

Grant Logan

SIMPLE-MIRROR-REACTOR MODEL

Table B-1 lists the parameters for a simple-mirror reactor (no cold plasma injection) as provided by Carlson,^{B1} along with appropriate plasma quantities from Fokker-Planck calculations.^{B2} Effective area (A_{eff}) and length (L_{eff}) can be defined such that

$$\frac{n_o^2 A_{\text{eff}} L_{\text{eff}}}{(n\tau)} = \frac{I_n}{q} \quad (\text{B.1})$$

and

$$\frac{L_{\text{eff}}}{\sqrt{\frac{A_{\text{eff}}}{\pi}}} = \frac{L}{r_p}, \quad (\text{B.2})$$

where $n\tau$ is the average ion-confinement parameter; which depends on only E_o , R , and kT_e . With $R = 10$ and $E_o = 200$ keV, a multispecies Fokker-Planck calculation by Futch *et al.*^{B2} gives $n\tau = 7.24 \times 10^{13}$ for the above model, with spatially-uniform plasma (no external gas source). Using the other parameters in Table B-1, Eqs. (B.1) and (B.2) give $A_{\text{eff}} = 7.7 \text{ m}^2$ and $L_{\text{eff}} = 4.3 \text{ m}$, useful information in calculating particle and energy losses from the plasma and for comparing results with spatially-independent Fokker-Planck calculations.

Table B-1. Simple-mirror-reactor parameters (no cold gas).

Injection energy, $E_o = 200$ keV
Fusion power, $P_{\text{fus}} = 1000$ MW
B_o , vac = 50 kG
$\beta = 0.9$
$R = R_{\text{vac}}/\sqrt{1 - \beta} = 9.5$
Plasma length, $L = 14.1$ m
Plasma radius, $r_p = 3.21$ m
Peak ion density, $n_o = 2.34 \times 10^{14} \text{ cm}^{-3}$
Reactor Q = 1.41
Injection current, $I_h = 3500$ A
Average ion energy (including α particles), $\bar{E}_i = 242$ keV
Electron temperature, $kT_e = 21.5$ keV
Ambipolar potential, $q\phi = 5 kT_e$

Figure B-1 (a and b) qualitatively shows the field, density, and potential profiles of the reactor model with exterior expander regions, where the field decreases to very low values before reaching the walls. A large expansion ratio (~ 100) is necessary to reduce the power flux density of escaping plasma to tolerable levels at the wall.

For the purposes of line tying, average line density over the length of the flux tube between the mirror and wall ($\int n \cdot ds$) is probably more important than average exterior density alone. For gas sources at the mirror, line density is roughly constant along the flux expander for any expansion ratio, simplifying the treatment of such expanders in line-tying theories. Placing a gas source at the end walls for $n_{\text{ext}}(z) = \text{const}$ along the expander will create a higher average line density, which may allow a lower n_{ext} value for the same degree of line tying. However, the potential drop in the expander, if any, would have to be provided by a discrete sheath at the wall. Whether or not such a sheath could exist depends on surface effects (e.g., secondary emission by electrons and ions), which are difficult to predict with a high neutral density at the wall. Besides, a line-tied, low-Q, simple-mirror reactor may require direct converters not compatible with a gas source at the ends. For electron heat loss to be manageable, some retardation of sheath potential is necessary. To insure at least a distributed potential drop, gas sources are located just outside the mirror throats, thereby allowing cold-plasma density to fall to a low value at the walls.

The expanding field converts the electrons' perpendicular energy to parallel energy, which sets up the distributed potential sufficient eventually to slow the electrons to the accelerated ion velocities before reaching the wall. Electrons collected at the wall at the total ion loss rate then have negligible kinetic energy and electrons injected with the ions at the midplane are counted as lost, carrying away an energy of $q\phi_{\text{tot}}$. The resulting loss surface, with expanders external to the mirrors, are portions of spherical surfaces intersected by the loss cones in velocity space, for electrons at the midplane (Fig. B-1c). Electrons at the midplane, occupying the double-hatched region of one loss cone between the energies $q\phi_m$ and $q\phi_{\text{tot}}$, occupy a hemisphere of radius $\sqrt{2\phi_{\text{ext}}/m_e}$ in velocity space at the mirror throat, with the other half-sphere being filled by electrons reflected by the exterior potential drop in the expander (Fig. B-1d). Electrons escape from the system by collisional diffusion, which increases their energy above the top of the potential barrier ϕ_{tot} in the expander. The average incremental energy above ϕ_{tot} with which an electron escapes is small compared to ϕ_{tot} (and kT_e) because, under reactor conditions, the electron scattering mean-free-path, especially for electrons in the Maxwellian tail, is much longer than the system. This point will be discussed further in the section on electron loss rates.

ION LOSS RATES WITH COLD-PLASMA SOURCE

Hamilton^{B3} has extended the Fokker-Planck calculations of Futch et al.^{B2} to include the effect of adding cold electrons at the midplane. The decrease in electron temperature resulting from injecting a given current of cold electrons at the midplane,

Fig. B-1. Field, density, and potential profiles of a reactor model.

in general, will differ from the drop associated with injecting the same cold-electron current at the mirror throat because, in the latter case, the electrons fall into the potential drop of the mirror (equivalent to injecting electrons of energy $q\phi_m$ at the midplane). Also, Hamilton assumed an absorbing boundary for particles at the mirror, while here the ambipolar potential profile is altered using external expanders. However, as long as cold ions from the gas source do not interact appreciably with the hot mirror-confined ions (penetration limited by $q\phi_m$) and since $n\tau$ for the hot ions is not substantially affected by the mirror potential drop for large mirror ratios,^{B1} the effect of either electron-injection method is mainly to change the drag on hot ions by lowering electron temperature in varying amounts. Therefore, Hamilton's calculations on reactor Q and $n\tau$ are plotted in Fig. B-2 as a function of kT_e rather than the electron-injection rate. The purpose of this work is to calculate the equilibrium electron temperature for injection of cold electrons at the mirror, and then use Fig. B-2 to obtain Q. Hamilton's calculations show that, for a fixed hot-ion injection energy E_o , lowering kT_e by cold-electron injection results in a lower average ion energy $\bar{E}_i$ as well as a lower $n\tau$ due to increased slowing of hot ions relative to side scatter. For the effective mirror ratio to remain high as T_e decreases, the plasma beta $\sim n_o \bar{E}_i$ must be held constant; therefore, n_o must be increased as $1/\bar{E}_i$, starting with n_o from Table B-1. The result of using Hamilton's data for $\bar{E}_i$ (kT_e) is shown in Fig. B-2. The loss rate of hot mirror-confined ions J_h is determined using Eq. B.1:

$$J_h (kT_e) = \frac{I}{q} = A_{eff} L_{eff} \left(\frac{n_o^2}{n\tau} \right), \quad (B.3)$$

where $(n_o^2/n\tau)$ is obtained from Fig. B-2.

Since hot ions escaping the mirror have parallel energies $= \phi_m$ at the mirror, the density of hot ions n_h at the mirror can be estimated from

$$n_h(z_m) \sqrt{\frac{2\phi_m}{m_i} \frac{A_{eff}}{R}} = J_h. \quad (B.4)$$

Using Eq. (B.3) with Table B-1, a minimum value of $n_h(z_m)/n_o = 4 \times 10^{-5}$ is obtained. For $z > z_m$, n_h falls roughly along with B for $\phi_{ext} \ll \phi_m$.

Injection of a density n_c of cold ions at a mirror raises the density not only at the source in the mirror throat, but also to some distance z_o into the mirror-confined plasma. This happens because the mirror drop ϕ_m is distributed over the hot-ion density profile (see Fig. 1b). A local minimum in total electron density and potential cannot exist if there is no local particle sink. Therefore, total density and potential must decrease monotonically away from the center. A small gradient in ϕ between z_o and z_m supports cold-ion pressure, so that the cold ions do not penetrate all the way between the mirrors unless external density is raised to the order of n_o and $\phi_m \rightarrow 0$.

The cold-ion-confinement time (a sound transit time over a distance $z_m - z_o$) is sufficiently short compared to equilibration times with hot ions and electrons that

Fig. B-2. Q , $n\tau$, and n_0 as a function of kT_e . $E_0 = 200$ keV, $R = 10$, $\beta = \text{constant}$.

kT_i (cold) $\ll kT_e$ and $q\phi_m$. Thus, total density and potential is nearly flat between z_o and z_m . The point z_o at which $n_{\text{ext}}(z_m) = n_c(z_m) + n_h(z_m) = n_h(z_o)$ approaches the midplane as $n_{\text{ext}} \rightarrow n_o$.

The rate of cold-ion injection J_c required to maintain a stagnant density $n_c(z_m)$ just behind the gas sources has been found by Lee and McNamara^{B4} to be given by

$$J_c \approx 2 \left(\frac{1}{2} n_c(z_m) \right) \left(\sqrt{\frac{kT_e}{m_i}} \right) \left(\frac{A_{\text{eff}}}{R} \right), \quad (\text{B.5})$$

where a flow velocity (just outside the mirrors) of $C_s = \sqrt{(kT_e + 5/3 kT_i)/m_i} \approx \sqrt{kT_e/m_i}$ is assumed to satisfy the Bohm sheath condition. A quantity that can be easily measured experimentally and related directly to line-tying theories is the ratio of external density at the mirror throat n_{ext} to the central density n_o , defined as

$$\epsilon \equiv \frac{n_{\text{ext}}}{n_o} = \frac{n_c(z_m) + n_h(z_m)}{n_c(o) + n_h(o)}. \quad (\text{B.6})$$

For $n_c = 0$, ϵ has a minimum of 4×10^{-5} , given by Eq. (B.4), due to the escaping hot ions. For $q\phi_m \gg kT_i$ (cold), the cold ions do not penetrate to the midplane and $n_c(o) \approx 0$. As ϵ approaches unity and $q\phi_m$ becomes sufficiently small compared to kT_i (cold), the cold plasma can dominate both inside and outside the mirror (the two-component limit). In this limit,^{B5} cold-ion drag on the hot ions is important, raising kT_i (cold) $\gtrsim kT_e$. However, over most of the range $4 \times 10^{-5} \ll \epsilon \lesssim 1$, $\epsilon \approx n_c(z_m)/n_h(o)$.

