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ELASTIC-PLASTIC STRESS-STRAIN RELATIONS USED IN CODE EPSOLA

M. Hartzman

April 20, 1973

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INTRODUCTION

The purpose of this note is to show the derivation of the elastic-plastic stress-strain relations which are used in the finite element program EPSOLA. The theory on which this program is based is described in a forthcoming report [1]*, and this note is in fact taken from the Appendix of this report. Due to interest by a number of users it is being published as a separate engineering note.

NOTATION

The index notation is used, a double index indicating summation. The derivation is performed in Cartesian coordinates. Symbols are defined as they occur.

DERIVATION

The basic assumptions on which the present derivation is based are listed as follows:

- a. Material is isotropic.
- b. A yield function f exists.
- c. The total strain increments are composed of elastic, plastic and thermal components.
- d. The plastic strain increments are proportional to the components of the gradient of the yield function (normality rule).
- e. The material is plastically incompressible.

Assumptions (a) and (b) imply that the yield condition is a function of the invariants of the stress tensor. In general, the yield function can be expressed as follows:

$$f = f(\sigma_{ij}, \epsilon_{ij}^p, T)$$

where

ϵ_{ij}^p = plastic strain components

T = temperature

*Numbers in brackets indicate references at the end of the note.

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or as

$$f = f(\sigma_{ij}, s)$$

where

s = yield strength in tension

$$= s(\epsilon_{ij}^p, T)$$

Yielding is specified by the conditions

$$f = 0$$

$$\dot{f} = \frac{\partial f}{\partial \sigma_{ij}} \dot{\sigma}_{ij} + \frac{\partial f}{\partial s} \dot{s} = 0 \quad (1)$$

where

$$\dot{s} = \frac{\partial s}{\partial \epsilon_{ij}^p} \dot{\epsilon}_{ij}^p + \frac{\partial s}{\partial T} \dot{T}$$

From assumption (c) we can express the total strain increments as follows:

$$\dot{\epsilon}_{ij} = \dot{\epsilon}_{ij}^E + \dot{\epsilon}_{ij}^p + \dot{\epsilon}_{ij}^T \quad (1)$$

where

$$\dot{\epsilon}_{ij}^E = \text{elastic strain increment}$$

$$\dot{\epsilon}_{ij}^p = \text{plastic strain increment}$$

$$\dot{\epsilon}_{ij}^T = \text{thermal strain increment}$$

(A dot above a symbol indicates an increment, i.e. $(\dot{\cdot}) \equiv \Delta(\cdot)$. These components can be further defined as follows:

The elastic strain increments are related to the stress increments by Hookes Law:

$$\dot{\epsilon}_{ij}^E = \frac{1}{2G} \dot{\sigma}_{ij} - \frac{\nu}{E} \dot{\sigma}_{kk} \delta_{ij} \quad (2)$$

The thermal strain increments are given as

$$\dot{\epsilon}_{ij}^T = \alpha \dot{T} \delta_{ij} \quad (3)$$

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According to assumption (d) the plastic strain increments are taken as

$$\dot{\epsilon}_{ij}^P = \dot{\Lambda} \frac{\partial f}{\partial \sigma_{ij}} \quad (4)$$

Here

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$\dot{\Lambda}$ = (unknown) proportionality coefficient

$\dot{\sigma}_{ij}$ = incremental stress tensor

σ_{ij} = total stress tensor

α = coefficient of thermal expansion

$\dot{T}$ = temperature rate

ν = Poissons ratio } assumed independent of temperature

E = Young's modulus

$G = E/2(1+\nu)$

Substituting Equations (2) - (4) into (1) we get

$$\dot{\epsilon}_{ij} = \frac{1}{2G} \dot{\sigma}_{ij} - \frac{\nu}{E} \dot{\sigma}_{kk} \delta_{ij} + \alpha \dot{T} \delta_{ij} + \dot{\Lambda} \frac{\partial f}{\partial \sigma_{ij}} \quad (5)$$

For practical application it is desirable to express the stress increments in terms of the total strain increments, since these are the only ones which can be expressed in terms of displacement increments.