ELECTRON LOSS RATES (NO COLD PLASMA)

For the reactor parameters assumed in this study, electron-electron scattering mean free path (for a test electron energy at the average electron energy) is $\bar{\lambda}_{ee} > 10^4$ m, much longer than the length of the high-density plasma region, where scattering can occur (collisions are negligible in the expander). Collision lengths and diffusion times in velocity space are even longer for electrons near the ambipolar potential energy $\gg kT_e$. Once an electron's energy exceeds the ambipolar potential, it can be lost in one bounce time $t_b \sim 10^{-8}$ sec. Therefore, if classical collisions are the only available mechanism for populating the high-energy tail of the electron distribution, the distribution of electrons above the ambipolar energy will be depleted from that of a Maxwellian and the electron loss rate will be controlled by the rate at which they diffuse to the edge of the ambipolar cutoff. In other words, if we could assume the distribution to be completely Maxwellian, the electron loss rate could be calculated by integrating (vf) over the volume of velocity space occupied by the loss region for electrons (Fig. B-1c). But, if the loss region is depleted, the loss rate should be calculated by

$$J_e = -A_{\text{eff}} L_{\text{eff}} \int D_v \nabla_v f d S_v, \quad (\text{B.7})$$

where the integral is over the loss surface in velocity space and where D_v is the velocity space diffusion coefficient and f is an appropriate distribution that decreases smoothly to zero at the loss surface.

Hall^{B6} has suggested that electrons Maxwellianize much faster than with classical collisions due to sheath oscillations. Hall quotes early experimental evidence by Langmuir^{B7} and later by Gabor et al.^{B8} in support of this claim. However, a more recent experiment by Harp^{B9} indicates that quiescent sheaths can exist without rf fluctuations. The fact that sheath oscillations sometimes do occur may be due to disturbances elsewhere in the plasma propagating to the sheath, and not to a fundamentally unstable nature of the sheath itself.

One result of a depleted electron distribution is a lower ambipolar potential to keep electron loss rates in balance. Futch et al.^{B1} find that $q\phi = 5 kT_e$, with a Fokker-Planck treatment for our model reactor parameters, while an assumption of Maxwellian distribution would give $q\phi = kT_e \ln(n_o/n_{ext}) = 10 kT_e$, for $n_{ext} = 4 \times 10^{-5} n_o$. With n_{ext} reduced by another factor of 10^3 in the expander, a total drop $q\phi \approx 17 kT_e$ would be calculated using a Boltzman factor. According to Hall, the higher energy loss per electron, using Maxwellian, is necessary to explain energy loss from the 2XII experiment. He assumes, however, that one hot electron is removed from the plasma per ion lost, which implies a nonemitting wall. Since plasma potentials were not directly measured in the 2XII experiment, the measured values of n_e , T_e , W_i , and τ can be accounted for equally well by assuming a smaller $q\phi/kT_e$ value, and using a nonzero secondary-emission coefficient that results in more than one hot electron lost per ion. Whether or not a cutoff electron distribution is more appropriate than a complete Maxwellian has yet to be verified by experiment. Nonetheless, in this study, a cutoff electron distribution is assumed.

Returning to Eq. (B.7), a reasonable model for the cutoff electron-distribution function is needed. Berk^{B10} has found an analytic solution to the Fokker-Planck equation for electrons at the midplane ($\phi = 0$); with a source at $v = 0$ and a boundary condition of $f \rightarrow 0$ as $v \rightarrow v_{max}$:

$$f(v) = e^{-v^2/\bar{v}^2} \int_v^{v_{max}} du e^{+u^2/\bar{v}^2}, \quad (B.8)$$

where

$$\bar{v} \equiv \sqrt{\frac{2kT_e}{m_e}}. \quad (B.9)$$

Equation (B.8) can be approximated to within 10% for $0 < v < v_{max}$ by

$$f(v) \approx e^{-v^2/\bar{v}^2} e^{-v_{max}^2/\bar{v}^2}. \quad (B.10)$$

Normalizing this f so that $n_o = \int_0^{v_{\max}} f(v) 4\pi v^2 dv$ gives

$$f(v) = \frac{n_o}{4\pi \bar{v}^3 \left[\frac{\pi}{4} - \left(\frac{\alpha}{2} + \frac{\alpha}{3} \right) e^{-\alpha^2} \right]} \left\{ e^{-v^2/\bar{v}^2} - e^{-\alpha^2} \right\}, \quad (\text{B.11})$$

where

$$\alpha^2 \equiv v_{\max}^2 / \bar{v}^2 = q\phi_{\text{tot}} / kT_e. \quad (\text{B.12})$$

This f approaches a Maxwellian for $\alpha^2 \gg 1$ and $v \ll v_{\max}$. It also is spherically symmetric, whereas the loss surface of Fig. B-1c tends toward an anisotropic distribution for energies near $q\phi_{\text{tot}}$. Consider the diffusion coefficients^{B11} that would result if f were Maxwellian; this should be a good approximation for the bulk of the electrons, if $q\phi_{\text{tot}} \gg kT_e$:

$$\langle \nabla_{\parallel}^2 \rangle \approx \frac{C}{v} G\left(\frac{v}{\bar{v}}\right), \quad (\text{B.13})$$

$$\langle \nabla_{\perp}^2 \rangle \approx \frac{C}{v} \left\{ \phi\left(\frac{v}{\bar{v}}\right) - G\left(\frac{v}{\bar{v}}\right) \right\}, \quad (\text{B.14})$$

where the constant $C = 3 \times 10^{19} n_o$, n is in cm^{-3} , and v is in cm/sec . For $v \gg \bar{v}$ near the loss surface, the functions G and $\phi - G$ can be approximated by

$$G \approx \frac{v^2}{2\bar{v}^2} \quad (\text{B.15})$$

$$\phi - G \approx 1, \quad (\text{B.16})$$

so the diffusion coefficients become

$$\langle (\nabla_{\parallel})^2 \rangle \approx 3 \times 10^{19} \frac{n_o}{v} \left(\frac{2\bar{v}^2}{v^2} \right) \left(\frac{(\text{cm/sec})^2}{\text{sec}} \right), \quad (\text{B.17})$$

$$\langle (\nabla_{\perp})^2 \rangle \approx 3 \times 10^{19} \frac{n_o}{v}. \quad (\text{B.18})$$

For $v \gg \bar{v}$, angular diffusion is much faster than radial diffusion in velocity space, which tends to keep the distribution below $q\phi_{\text{tot}}$ fairly isotropic. Above $q\phi_{\text{tot}}$ and at perpendicular velocities well outside the loss cone, Fokker-Planck calculations^{B12} indicate that side diffusion is not sufficient to keep the distribution cutoff at $q\phi_{\text{tot}}$ at angles $\theta \gg \theta_c$ (see Fig. B-1c). However, electrons diffusing beyond $q\phi_{\text{tot}}$ outside the loss cone are balanced primarily by electrons beyond $q\phi_{\text{tot}}$ slowing down due to dynamical friction, as in a steady-state Maxwellian. Because of the decrease in diffusion rates at higher velocities, those portions of loss-cone surfaces nearest the origin

in velocity space will collect most of the escaping electrons. If one neglects electron loss across lateral loss-cone surfaces (Fig. B-1c) compared to diffusion across the spherical surfaces intersected by the loss cones, Eq. (B.7) can be approximated by

$$J_e = - \left[\langle (\nabla v_{\parallel})^2 \rangle \frac{df}{dv} \right]_{v=v_{\max}} 4\pi v_{\max}^2 \Psi A_{\text{eff}} L_{\text{eff}} \quad (\text{B.19})$$

where the diffusion coefficient is obtained from Eq. (B.7), for $v = v_{\max}$, and Ψ is the fraction of the velocity space surface within the loss cones given by

$$\Psi = 1 - \cos \theta_c = 1 - \sqrt{1 - \frac{1}{R}}. \quad (\text{B.20})$$

For $R = 10$ used in this model, Eq. (B.20) gives $\Psi = 0.054$. The distribution gradient at $v = v_{\max}$ is computed from Eq. (B.11):

$$\frac{df}{dv} \approx -2 \left(\frac{n_o}{\pi^{3/2} \bar{v}^3} \right) \left(\frac{v}{\bar{v}} \right) e^{-v^2/\bar{v}^2}, \quad (\text{B.21})$$

where we have made the approximation that $\alpha^2 \gg 1$ in the normalization constant. Thus, at $v = v_{\max}$ we have

$$\left. \frac{df}{dv} \right|_{v=v_{\max}} \approx - \left(\frac{2n_o \alpha}{\pi^{3/2} \bar{v}^4} \right) e^{-\alpha^2}. \quad (\text{B.22})$$

Substituting into Eq. (B.19), we get

$$J_e = \left(\frac{3 \times 10^{19} n_o \bar{v}^2}{2 v_{\max}^3} \right) \left(\frac{2n_o \alpha e^{-\alpha^2}}{\pi^{3/2} \bar{v}^4} \right) (4\pi v_{\max}^2) \Psi A_{\text{eff}} L_{\text{eff}},$$

or

$$J_e = \frac{1.2 \times 10^{20} n_o^2 e^{-\alpha^2} \Psi A_{\text{eff}} L_{\text{eff}}}{\sqrt{\pi} \bar{v}^3}. \quad (\text{B.23})$$

Equating the electron loss rate to the hot-ion loss rate gives the following equation for α (no cold gas), in terms of kT_e and $n\tau$,

$$J_e = J_h \quad (\text{B.24})$$

or

$$e^{-\alpha^2} = \frac{\sqrt{\pi} \bar{v}^3}{1.2 \times 10^{20} n\tau \Psi}, \quad (\text{B.25})$$

where Eq. (B.23) is used for J_e and Eq. (B.3) for J_h . Taking the value $kT_e = 21.5$ keV and $n\tau = 7.36 \times 10^{13}$ cm⁻³/sec from Fig. B-2 (corresponding to no cold-electron

injection), Eq. (B.24) yields $\alpha^2 = 5.96$, a value somewhat greater than the $\alpha^2 = 5.0$ obtained by Fokker-Planck calculation.^{B2} This is to be expected since electron-loss surface area at $q\phi_{\text{tot}}$ is larger with than without an expander. We shall now show that this larger value of α with an expander is also consistent with the assumed value of kT_e . The heat input rate to the electrons from the slowing down ions is given by

$$P_{i \rightarrow e} = \frac{A_{\text{eff}} L_{\text{eff}} n_o \bar{E}_i}{\tau_{ie}}, \quad (\text{B.26})$$

where τ_{ie} is the energy-exchange time^{B11} for ions on electrons, which can be written

$$\tau_{ie} = 1.4 \times 10^{12} \frac{kT_e^{3/2}}{n_o}, \quad (\text{B.27})$$

where kT_e is in keV and n_o is in cm^{-3} . Since, with an expander, each electron lost carries away no more energy than $q\phi_{\text{tot}}$, the power balance [using Eqs. (B.3), (B.24), (B.26), and (B.27)] becomes

$$\frac{A_{\text{eff}} L_{\text{eff}} n_o^2 \bar{E}_i}{1.4 \times 10^{12} kT_e^{3/2}} = (J_e)(q\phi_{\text{tot}}) = \frac{A_{\text{eff}} L_{\text{eff}} n_o^2 q\phi_{\text{tot}}}{n\tau}, \quad (\text{B.28})$$

or

$$kT_e^{5/2} = \frac{n\tau}{1.4 \times 10^{12}} \frac{\bar{E}_i}{\alpha^2}, \quad (\text{B.29})$$

where kT_e is in keV. Inserting $n\tau$ and $\bar{E}_i$ from Table B-1 and letting $\alpha^2 = 5.96$, Eq. (B.29) gives $kT_e = 22$ keV. To understand why this model, with an expander, should give the same temperature with a higher value of $q\phi_{\text{tot}}$ than obtained in the Fokker-Planck calculation,^{B2} consider a power balance in the form of Eq. (B.28) for the Fokker-Planck model. In this case, one replaces the energy lost per electron $q\phi_{\text{tot}}$ with $(q\phi_{\text{FP}} + \bar{E}_{e1})$, where $\bar{E}_{e1}$ is the mean perpendicular energy of escaping electrons. A reasonable estimate for $\bar{E}_{e1}$ is $\bar{E}_{e1} \approx kT_e$, hence, $q\phi_{\text{FP}} + \bar{E}_{e1} \approx 6 kT_e$, in agreement with the energy lost per electron with an expander.

ELECTRON ENERGY LOSS WITH COLD PLASMA

In considering the injection of cold plasma at the mirrors, we have the problem that the solution for f given by Eq. (B.8) can no longer be used because it develops from a source of electrons at zero energy at the midplane, whereas electrons injected at the mirrors have energy $(q\phi_m)$ at the midplane. This probably will not change f very much at low velocities $v \ll v_{\text{max}}$, in that f is Maxwellian at these velocities. However, at energies $\phi_m \sim \phi_{\text{tot}} \gg kT_e$, electron injection at the mirrors will peak density in velocity space at ϕ_m , giving rise to a steeper distribution gradient at $q\phi_{\text{tot}}$.

In response, the electron loss rate at $q\phi_{\text{tot}}$ will increase to accommodate the additional injected electrons, partly because of a steeper $\frac{df}{dv}$ at v_{max} and partly because of a lower $q\phi_{\text{tot}}$. However, for small cold-plasma currents J_c compared to J_h , we may make the approximation that $q\phi_{\text{tot}}$ is constant and, for large J_c compared to J_h , kT_e is relatively insensitive to $q\phi_{\text{tot}}$ — and it is kT_e that we are after.

Now, considering the limit $\epsilon \rightarrow 1$ (corresponding to $q\phi_m \rightarrow 0$), where electrons injected from both J_c and J_h have zero energy at the midplane, Berk's distribution function [Eq. (B.11)] is again valid. Neglecting J_h compared to J_c in this limit, we can find the minimum value of $q\phi_{\text{tot}}$ by setting J_e from Eq. (B.23) equal to J_c from Eq. (B.5), with $n_c = n_o$. Particle balance, then, is:

$$J_e = J_c,$$

or

$$\frac{1.2 \times 10^{20} n_o^2 e^{-\alpha^2} \Psi A_{\text{eff}} L_{\text{eff}}}{\sqrt{\pi} v^3} = n_o \sqrt{\frac{kT_e}{m_i}} \frac{A_{\text{eff}}}{R},$$

or

$$e^{-\alpha^2} = 9.66 \times 10^{12} \frac{kT_e}{n_o}, \quad (kT_e \text{ in keV}), \quad (\text{B.30})$$

where values of constants other than n_o and kT_e are from Table B-1. Also, we must have energy balance:

$$P_{i \rightarrow e} = (J_c)(q\phi_{\text{tot}}) = J_c \alpha^2 kT_e,$$

or

$$\frac{A_{\text{eff}} L_{\text{eff}} (n_o \bar{E}_i) n_o}{1.4 \times 10^{12} kT_e^{3/2}} = n_o \sqrt{\frac{kT_e}{m_i}} \frac{A_{\text{eff}}}{R} \alpha^2 kT_e, \quad (\text{B.31})$$

or

$$kT_e = \left(\frac{7.945}{\alpha^2} \right)^{1/3} kT_e \text{ in keV}, \quad (\text{B.32})$$

where the value $n_o \bar{E}_i = \beta B_o^2 / 2\mu_o = \text{const.}$ is used for the parameters in Table B-1. Equations (B.30) and (B.32) give two equations for the two unknowns α^2 and kT_e [$n_o(kT_e)$ is given in Fig. B-2]. The solutions, by reiterative substitution, are $\alpha^2 = 3.9$ and $kT_e = 1.26$ keV. The decrease in α^2 , or decrease in $q\phi_{\text{tot}}$ relative to kT_e , is due to the relatively poor cold-ion confinement. The value of α^2 can be expected to decrease monotonically from 5.96 at $J_c = 0$ to 3.9 at $\epsilon \rightarrow 1$, due to steadily decreasing average total ion-confinement time (hot and cold). This slow α^2 decrease also results in Hamilton's calculations,^{B2} with decreasing $n\tau$ due to cold-electron

injection at the midplane. Since kT_e varies slowly with α^2 , according to Eq. (B.32), only small errors in kT_e result from assuming α^2 to be constant. For example, taking $\alpha^2 = 5.96$ in Eq. (B.32) results in $kT_e = 1.1$ keV, a 15% error in kT_e . For smaller ϵ , the error is even less. For our purposes this amount of error is acceptable. Taking $\alpha^2 = 5.96 = \text{const.}$ eliminates the need to calculate J_e explicitly in terms of α , in that conservation of charge requires $J_e = J_c + J_h$ and we have the ion loss rates from Eqs. (B.3) and (B.5) to use in the energy-balance equation for arbitrary J_c/J_h ratios:

$$P_{i \rightarrow e} = (J_c)(q\phi_{\text{ext}}) + (J_h)(q\phi_{\text{tot}}). \quad (\text{B.33})$$

Now that we have fixed $q\phi_{\text{tot}}$ in terms of kT_e , we need to determine the variation of $q\phi_{\text{ext}}$ with ϵ . Since the electron distribution departs from Maxwellian for energies approaching $q\phi_{\text{tot}}$, the Boltzman factor cannot be used to calculate $q\phi_m$ for small values of ϵ at the mirror, where $q\phi_m \lesssim q\phi_{\text{tot}}$. Therefore, we must resort to the cut-off distribution, Eq. (B.11). The limitation of f to the case $J_c = 0$ does not restrict us for potentials below $q\phi_m$, assuming that the presence of cold plasma for $J_c \neq 0$ does not significantly alter the hot-ion density distribution. The density and, therefore, potential profile between points $\pm z_o$, where $n_h(z_o) = n_{\text{ext}}(z_m)$ (see Fig. B-1b), is independent of J_c ; only the point z_o changes with n_{ext} . Assuming potential at $z = \pm z_o$ is nearly the same as that at $z = \pm z_m$, ($kT_i \text{ cold} \approx 0$), we need only calculate

$$n_e(\phi_m) = n_h(z_o) \approx n_{\text{ext}}(z_m). \quad (\text{B.34})$$

Hamilton and Moir^{B13} find that magnetic effects on electron density can be ignored; that is, $n_e(\phi, B) = n_e(\phi)$. This can be seen as a consequence of Liouville's Theorem on the constancy of density in phase space. Considering electrons in the loss cones (Figs. B-1, c and d), the volume of velocity space within the cones expands going from the midplane toward the mirror (due to the increase in θ_c), but the volume of an element in configuration space correspondingly decreases along a flux tube. Only the potential gradient changes the electron density, because the electric field changes the electron's energy.