To invert Equations (5) we first rewrite it as follows:

$$\dot{\sigma}_{ij} - \frac{\nu}{1+\nu} \dot{\sigma}_{kk} \delta_{ij} = 2G \dot{\epsilon}_{ij} - 2G \alpha \dot{T} \delta_{ij} - 2G \dot{\Lambda} \frac{\partial f}{\partial \sigma_{ij}} \quad (6)$$

We solve for $\dot{\sigma}_{kk}$ by letting $j=1$ and summing. Thus

$$\dot{\sigma}_{kk} = \left(\frac{E}{1-2\nu} \right) \left[\dot{\epsilon}_{kk} - 3\alpha \dot{T} - \dot{\Lambda} \frac{\partial f}{\partial \sigma_{kk}} \right] \quad (7)$$

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Substituting back into (6) we get:

$$\dot{\sigma}_{ij} = 2G \dot{\epsilon}_{ij} + \frac{2G\nu}{(1-2\nu)} \dot{\epsilon}_{kk} \delta_{ij} - \frac{E\alpha\dot{T}}{1-2\nu} \delta_{ij} - 2G\dot{\Lambda} \left[\frac{\nu}{1-2\nu} \frac{\partial f}{\partial \sigma_{kk}} \delta_{ij} + \frac{\partial f}{\partial \sigma_{ij}} \right] \quad (7)$$

The solution for $\dot{\Lambda}$ is obtained as follows:

Multiply (5) by $\partial f / \partial \sigma_{ij}$ and add:

$$\frac{\partial f}{\partial \sigma_{ij}} \dot{\epsilon}_{ij} = \frac{1}{2G} \frac{\partial f}{\partial \sigma_{ij}} \dot{\sigma}_{ij} - \frac{\nu}{E} \dot{\sigma}_{kk} \frac{\partial f}{\partial \sigma_{mm}} + \alpha \dot{T} \frac{\partial f}{\partial \sigma_{mm}} + \dot{\Lambda} \frac{\partial f}{\partial \sigma_{ij}} \frac{\partial f}{\partial \sigma_{ij}} \quad (8)$$

From (1) we get at yielding

$$\frac{\partial f}{\partial \sigma_{ij}} \dot{\sigma}_{ij} = - \frac{\partial f}{\partial s_*} \left(\frac{\partial s}{\partial \epsilon_{ij}^P} \dot{\epsilon}_{ij}^P + \frac{\partial s}{\partial T} \dot{T} \right) = - \frac{\partial f}{\partial s} \left(\frac{\partial s}{\partial \epsilon_{ij}^P} \dot{\Lambda} \frac{\partial f}{\partial \sigma_{ij}} + \frac{\partial s}{\partial T} \dot{T} \right) \quad (9)$$

Substituting (6a) and (9) into Equations (8) we get:

$$\begin{aligned} \frac{\partial f}{\partial \sigma_{ij}} \dot{\epsilon}_{ij} = & - \frac{1}{2G} \frac{\partial f}{\partial s} \left(\frac{\partial s}{\partial \epsilon_{ij}^P} \dot{\Lambda} \frac{\partial f}{\partial \sigma_{ij}} + \frac{\partial s}{\partial T} \dot{T} \right) - \frac{\nu}{1-2\nu} [\dot{\epsilon}_{kk} - 3\alpha\dot{T} - \dot{\Lambda} \frac{\partial f}{\partial \sigma_{kk}}] \frac{\partial f}{\partial \sigma_{mm}} \\ & + \alpha\dot{T} \frac{\partial f}{\partial \sigma_{mm}} + \dot{\Lambda} \frac{\partial f}{\partial \sigma_{ij}} \frac{\partial f}{\partial \sigma_{ij}} \end{aligned}$$

Collecting terms and solving for $\dot{\Lambda}$ we get:

$$\begin{aligned} \dot{\Lambda} = & \frac{\frac{\partial f}{\partial \sigma_{ij}} \dot{\epsilon}_{ij} + \frac{\partial f}{\partial \sigma_{mm}} \left[\left(\frac{\nu}{1-2\nu} \right) \dot{\epsilon}_{kk} - \frac{1+\nu}{1-2\nu} \alpha\dot{T} \right] + \frac{1}{2G} \frac{\partial f}{\partial s} \frac{\partial s}{\partial T} \dot{T}}{\frac{\partial f}{\partial \sigma_{ij}} \frac{\partial f}{\partial \sigma_{ij}} + \frac{\nu}{1-2\nu} \frac{\partial f}{\partial \sigma} \frac{\partial f}{\partial \sigma_{mm}} - \frac{1}{2G} \frac{\partial f}{\partial s} \frac{\partial s}{\partial \epsilon_{ij}^P} \frac{\partial f}{\partial \sigma_{ij}}} \quad (10) \end{aligned}$$

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The quantity $\frac{\partial s}{\partial \epsilon_{ij}^P}$ is obtained as follows:

From (1):

$$\begin{aligned}\dot{s} &= \frac{\partial s}{\partial \epsilon_{ij}^P} \dot{\epsilon}_{ij}^P + \frac{\partial s}{\partial T} \dot{T} \\ &= \frac{\partial s}{\partial \epsilon^P} \frac{\partial \epsilon^P}{\partial \epsilon_{ij}^P} \dot{\epsilon}_{ij}^P + \frac{\partial s}{\partial T} \dot{T}\end{aligned}$$

where ϵ^P = equivalent plastic strain.

The quantity $\frac{\partial s}{\partial \epsilon^P}$ is therefore the slope H' of the s - ϵ^P curve which can be obtained from a uni-axial stress-strain curve at constant temperature. Therefore

$$\dot{s} = H' \frac{\partial \epsilon^P}{\partial \epsilon_{ij}^P} \dot{\epsilon}_{ij}^P + \frac{\partial s}{\partial T} \dot{T} \quad (11)$$

However, we can also write $\dot{s}$ as follows:

$$\begin{aligned}\dot{s} &= \frac{\partial s}{\partial w^P} \dot{w}^P + \frac{\partial s}{\partial T} \dot{T} \\ &= \frac{\partial s}{\partial \epsilon^P} \frac{\partial \epsilon^P}{\partial w^P} \dot{w}^P + \frac{\partial s}{\partial T} \dot{T}\end{aligned}$$

where

$$\begin{aligned}\dot{w}^P &= \text{plastic work increment} \\ &= \sigma'_{ij} \dot{\epsilon}_{ij}^P \\ &= s \dot{\epsilon}^P\end{aligned}$$

where $\sigma'_{ij} = \sigma_{ij} - \frac{1}{3} \sigma_{kk} \delta_{ij}$ and $\dot{\epsilon}_{kk}^P = 0$

Thus

$$\dot{s} = \frac{H'}{s} \sigma'_{ij} \dot{\epsilon}_{ij}^P + \frac{\partial s}{\partial T} \dot{T} \quad (12)$$

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Comparing (11) and (12) we get

$$\frac{\partial \epsilon^P}{\partial \epsilon_{ij}^P} = \frac{\sigma'_{ij}}{s}$$

and

(13)

$$\frac{\partial s}{\partial \epsilon_{ij}^P} = \frac{H'}{s} \sigma'_{ij}$$

Equation (7), (10) and (13) define the desired stress-strain relations. For the Von Mises yield condition we take

$$f = J_2 - \frac{s^2}{3}$$

where

(14)

$$J_2 = \frac{1}{2} \sigma'_{ij} \sigma'_{ij}$$

At yielding

$$J_2 = \frac{s^2}{3}$$

From (14) we get

$$\frac{\partial f}{\partial \sigma'_{ij}} = \sigma'_{ij}$$

$$\frac{\partial f}{\partial s} = -\frac{2}{3} s$$

$$\frac{\partial f}{\partial \sigma_{kk}} = 0$$

∴ the expression for $\dot{\lambda}$ becomes

$$\dot{\lambda} = \frac{\sigma'_{ij} \dot{\epsilon}_{ij} - \frac{1}{3G} s \frac{\partial s}{\partial T} \dot{T}}{\sigma'_{ij} \sigma'_{ij} + \frac{s}{3G} \frac{H'}{s} \sigma'_{ij} \sigma'_{ij}} = \frac{\sigma'_{ij} \dot{\epsilon}_{ij} - \frac{1}{3G} s \frac{\partial s}{\partial T} \dot{T}}{\frac{2}{3} s^2 \left[1 + \frac{H'}{3G} \right]}$$