The distribution given by Eq. (B.11) applies only to electrons at the midplane. By making the transformation

$$\frac{1}{2}mv^2 = q\phi + \frac{1}{2}mu^2, \quad (\text{B.34})$$

or

$$u^2 = v^2 - \frac{2q\phi}{m}, \quad (\text{B.35})$$

and

$$du = \frac{v dv}{\sqrt{v^2 - \frac{2q\phi}{m}}}, \quad (\text{B.36})$$

where u is velocity at potential ϕ , we have:

$$n_e (\phi = \phi_m) = n_{\text{ext}} = \int_{u=0}^{u=u_{\text{max}}} f(u) 4\pi u^2 du,$$

or

$$n_{\text{ext}} = \int_{\sqrt{\frac{2q\phi_m}{m}}}^{\sqrt{\frac{2q\phi_{\text{tot}}}{m}}} f(v) 4\pi \left(v^2 - \frac{2q\phi_m}{m} \right) \left(\frac{v dv}{\sqrt{v^2 - \frac{2q\phi_m}{m}}} \right). \quad (\text{B.37})$$

Substituting for $f(v)$ from Eq. (B.11), we get

$$n_{\text{ext}} = \frac{4\pi n_0}{\pi^{3/2} \bar{v}^3} \int_{v_{\text{min}}}^{v_{\text{max}}} \left[e^{-v^2/\bar{v}^2} - e^{-\alpha^2} \right] \left(\sqrt{v^2 - v_{\text{min}}^2} v dv \right). \quad (\text{B.38})$$

Defining $y = \frac{v}{\bar{v}}$, $\beta = \frac{v_{\text{min}}}{\bar{v}}$, and $v_{\text{min}} = \sqrt{\frac{2q\phi_m}{m}}$, Eq. (B.38) can be written

$$\epsilon = \frac{4}{\sqrt{\pi}} I(\alpha, \beta), \quad (\text{B.39})$$

where

$$I(\alpha, \beta) = \int_{\beta}^{\infty} \left(e^{-y^2} - e^{-\alpha^2} \right) \left(\sqrt{y^2 - \beta^2} \right) y dy. \quad (\text{B.40})$$

The integral I was numerically evaluated for $\alpha^2 = 5.96, 5, 4$, and 2.25 and for several values of β (corresponding to values of ϕ_m/ϕ_{tot} between zero and one). The results are shown in Fig. B-3. Note that $\phi_m/\phi_{\text{tot}} \rightarrow 1$, as $\epsilon \rightarrow 0$, and $\phi_m/\phi_{\text{tot}} \rightarrow 0$, as $\epsilon \rightarrow 1$. The values of ϕ_m/ϕ_{tot} for a given ϵ are rather insensitive to α^2 .

The variation of Q with ϵ can now be determined by choosing a value of ϵ between 4×10^{-5} and 1, using Fig. B-3 to obtain ϕ_m/ϕ_{tot} , and then obtaining ϕ_{ext} from:

$$q\phi_{\text{ext}} = q\phi_{\text{tot}} \left(1 - \frac{\phi_m}{\phi_{\text{tot}}} \right),$$

or

$$q\phi_{\text{ext}} = \alpha^2 kT_e \left(1 - \frac{\phi_m}{\phi_{\text{tot}}} \right). \quad (\text{B.41})$$

Taking $\alpha^2 = 5.96$, this $q\phi_{\text{ext}}$ is substituted into the energy balance equation, Eq. (B.33):

Fig. B-3. Ratio of mirror drop (ϕ_m) to total drop (ϕ_{tot}) as a function of ϵ .

$$\frac{A_{\text{eff}} L_{\text{eff}} (n_o \bar{E}_i) n_o}{1.4 \times 10^{12} kT_e^{3/2}} = n_o (\epsilon - 4 \times 10^{-5}) \left(\sqrt{\frac{kT_e}{m_i}} \right) \left(\frac{A_{\text{eff}}}{R} \right) \alpha^2 kT_e \left(1 - \frac{\phi_m}{\phi_{\text{tot}}} \right)$$

$$+ \frac{A_{\text{eff}} L_{\text{eff}} n_o^2}{n\tau} (\alpha^2 kT_e). \quad (\text{B.42})$$

Equation (B.42) is then solved for kT_e by reiterative substitution, using values for n_o and $n\tau$ from Fig. B-2 for each value of kT_e . The value of Q , then, is given in Fig. B-2 for the value of kT_e that satisfied Eq. (B.42). Results for kT_e and Q are plotted in Fig. 6 of the main text.

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APPENDIX C

CURRENT STABILIZATION OF THE CURVATURE-DRIVEN $m = 1$ MODE

G. Logan

INTRODUCTION

This report presents a short review of the $m = 1$ mode instability of long, thin, high-beta plasma columns in axisymmetric mirror fields, followed by a δW calculation of the magnitude of longitudinal plasma current required to stabilize the curvature-driven instability by the image currents returning in a conducting wall. The parameter regimes explored are those useful for proposed two-component mirror fusion reactors^{C1} and multiple-mirror reactors.^{C2} Finally, the possible excitation of current-driven helical modes and their suppression by alternating the current are discussed.

CURVATURE-DRIVEN INSTABILITY OF THE EQUILIBRIUM

Typical parameters assumed for the equilibrium are shown in Fig. C-1. Only half of distance L ($L = 12$ m) between two mirrors is considered due to symmetry. The effect of end regions outside the mirrors (possibly stabilizing) is neglected by doing the calculation for an infinitely long periodic system. Plasma pressure P is assumed to be constant with z and $r < R_p$, with currents flowing only in the surface of a sharp boundary. For real systems with a diffuse radial pressure gradient, the radially averaged beta is appropriate to use in the sharp boundary model. The plasma radius R_{po} and beta β_o are specified at the midplane, with $R_p(z)$ and $\beta(z)$ determined elsewhere by zero-order pressure balance and flux conservation:

$$R_p(z) = R_{po} \left[(1 - \beta_o) / \left((B_e^2(z)/B_{eo}^2) - \beta_o \right) \right]^{1/4}, \quad (C.1)$$

$$\beta(z) = P / (B_e^2(z)/2\mu_o) = \beta_o (B_e(z)/B_{eo})^2, \quad (C.2)$$

where $B_e(z)$ is the external field. The external mirror field is assumed to be created by a field of current rings (radius $R_m \gg R_p$, spaced a distance $L \gg R_m$ apart) superimposed on the uniform field of a current solenoid. The ratio ϵ of plasma radius R_p to the characteristic distance L , over which the equilibrium field varies, is small compared to unity so that the zero-order field along the plasma surface is not appreciably different from the vacuum field in the absence of plasma. Perturbations of plasma equilibrium are described in terms of the amplitude of the normal displacement component of the plasma surface

$$\xi(z) \equiv \xi_r(z), \quad (C.3)$$

Fig. C-1. Equilibrium (typical).

where the normal component is taken to vary as $\xi_r(z) \exp(im\theta)$ and m is the azimuthal mode number. The worst displacements are assumed to be incompressible ($\nabla \cdot \vec{\xi} = 0$) to minimize increases in plasma internal energy. It will be shown that the most unstable displacements have parallel wavelengths on the order or of longer than L , resulting in almost purely transverse motion ($\xi_z \approx 0$).

Although the plasma betas considered here are sufficient to reduce the internal field significantly, an evaluation of stability properties based on low-beta theory is still instructive. Consider the plasma surface to have some average radius of unfavorable curvature $R_c \sim L^2/2R_p$ due to the mirror field, this results in an outward acceleration $g \sim 2\bar{V}_i^2/R_c$ of the ions, where $\bar{V}_i$ is ion thermal velocity. This outward acceleration, in turn, leads to Rayleigh-Taylor growth rates ($k_{\parallel} = 0$)

$$\gamma (\text{Low } \beta) \approx \sqrt{gm}/R_p = 2\sqrt{m}(\bar{V}_i/L), \quad (\text{C.4})$$

which increase with mode number m . Equation (C.4) applies when the ion Larmor radius ρ_i is sufficiently small compared to the azimuthal wavelength $2\pi R_p/m$. Rosenbluth *et al.*^{C3} found that, for a perpendicular density-gradient scale length $\Delta \gg \rho_i$, all mode numbers

$$m > 5(R_p/\rho_i) \sqrt{\Delta/R_c} \quad (\text{C.5})$$

are stable due to finite Larmor radius (FLR). Assuming $\Delta \sim R_p$ in a real system, the parameters assumed for the system in Fig. C-1 satisfy inequality [Eq. (C.5)] for $m > 1$ by more than an order of magnitude. Equation (C.5), based on a slab model, breaks down for the $m = 1$ mode of a cylindrical plasma, which is not expected to be FLR stabilized.^{C3} As for the effects of finite beta on the unstable mode numbers, Pearlstein and Krall^{C4} have shown that the low-beta FLR-stabilization criterion [Eq. (C.5)] is pessimistic; that is, stability is improved with β .

Several experiments^{C5-C7} with plasma parameters similar to those assumed in the model have confirmed the prediction of long wavelength $m = 1$ instability with the absence of higher m modes. Therefore, only the $m = 1$ mode is considered in the following calculations.

Haas and Wesson^{C8} have used the hydromagnetic-energy principle^{C9} to investigate the stability of high-beta mirror equilibria (such as those in Fig. C-1) for plasmas carrying no longitudinal current. Their results (Eq. 3.13 of Ref. C8) for the potential energy change δW_m of an $m = 1$ displacement $\xi(z)$ of a sharp-boundary $\beta < 1$ plasma enclosed by conducting walls of radius R_w can be written

$$\begin{aligned} \delta W_m = & \frac{1}{2\mu_0} \int_0^{L/2} \xi^2 B_e^2 \beta \frac{d^2 R_p}{dz^2} R_p dz + \frac{1}{2\mu_0} \int_0^{L/2} B_e^2 G \left[\frac{d}{dz} (R_p \xi) - 2\xi \frac{dR_p}{dz} (1 - \beta) \right]^2 dz \\ & + \frac{1}{2\mu_0} \int_0^{L/2} B_e^2 (1 - \beta) \left[R_p \frac{d\xi}{dz} - \xi \frac{dR_p}{dz} \right]^2 dz, \quad (\text{C.6}) \end{aligned}$$

where

$$G = (1 + (R_p/R_w)^2) / (1 - (R_p/R_w)^2).$$

The first term can be identified as the change in plasma-column potential energy due to field curvature (the so-called "surface energy" term of Bernstein *et al.*^{C9}). The second term is the change in vacuum field energy between the plasma and the conducting wall (the "vacuum energy"), and the third term is due to changes in the plasma internal field energy (the "fluid energy"). No contribution to plasma energy change appears in the third term because $\nabla \cdot \vec{\xi}$ is equal to zero. The second and third terms give energy change due to field-line bending external and internal to the plasma; these contributions are always positive and stabilizing. The first term can be negative due to unfavorable curvature ($d^2 R_p / dz^2 < 0$). Instability arises when this contribution exceeds the line-bending contributions. For the limit $\beta \rightarrow 1$, consider a displacement $\xi = \text{constant}$ corresponding to a rigid, transverse motion of the plasma column. The second derivative of the plasma radius in the first integral of Eq. (C.6) is related to the normal component of field pressure gradient at the plasma surface $r = R_p(z)$:

$$\frac{d^2 R_p}{dz^2} = \frac{1}{R_c(z)} = \frac{\vec{n} \cdot \nabla (B_e^2 / 2\mu_0)}{2B_e^2 / 2\mu_0}. \quad (C.7)$$

Thus, in the midplane region of unfavorable curvature, a radial displacement ξ results in a net unbalanced force F per unit length in the direction of ξ , assuming constant plasma pressure across the plasma column. In the mirror region, where $d^2 R_p / dz^2 > 0$, the force is restoring. Therefore, energy change ($\delta W = -\int F \cdot \xi dz$ due to $\xi = \text{const.}$) in a mirror field is positive in the mirror regions and negative in the midplane regions. However, since plasma surface area ($\sim R_p$) is smaller in the mirror than in the midplane, the negative contribution of the regions of unfavorable curvature always dominate in the first term of Eq. (C.6).

The presence of conducting walls has a stabilizing effect, which appears in the factor G of the second term in Eq. (C.6), but this factor is appreciable only when $R_w = R_p$, which is usually not practical. Since fusion α - particles make 5-cm excursions from the plasma surface, a wall radius $R_w = 5 R_p$ is taken in this model, for which $G \approx 1$. Although the first term of Eq. (C.6) goes to zero as $\beta \rightarrow 0$ while the line-bending terms generally do not, there can be displacements satisfying

$$\xi \frac{dR_p}{dz} = R_p \frac{d\xi}{dz},$$

or

$$\xi(z) = \xi_0 \exp \int_{z_0}^z \frac{1}{R_p} \frac{dR_p}{dz} dz, \quad (C.8)$$

which make the line-bending terms go to zero. Thus, Eq. (C.8) gives the most unstable $\xi(z)$ for low beta.

At high beta, the most unstable $\xi(z)$ is found by maximizing the dominant first term in Eq. (C.6) rather than by minimizing the line-bending terms:

$$\xi(z) = \xi_0 \exp(-AB_e^2 \beta R_p / R_c), \quad (C.9)$$

where A is an adjustable constant. As a check, a trial function satisfying $R_p \xi = \text{const.}$ (which makes the vacuum energy term zero as $\beta \rightarrow 1$) is found to make $\delta W_m > 0$, while Eq. (C.9) gave the most negative values of δW_m . Results for $\xi(z)$, which minimize δW_m for the model of Fig. C-1, are shown in Fig. C-2, along with the radius of curvature. Note that the most unstable ξ for the high beta considered ($\beta_0 = 0.9$) is nearly constant with z, with a slight peak in the region of maximum unfavorable curvature.

For an infinite periodic system with an equilibrium plasma radius given by $R_p = R_{av} (1 + \delta \cos kz)$, Haas and Wesson^{C8} find a maximum $m = 1$ growth rate

$$\gamma = \frac{1}{\sqrt{2}} k \delta \bar{V}_A \left\{ \left[\frac{\beta(3 - 2\beta)}{(1 - \beta + G)} \right] [1 + (1 - 2\beta)G] \right\}^{\frac{1}{2}}, \quad (C.10)$$

where $\bar{V}_A$ is the Alfvén velocity $\sqrt{B_e^2 / \mu_0 \rho_m}$ and the mass density ρ_m , B_e , β and G are averaged over z. With $G = 1$, an average beta $\bar{\beta} = 0.5$, $\bar{V}_A \approx 1.5 \bar{V}_i$, $\delta \approx 0.5$, and $k \approx 2\pi/L$, Eq. (C.10) gives

$$\gamma (\text{high } \beta) \approx 3(\bar{V}_i / L), \quad (C.11)$$

which is nearly the same as the low beta-growth rate for $m = 1$ given by Eq. (C.4).

Haas and Wesson^{C8} also investigated the stability of a plasma with a diffuse radial pressure profile, for peak betas on the axis $\beta_A > 0.75$ everywhere, and $\xi(z) = \text{const.}$ For the same value of β_A , they found diffuse profiles slightly more unstable than for sharp boundaries, which they attribute to lower radially averaged $\langle \beta \rangle_r$ for the diffuse plasma. Therefore, use of betas in a sharp-boundary model equal to the $\langle \beta \rangle_r$ rather than β_A of a diffuse profile should give adequate results for δW_m .

IMAGE CURRENT STABILIZATION OF THE $m = 1$ MODE

Now, the expression for the energy-change δW_m [Eq. (C.6)] is extended to include the effect of a longitudinal current I_z flowing down the plasma and returning via a conducting wall in a closed circuit. The work done by a displacement of the form given by Eq. (C.3) against the repulsive force between the plasma and image current appears as an additional stabilizing term in Eq. (C.6). Only lateral displacements $\xi_r(z) \cos \theta$ are considered, as the object is to find the minimum current required to stabilize the most unstable curvature-driven modes. Helical displacements of the form $\xi_r \cos(\theta - hz)$ are most unstable in the presence of a dc current, which causes

Fig. C-2. Curvature-driven unstable motion.

the field lines to twist. However, for these current-driven modes, there is an additional means of stabilization available – namely, alternating the current (to be discussed later).

The restoring force F per unit length on a perfectly conducting plasma initially carrying a current I_z and making a lateral $m = 1$ displacement $\xi(z)$ is given by

$$F(z) = -\frac{\mu_0 I_z^2}{2\pi} \left(\frac{\xi}{R_w^2 - \xi^2} \right), \quad (C.12)$$

where

$$I_z' = \frac{I_z}{(L/2)} \int_0^{L/2} \frac{\ln(R_w/R_p) dz}{\ln \left(\frac{R_w^2 - \xi^2 - R_p \xi}{R_w R_p} \right)} \quad (C.13)$$

is the current after making displacement $\xi(z)$ toward a conducting wall of radius R_w carrying $-I_z'$. For $\xi > 0$, I_z' is greater than I_z , due to conservation of azimuthal flux between the plasma and wall and the fact that the inductance of the circuit decreases. As $\xi \rightarrow (R_w - R_p)$, the restoring force for any current becomes arbitrarily large, and therefore the instability amplitude will be limited within the conducting wall radius for nonzero currents. To prevent motion completely, however, requires a minimum current. In the limit of small displacements $\xi \ll (R_w - R_p)$, $I_z' \rightarrow I_z$, and F reduces to

$$F(z) = -\frac{\mu_0 I_z^2 \xi}{2\pi R_w^2}. \quad (C.14)$$

Since $F(z)$ is linear with ξ in this limit, the corresponding work done by a small displacement is given by

$$\delta W_I = -\frac{1}{2} \int_0^{L/2} F \xi dz. \quad (C.15)$$

The total energy change due to a displacement $\xi(z)$ given by Eq. (C.3) for a current-carrying plasma is given by adding δW_I [given by Eq. (C.15)] to δW_m [given by Eq. (E.6)]:

$$\delta W = \delta W_m + \delta W_I. \quad (C.16)$$

Equation (C.16), normalized to $\int_0^{L/2} \xi^2 dz$, is numerically minimized with respect to parameter A in Eq. (C.3), for increasing currents I_z . The value of I_z for which the stationary, or minimum, value of the normalized δW just becomes positive is the minimum current to stabilize the $m = 1$ curvature mode. The results of such a minimization for the model parameters in Fig. C-1 are shown in Fig. C-3. Note that the

Fig. C-3. Normalized energy δW as a function of variation parameter A for $I_z = 0, 3, 4, 5,$ and 6 kA .

minima of δW occur at the same ξ -variation parameter A for different currents. This $A \approx 45$ corresponds to $\xi(z)$, as given in Fig. C-2. Note also that the plasma is stable with no current for $A \rightarrow 0$, corresponding to a rigid displacement $\xi = \text{constant}$. Figure C-3 shows that a current slightly above 4.5 kA is required for the stability of any parameter A .

The dependence of the minimum current for stabilization on the vacuum mirror ratio M , the ratio α of the mirror coil radius R_m to the plasma radius R_{po} , the ratio η of wall radius R_w to plasma radius R_{po} , and on β_o was investigated by varying B_{max} , R_m , R_p , and plasma pressure P , respectively, about the nominal values assumed in Fig. C-1. Results are presented in Table C-1 (a, b, c, and d). Table C-1 (a) shows that $I_z \rightarrow 0$ and $M \rightarrow 1$, corresponding to the case of a uniform field. I_z increases slowly with $M > 2$ due to decreasing effects on field curvature. Table C-1 (b) indicates that localized mirrors (small R_m) are more destabilizing than gradual mirror fields for the same M . Increasing L for fixed R_m increases δW_I relative to δW_m , and, in the limit $L \gg R_m$, $I_z \sim 1/\sqrt{L}$. Table C-1 (c) illustrates the stringent requirement $(R_w - R_{po}) \approx 0$ for stability without current by placing conducting walls very close to the plasma.^{C8} For large η , reducing R_p while maintaining R_w reduces I_z proportionately. In Table C-1 (d), Eq. (C.8) is used for $\xi(z)$ for the lower betas 0.1, 0.2 and 0.3, and Eq. (C.9) for the higher betas. For low betas, I_z increases as $\sqrt{\beta}$ as expected, but for betas > 0.9 , I_z decreases slightly due to increasing line-bending energy with beta. To summarize, for $R_{po} \ll R_m \ll L$, $R_w \gg R_{po}$, $\beta_o \lesssim 0.9$, the stabilizing current scales approximately as

$$I_z \sim B_{eo} (M - 1)^{0.25} (\beta_o / \alpha L)^{0.5} / \eta. \quad (\text{C.17})$$

Having determined the required stabilizing currents I_z to be between 2 and 10 kA depending on the reactor parameters M , α , η and β_o , questions arise as to whether such currents will excite turbulence in the plasma and how such currents can be fed to the plasma without incurring large heat loss by electron conduction. Assuming plasma temperature to be $KT_i = kT_e = 5 \text{ keV}$, β_o to be 0.9, and B_{eo} to be 100 kG, results in a plasma density of $n \approx 2 \times 10^{16} \text{ cm}^{-3}$. The electron-drift velocity V_{ed} associated with a 5-kA current carried within a minimum skin depth c/ω_{pe} around a 3-cm-diam plasma is $V_{ed} \approx 3 \times 10^7 \text{ cm/sec}$. Since this drift velocity is less than the ion sound speed, no excitation of ion acoustic turbulence can be expected. Also, since the component currents associated with normal ion and electron particle loss through ambipolar sheaths at the ends are an order of magnitude larger than I_z , slightly biasing the end walls or electrode potentials asymmetrically can produce a net current of I_z out of electrons and ions leaving the plasma rather than by having the electrodes emitting thermionically. Thus, the heat insulation provided by ambipolar sheaths can be maintained.

Table C-1. Minimum stabilizing currents.

(a)		(b)		(c)		(d)	
M	I_z (kA)	α	I_z (kA)	η	I_z (kA)	β_o	I_z (kA)
1.0	0.0	16.7	10.0	7.5	3.4	0.1	1.8
1.1	1.6	33.3	6.6	5.0	4.6	0.2	2.4
1.3	3.2	66.7	4.6	2.5	9.3	0.3	3.0
1.7	4.1	133	2.8	1.5	14.0	0.6	3.4
2.0	4.6	267	2.4	1.2	13.0	0.8	4.2
2.3	5.2			1.1	6.0	0.9	4.6
3.0	5.7			1.05	0.0	0.95	4.2
4.0	6.0					0.975	3.6
$M \equiv \frac{B_e(\text{max})}{B_{eo}}$		$\alpha \equiv \frac{R_m}{R_{po}}$		$\eta \equiv \frac{R_w}{R_{po}}$		$\beta_o \equiv \frac{P}{(B_{eo}^2/2\mu_o)}$	
$B_{eo} = 120 \text{ kG}$		$R_{po} = 1.5 \text{ cm}$		$R_w = 7.5 \text{ cm}$		$B_{eo} = 120 \text{ kG}$	
$\alpha = 66.7$		$M = 2$		$M = 2$		$M = 2$	
$\eta = 5$		$\eta = 5$		$\alpha = 66.7$		$\alpha = 66.7$	
$\beta_o = 0.9$		$\beta_o = 0.9$		$\beta_o = 0.9$		$\eta = 5$	

CURRENT-DRIVEN KINK MODES AND THEIR DYNAMIC STABILIZATION

The growth rate for helical $m = 1$ displacements of the form $\xi_r \cos(\theta - h_z)$, (where $\xi_r = \text{const.}$) excited by the current I_z , will now be estimated. In the limit $h R_w \ll 1$ and $\xi_r \ll (R_w - R_p)$, Ribe and Reisenfeld^{C10} derive the transverse helical force per unit length of a current-carrying plasma column [Eq. (16) of Ref. C10], which includes the helical wall-image force given by Eq. (C.14):

$$F(\text{helical}) = - \left[\frac{\mu_o I_z^2}{2\pi R_w^2} + (2 - \beta)(h R_p) \left(\frac{\pi}{\mu_o} \right) B_e^2 - h B_e I_z \right] \xi_r. \quad (\text{C.18})$$

The third term in Eq. (C.18) is a destabilizing term that can be interpreted as a $\vec{J} \times \vec{B}_e$ force in the direction of ξ_r . This force increases with the pitch angle $\psi \propto h$ between the current column and the z-directed mirror field B_e . The second term represents the stabilizing effect of field-line bending, since the helical displacement must warp the zero-order mirror field B_e lines. The field-line bending scales as h^2 , exceeding the destabilizing term for sufficiently large h . Setting the derivative of Eq. (C.18), with respect to h , to zero results in the most unstable perturbation wave number:

$$h = \frac{g}{(2 - \beta)}, \quad (C.19)$$

where

$$g = \frac{\mu_o I_z^2}{2\pi R_p^2 B_e} \quad (C.20)$$

is the field-line pitch-wave number ($B_\theta/(R_p B_e)$). Substituting Eq. (C.19) and Eq. (C.20) into Eq. (C.18)

$$F(\text{helical}) \approx - \left[\frac{\mu_o I_z^2}{2\pi R_w^2} + \frac{\mu_o I_z^2}{4\pi R_p^2 (2 - \beta)} - \frac{\mu_o I_z^2}{2\pi R_p^2 (2 - \beta)} \right] \xi_r$$

or

$$F(\text{helical}) \approx - \left[\frac{\mu_o I_z^2}{2\pi R_w^2} - \frac{\mu_o I_z^2}{4\pi R_p^2 (2 - \beta)} \right] \xi_r. \quad (C.21)$$

Since $(R_w/R_p)^2/4 \approx 6$ with our model parameters, the destabilizing force exceeds the wall-image force for small ξ_r , and the kink mode grows.

Using Rayleigh's principle,^{C11} kink-mode growth rate is determined by

$$\omega^2 = - \frac{\frac{1}{2} \int_0^{L/2} F(\text{helical}) \xi_r dz}{\frac{1}{2} \int_0^{L/2} \rho_m \xi_r^2 \pi R_p^2 dz}. \quad (C.22)$$

Using Eq. (C.21) and neglecting the wall-image force, letting $\xi_r = \text{const}$, and mass density $\rho_m = \text{const}$, one obtains

$$\gamma(\text{helical}) \approx \left[\frac{\mu_o I_z^2}{(2\pi)^2 \rho_m (2 - \beta) R_{av}^4} \right]^{\frac{1}{2}}, \quad (C.23)$$

where $\bar{\beta}$ and R_{av} are appropriate average beta and plasma radius, respectively. Using Eq. (C.20), Eq. (C.23) can be written

$$\gamma(\text{helical}) \approx g \bar{V}_A / \sqrt{2 - \bar{\beta}}, \quad (C.24)$$

where $\bar{V}_A$ is an average Alfvén velocity $\sqrt{B_e^2/\mu_o \rho_m}$. With $\bar{\beta} \approx 0.5$, and $\bar{V}_A \approx 1.5 \bar{V}_i$ Eq. (C.24) becomes

$$\gamma(\text{helical}) \approx 9 \bar{V}_i / \lambda_B, \quad (C.25)$$

where $\lambda_B = 2\pi/g$ is the field-line-pitch wavelength. With $I_z = 5$ kA, $R_{av} \approx 1.2$ cm, $\bar{B}_e \approx 150$ kG, and $\lambda_B \approx 12$ m. Since $\lambda_B \approx L$ in this case, the current-driven kink-mode growth rate is about 3 times the curvature-driven growth rate in the absence of current, as given by Eq. (C.10). Thus, by passing current through the plasma one replaces the curvature-driven instability with a faster current-driven instability. However, the amplitude of the kink mode can be suppressed by alternating the current in a time less than the kink-mode growth rate. The required frequency $f \gtrsim \gamma(\text{helical})/2\pi$ is of the order of 100 kHz for our parameters, easily within the range of power oscillators. Considering the plasma column within a conducting wall as a coaxial transmission line, one finds a wave velocity of 10^8 m/sec, assuming that most of the annular space between the current-carrying plasma and the conducting wall is occupied by a relatively high dielectric constant α -particle plasma. At 100 kHz , the current-amplitude wavelength is $\lambda_c \approx 10^3$ m, much longer than the L s considered. Since $I_z = \text{const}$ with z , impedance of the system is low. As an example of a simple, pulsed circuit that might be employed, a 10-kV, 1- μ F capacitor discharged into the 12-m, 3- μ H-load system shown in Fig. C-1 supplies the required 100-kHz , 5-kA current.

Finally, two effects not considered here may reduce the magnitude or frequency of oscillating current required for stabilization. One of these effects, investigated by Ribe and Riesenfeld,^{C10} is the reduction of current required to stabilize the forces of unfavorable curvature owing to the oscillating nature of the current itself. Ribe and Riesenfeld have found that dynamic stabilization may occur for oscillating currents roughly R_w/R_p times smaller than the required dc image currents. Thus, an oscillating current as low as 1 kA may be sufficient for our model. The other potentially important effect is the damping of kink-mode amplitudes at the end electrodes and additional favorable curvature regions outside the end mirrors providing a current threshold for kink modes. The lack of kink modes in the DCX arc,^{C12} which carried 300 A dc in a 10-kG mirror field, may have been due to end effects. In conclusion, the low image currents ($\lesssim 5$ kA) required to stabilize long, thin plasma columns^{C1, C2} should provide a particularly simple method of confining such plasmas in axisymmetric mirror fields.

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APPENDIX D

COMPARISON OF IMAGE-FORCE STABILIZATION OF SCYLLAC AND A TWO-COMPONENT MIRROR (TCM)

Table D-1 summarizes the relevant parameters for current stabilization of Scyllac^{D1} and a TCM reactor.^{D2} The greatest parameter differences are in the scale length for modulating the longitudinal, dc, continuing field (length L) and for the ratio of mirror coil to plasma radius (parameter α), which makes unfavorable curvature much weaker for the TCM than for the necessary equilibrium field of the scyllac. The result is that an order of magnitude more current is required for scyllac, with two orders more oscillating field energy to be provided by the stabilizing power supplies. Moreover, since the helical growth rate γ_{helical} scales with I_z , the required frequency ω is greater than mHz for scyllac compared to 100 kHz for the TCM. For these reasons, dynamic, image-current stabilization may be prohibitive (in terms of cost) for scyllac, but may be quite reasonable for the TCM.

Table D-1. Stabilization parameters.

Parameter	Scyllac ^a	TCM ^b
L	15 cm	12 m
R _p	0.7 cm	1.5
R _w	5 cm	7.5
R _m	4 cm	100 m
B _{zo}	90 kG	120 kG
M	1.56	2
(M - 1) ^{0.25}	0.87	1
β_o	0.5	0.9
α	5.7	67
η	7	5
I _z	≈55 kA	5 kA

^aRef. D1.

^bRef. D2.

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APPENDIX E

STABILIZATION OF LOW M-NUMBER INTERCHANGE MODES IN SIMPLE MIRRORS BY AXIALLY INJECTED ELECTRONS

D. E. Baldwin and G. Rowlands

There have been a number of investigations of line tying as a means of stabilizing interchange modes in simple mirror machines. Generally, however, there has been a lack of agreement between theory and experiment, sometimes differing by orders of magnitude in the plasma density external to the trap necessary for line-tying stabilization. A common result of a number of theories is that the greatest contribution to the impedance seen by axial electron flow occurs at the electron sheath that forms where plasma-loaded magnetic lines enter a conductor. This sheath can be reduced by injecting an electron current comparable to the natural escaping electron current. However, doing so also removes the thermal insulation provided by the sheath, which would be vital to any energy-economic mirror reactor. We wish to point out that earlier investigators considered only the effect of current injection on the dc sheath and the concomitant effect on the escape of otherwise trapped electrons. They did not consider the direct electrical effect of the injected electrons on the interchange mode itself. Including this direct effect, we find that a relatively weak injected current is sufficient to stabilize an $m = 1$ interchange mode in a simple mirror. Whether this effect ultimately could be considered a means of stabilizing a simple mirror depends, of course, on the results of further investigation. Here, we merely wish to demonstrate how this stabilization comes about and, in doing so, stimulate discussion on its potential utility.

Because of the thermal insulation of the sheath, we will take as a requirement that any injected current be sufficiently weak compared to $n_w v_{Te} |e|$ that the magnitude of the sheath not be significantly affected (n_w is plasma density near the wall just outside the sheath and v_{Te} is electron thermal velocity). Under normal circumstances, ambipolar potential increases monotonically from the wall to the midplane. Thus, the injected electrons are accelerated to the midplane where, for low injection energy compared to the ambipolar potential ϕ , they have an axial velocity $v_{||} \propto \sqrt{\phi}$ ($\phi = 0$ at the wall). At the midplane, the weak-current requirement translates into a requirement that the density of injected electrons be small compared with that of the ions, or that of the trapped electrons. (Since injected electrons leave a symmetric plasma with their original velocity, we refer to them as "passing electrons.")

To investigate the effect of passing electrons, we consider an interchange instability in a three-component plasma: ions, trapped electrons, and passing electrons. It is necessary to treat the passing electrons differently from the trapped particles. Because of the low frequencies involved, we can treat the trapped particles in the drift kinetic description, i. e., a particle of species σ has only motion parallel to $\underline{b} \equiv \underline{B}/B$ and a perpendicular drift

$$\underline{v}_{D\sigma} = \frac{1}{\Omega_\sigma} \underline{b} \times \left[\left(\frac{v_\perp^2 + 2v_\parallel^2}{2B} \right) \nabla B \frac{e_\sigma}{m_\sigma} \nabla \phi + \frac{e_\sigma}{m_c \Omega_\sigma} \underline{b} \times \nabla \frac{\partial \phi}{\partial t} \right], \quad (\text{E.1})$$

where Ω_σ is the (signed) cyclotron frequency. We neglect the velocity dependence of the ∇B drift terms by replacing $v_\perp^2$ and $v_\parallel^2$ with appropriate thermal velocities. This approximation is valid as long as there is not a resonance between a significant number of drifting particles and the phase velocity of the wave. As a model, we treat an otherwise uniform plasma with density and B variation in x only. More properly, this model stems from a bounce average of the drift kinetic equation, in which the perturbed ϕ is taken as constant over the extent of the bounce of the trapped particles and the density variable becomes the number of particles on a flux tube — thus, motion parallel to $\underline{b}$ drops out. Introducing a perturbation of the form $\exp i(k_x x + k_y y - \omega t)$ and neglecting the electron ∇B drift and dielectric contributions, the perturbed charge density for trapped electrons and ions becomes

$$4\pi\rho_t = -k_{Dt}^2 \frac{\omega_{*t}}{\omega} \phi \quad (\text{E.2})$$

$$4\pi\rho_i = -\frac{\omega_{pi}^2}{\Omega_i^2} k^2 \phi - k_{Di}^2 \frac{\omega_{*i}}{(\omega - \omega_{Di})} \phi, \quad (\text{E.3})$$

where

$$\omega_{Di} \equiv \frac{k}{\Omega_i} \cdot \underline{b} \times \frac{(v_\perp^2 + 2v_\parallel^2)}{2B} \nabla B \quad (\text{E.4})$$

$$\rightarrow \frac{3}{2} k_y \frac{v_{Ti}}{\Omega_i} e_{\underline{x}} \cdot \nabla \ln B,$$

$$\omega_{*\sigma} \equiv v_{T\sigma}^2 \frac{k}{\Omega_\sigma} \cdot \underline{b} \times \nabla \ln \omega_{p\sigma}^2 \rightarrow k_y \frac{v_{T\sigma}^2}{\Omega_\sigma} e_{\underline{x}} \cdot \nabla \ln \omega_{p\sigma}^2, \quad (\text{E.5})$$

and

$$k_{D\sigma}^2 \equiv \omega_{p\sigma}^2 / v_{T\sigma}^2.$$

The subscript t refers to trapped electrons; the subscript p to passing electrons.

The response of passing electrons to a perturbing field is quite different. Trapped electrons and ions move in regions of no axial dependence of ϕ , hence, a change in ϕ magnitude does not give rise to a change in particle energy due to parallel motion. However, the mode cannot be flute all the way to the wall; all of the axial variation required to satisfy the $\phi = 0$ boundary condition will take place in the sheath near the wall. Thus, passing electrons drawn into the positive ambipolar potential are accelerated by the instantaneous value of the perturbed potential. In fact, because

of the short transit time of the passing electrons compared to the MHD growth time, they see an effectively static perturbed potential. Thus, the number of such passing electrons on a flux tube varies as the time of transit. The linearized charge density resulting from a perturbed ϕ is

$$4\pi\rho_p = -\frac{m_e \omega_{pp}^2}{|e|\phi} \phi, \quad (\text{E.6})$$

where ω_{pp} is the plasma frequency of passing electrons.

Notice that the sign of the perturbed passing-electron charge density is opposite to the familiar expression for the change in charge density by Boltzman distributed electrons in a positive perturbed potential ϕ . Because the assumptions leading to Eq. (E.6) are equivalent to a constant current source, an increase in $\phi \rightarrow \phi + \phi$ physically results in higher parallel velocity and, therefore, a lower density of passing electrons (i. e., a positive change in charge density). It is this passing-electron behavior that leads to the new physical effects described here.

Introducing Eq. E.1 through E.3 into Poisson's equation gives the dispersion relation

$$k^2 \left(1 + \frac{\omega_{pi}^2}{\Omega_i^2} \right) = -k_{Di}^2 \frac{\omega_{*i}}{\omega - \omega_{Di}} - k_{Dt}^2 \frac{\omega_{*t}}{\omega} + \frac{m_e \omega_{pp}^2}{|e|\phi}. \quad (\text{E.7})$$

Introducing $\epsilon = \omega_{pi}^2/\Omega_i^2$ and $\alpha = n_p/n_i$ and using $\omega_{*t} = -\omega_{*i} T_e/T_i$ and neutrality condition $n_i = n_t + n_p$, Eq. (E.7) becomes

$$k^2 a_i^2 \left(1 + \frac{1}{\epsilon_i} \right) = -\frac{\omega_{*t}}{\omega - \omega_{Di}} + (1 - \alpha) \frac{\omega_{*i}}{\omega} + \alpha \frac{T_i}{|e|\phi}, \quad (\text{E.8})$$

with the solution

$$\omega = \frac{1}{2 \left(k^2 a_i^2 \left(1 + \epsilon_i^{-1} \right) - \alpha \frac{T_i}{|e|\phi} \right)} \left\{ \omega_{Di} \left(k^2 a_i^2 \left(1 + \epsilon_i^{-1} \right) - \alpha \frac{T_i}{|e|\phi} \right) - \alpha \omega_{*i} \right. \\ \left. \pm \left[\left(\omega_{Di} \left(k^2 a_i^2 \left(1 + \epsilon_i^{-1} \right) - \alpha \frac{T_i}{|e|\phi} \right) - \alpha \omega_{*i} \right)^2 \right. \right. \\ \left. \left. - 4 \omega_{Di} \omega_{*i} (1 - \alpha) \left(k^2 a_i^2 \left(1 + \epsilon_i^{-1} \right) - \alpha \frac{T_i}{|e|\phi} \right) \right]^{1/2} \right\}. \quad (\text{E.9})$$

The effects of passing electrons are reflected in two distinct ways: (a) the final term of Eq. (E.8) results from the direct contribution of passing electron to Poisson's equation, and (b) the α in the second term of Eq. (E.8) arises from the reduction of trapped electrons to maintain equilibrium charge neutrality. These effects can be distinguished in Eq. (E.9) by whether or not α is multiplied by T_i/ϕ . From Eq. (E.9), the first effect gives a sufficient condition for stability in a simple mirror, where

$\omega_{Di} \omega_{*i} > 0$, namely

$$\alpha > \frac{|e|\Phi}{T_i} k^2 a_i^2 (1 + \epsilon_i^{-1}). \quad (\text{E.10})$$

To apply this result to cylindrical geometry we take $k \rightarrow m/R_p$, where m is the azimuthal mode number and R_p the plasma radius. Then, Eq. (E.9) gives the possibility of stabilizing low m values. For high m values, finite Larmor radius (FLR) effects have to be invoked. Under reactor conditions, the right side of Eq. (E.10) might be 10^{-3} .

The mechanism by which passing electrons bring about this dramatic effect can be better understood by rewriting Poisson's equation as

$$\nabla_{\perp}^2 \left(1 + \frac{\omega_{pi}^2}{\Omega_i^2} \right) \nabla \phi + \frac{m_e \omega_{pe}^2}{|e|\Phi} \phi = -4\pi(\rho_t + \rho_i) + \nabla \cdot \frac{\omega_{pi}^2}{\Omega_i^2} \nabla \phi.$$

In this form, contributions to the perturbed charge density from density- and magnetic-gradient drifts are placed on the right-hand side and the trapped-ion dielectric and passing-electron contributions are on the left. If we consider a sinusoidal perturbed charge density, the resulting perturbed potential will be reversed in sign, if the second term on the left is greater than the first. In fact, this simple condition leads to Eq. (E.10). Similarly, electron fields resulting from a given charge separation of trapped particles will be reversed. For example, in the Rosenbluth-Longmire picture of the simple interchange, the electric field resulting from the ∇B drift of particles in a perturbation tends to increase the perturbation. However, passing electrons will reverse the sign of the electric field and the associated drift, provided the wavelengths of the perturbation are long enough and can lead to stability. This effect of passing electrons changes the Poisson equation from elliptic to hyperbolic.

The second effect of passing electrons, due to a reduction in the number of trapped electrons, is related to FLR effects in conventional theory.

In fact, were we to include FLR terms, Eq. (E.9) would become

$$\omega = \frac{1}{2 \left[k^2 a_i^2 (1 + \epsilon_i^{-1}) - \alpha \frac{T_i}{|e|\Phi} \right]} \left\{ \omega_{Di} \left[k^2 a_i^2 (1 + \epsilon_i^{-1}) - \alpha \frac{T_i}{|e|\Phi} \right] - (\alpha - k^2 a_i^2) \omega_{*i} \right. \\ \left. \pm \left[\left(\omega_{Di} \left(k^2 a_i^2 (1 + \epsilon_i^{-1}) - \alpha \frac{T_i}{|e|\Phi} \right) - (\alpha - \underline{\underline{k^2 a_i^2}}) \omega_{*i} \right)^2 \right. \right. \\ \left. \left. - 4 \omega_{*i} \omega_{Di} (1 - \alpha) \left(k^2 a_i^2 (1 + \epsilon_i^{-1}) - \alpha \frac{T_i}{|e|\Phi} \right) \right]^{1/2} \right\}. \quad (\text{E.11})$$

It should be noted that the FLR terms double underlined in Eq. (E.11) enter only in the combination $\alpha - k^2 a_i^2$. This has a simple physical explanation. FLR stabilization occurs because the ion diamagnetic-drift current, which is reduced due to FLR by a factor $(1 - k^2 a_i^2)$, no longer cancels the electron diamagnetic current. In our model, trapped-electron current is reduced by a factor $(1 - \alpha)$ so that these two reduced currents may again be equal, if $\alpha = k^2 a_i^2$; and the FLR effect would disappear. However,

examination of Eq. (E.11) shows that there is a value of α such that stability results for all k . (In this demonstration, k in the FLR terms properly should be replaced by $(m - 1)/R_p$, so that they do not affect the $m = 1$ mode.)

To achieve a 0.1% steady-state density of passing electrons under reactor conditions requires rather large currents. For example, along a tube of constant field strength a density of 10^{11} cm^{-3} of electron accelerated through a potential of 10^4 keV , requires a current of about 10^2 A/cm^2 . Assuming this current is injected where the magnetic field is reduced by a factor of 100, the current is only 1 A/cm^2 . (The power required is fairly low because the electrons can be injected at low energy and recovered after passing through the machine.) To reduce the total current required, it might be sufficient to pass current only along those flux tubes on the surface of the plasma.

Finally, we have not considered other effects of these passing electrons, (e.g., two-stream instabilities). Presumably, growth rates for these instabilities could be minimized by spreading the distribution of injected electrons. We also have not considered finite β effects. The simplest, that of the direct magnetic field due to the current of the passing electrons, could be removed by counter-streaming.

Perhaps even more startling, is the effect of passing electrons in a good well, where $\omega_{Di} \omega_{*i} < 0$. In this case, the well is destabilized when Eq. (E.11) is satisfied. One good well known to be unstable with drift-type frequencies is BBII, with the neutral beam on and ϵ_i above a certain threshold. Although passing electrons are not injected into this device, it is interesting to speculate whether there may be some mechanism that provides an effective source of passing electrons. Such a source, with an α of a few percent, would, by Eq. (E.11), give rise to instability in the observed ϵ_i range that is independent of ion energy. In addition, by the above showing electric-field-reversal argument, the plasma would drift upstream in the neutral beam rather than downstream as is usual. This is consistent with Porter's measurements.

The source of electrons in BBII that could give rise to an effective α of a few percent is not at all clear. Using reasonable secondary-emission coefficients, it appears that both the axial loss of hot ions by scattering and the direct flux of hot ions ionized with large $v_{||}$ are insufficient in number. Also, it is hard to imagine even pointed structures in the fans creating the electric field strengths (order of 10^5 V/cm) necessary for cold-field emission. It is possible that a copious source of electrons initially generated by the ionization of a cold background gas might have an electrical effect similar to that of passing electrons, but a detailed mechanism is not apparent at this time.

In summary, the physics of passing electrons merit study as a potential means of stabilizing simple mirrors and as a possible cause of anomalous behavior in minimum-B wells.

APPENDIX F

FEEDBACK STABILIZATION CRITERIA

R. W. Moir

The following derivations were related to the author by V. A. Chuyanov of the Kurchatov Institute in December 1974. For feedback stabilization to be possible, the current due to the growth of the mode $\tilde{j}$ must be less than the end loss current j :

$$j = \frac{nLds}{t}, \quad (F.1)$$

where ds = area of flux tube, and

$$\tilde{j} = \tilde{n}\gamma Lds,$$

where γ = growth rate and $\tilde{n}$ = perturbed density.

For stabilization to be possible, we require

$$\tilde{j} < j. \quad (F.2)$$

We now estimate the maximum growth rate

$$\gamma_{\max} = \sqrt{\omega_{ci}\omega^*}, \quad (F.3)$$

where ω^* is drift frequency due to magnetic inhomogeneity:

$$\gamma_{\max} = \nu_{i\perp} \sqrt{\xi} \quad (\text{rough and pessimistic}) \quad (F.4)$$

$$B = B_0 \left(1 + \frac{Z^2}{L^2}\right) \quad \xi = \frac{1}{2L^2} \quad (F.5)$$

$$\gamma_{\max} = \frac{\nu_{i\perp}}{\sqrt{r_{\text{Radius}} R_{\text{Radius of plasma curvature}}}} \quad (\text{more correct, less pessimistic}) \quad (F.6)$$

$$\nu_{\text{drift}} = C \frac{E}{B} \leq \nu_{\text{Thi}}. \quad (F.7)$$

We get the perturbed density from the expression

$$\nabla \cdot \bar{E} = k_{\perp} E = 4\pi \tilde{n}_e. \quad (F.8)$$

Then, using the pessimistic value of γ ($\gamma = \nu_{i\perp} \sqrt{\xi}$) we get $\tilde{n}$ as follows:

$$\tilde{n} \leq \frac{k_{\perp} B \nu_{\text{Thi}}}{4\pi e C} = \frac{n k_{\perp} \rho_i}{\epsilon}, \quad (F.8)$$

$$\epsilon = \frac{\omega_{pi}^2}{\omega_{ci}^2} \sim 10^4. \quad (F.9)$$

From ideal gas thermal fluctuations, we can overestimate $\tilde{n}$ as follows:

$$\tilde{n} = \frac{\Delta N}{V} = \frac{\sqrt{N}}{V} = \frac{\sqrt{\frac{Ln}{k_1^2}}}{V} = \frac{\sqrt{\frac{Ln}{k_1^2}}}{\frac{L}{k_1^2}} \quad (\text{underestimate, "min"}), \quad (\text{F.10})$$

giving

$$\tilde{n}_{\max} = k_1 \sqrt{\frac{n}{L}} \quad (\text{F.11})$$

$$\frac{\tilde{n}_{\max}}{\tilde{n}_{\min}} = \frac{\sqrt{n_o L} \rho_i}{\epsilon} \sim 10^2. \quad (\text{F.12})$$

Using the minimum estimate of $\tilde{n}$ we get

$$j = \frac{n_o L ds}{\tau} \gg \tilde{n} \gamma L ds \quad (\text{for feedback stabilization}) \quad (\text{F.13})$$

$$\gg \tilde{n} \nu_{i1} \sqrt{\xi} \quad (\text{F.14})$$

$$\frac{1}{\tau} \gg \frac{\nu_{i1} \sqrt{\xi} k_1 \rho_i}{\epsilon} \quad (\text{F.15})$$

$$k_1 = \frac{\sqrt{m^2 + n^2 \pi^2}}{a}, \quad (\text{F.16})$$

where a = plasma radius, m = azimuthal wave number, and n = radial wave number.

Coulomb time:

$$\tau < \frac{a}{\nu_{Ti} \sqrt{\xi} \rho_i \sqrt{m^2 + n^2 \pi^2}}. \quad (\text{F.17})$$

Example from LLL FERF:

$$\nu_{i1} = 2.2 \times 10^8 \text{ cm sec}^{-1}$$

$$\sqrt{\xi} = 5 \times 10^{-3} \text{ cm}^{-1}$$

$$\epsilon = 2.8 \times 10^4$$

$$a = 25 \text{ cm} \quad \rho_i = 2.5 \text{ cm}$$

$$\tau = 19 \text{ } \mu\text{sec}$$

$$\tau < \frac{0.25 \text{ sec}}{\sqrt{m^2 + n^2 \pi^2}}$$

$$19 \mu\text{sec} \leq \frac{250 \mu\text{sec}}{\sqrt{m^2 + n^2\pi^2}}$$

$\therefore$ A number of modes can be stabilized.

Applications and problems:

- Need diagnostic to observe mode
- How to feed back signal
- Transmitting coefficient (ω , $k_{\perp}$) is different from usual control theory. Therefore, need to know plasma theory. Also, weak instabilities can be driven by the feedback system.
- Distributed feedback electrode not perfect and can get nonhomogeneous feedback.

NOTICE

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