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Equations (7) thus become

$$\dot{\sigma}_{ij} = 2G \dot{\epsilon}_{ij} + \left(\frac{2G\nu}{1-2\nu} \dot{\epsilon}_{kk} - \frac{E\alpha\dot{T}}{1-2\nu} \right) \delta_{ij} - \beta 2G \sigma'_{ij} \left[\frac{\sigma'_{mn} \dot{\epsilon}_{mn} - \frac{1}{3G} s \frac{\partial s}{\partial T} \dot{T}}{\frac{2}{3} s^2 [1 + H'/3G]} \right]$$

or

$$\dot{\sigma}_{ij} = 2G \left[\dot{\epsilon}_{ij} + \frac{\nu \delta_{ij}}{1-2\nu} \dot{\epsilon}_{kk} - \beta \frac{\sigma'_{ij} \sigma'_{mn} \dot{\epsilon}_{mn}}{\frac{2}{3} s^2 [1 + H'/3G]} \right] - \left[\frac{E\alpha \delta_{ij}}{1-2\nu} - \frac{\beta \frac{\partial s}{\partial T} \sigma'_{ij}}{s[1 + H'/3G]} \right] \dot{T} \quad (15)$$

These are the same relations as developed by Yamada [2], except for the inclusion of the thermal terms.

In Equations (15) the factor β was introduced to indicate loading or unloading, and is specified as follows:

$$\beta = 1 \quad \text{if } f = 0 \text{ and } df > 0$$

$$\beta = 0 \quad \text{if } f = 0 \text{ and } df = 0$$

$$\text{or } f < 0$$

Hence the plastic terms are included only upon loading, and unloading occurs elastically.

The equivalent plastic strain increment is obtained as follows:

$$\begin{aligned} \dot{\epsilon}^P &= \frac{\sigma'_{ij} \dot{\epsilon}^P_{ij}}{s} = \frac{\sigma'_{ij}}{s} \dot{\lambda} \frac{\partial f}{\partial \sigma_{ij}} = \frac{2}{3} s \dot{\lambda} \\ &= \frac{1}{s} \frac{(\sigma'_{ij} \dot{\epsilon}_{ij} - \frac{s}{3G} \frac{\partial s}{\partial T} \dot{T})}{[1 + H'/3G]} \end{aligned}$$

Also obtained by Yamada except for the thermal term.

The term H' can be shown (see the Appendix) to be related to the uniaxial stress-strain curve by the relation

$$H' = \frac{E E^*}{E - E^*}$$

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where

E^* = slope of the uniaxial stress-strain curve at the current value of s

Equations (15) can also be expressed in the following form:

$$\dot{\sigma}_{ij} = C_{ijkl} \dot{\epsilon}_{kl} - B_{ij} \dot{T} \quad (16)$$

where

$$C_{ijkl} = G (\delta_{ik} \delta_{jl} + \delta_{jk} \delta_{il}) + \lambda (\delta_{ij} \delta_{kl}) - 3G \beta \frac{\sigma'_{ij} \sigma'_{kl}}{s^2 [1+H'/3G]}$$

$$B_{ij} = \frac{E \alpha}{1-2\nu} \delta_{ij} - \beta \frac{\frac{\partial s}{\partial T} \sigma'_{ij}}{s^2 [1+H'/3G]}$$

$$\lambda = \frac{2G\nu}{1-2\nu}$$

Equations (16) are valid in Cartesian coordinates only.

For practical application, the coefficients C_{ijkl} are shown in matrix form as follows:

$$[C_{ijkl}] = 2G \begin{bmatrix} A - (\sigma'_{xx})^2/D & B - \sigma'_{xx} \sigma'_{yy}/D & B - \sigma'_{xx} \sigma'_{zz}/D & -\sigma'_{xx} \sigma'_{xy}/D & -\sigma'_{xx} \sigma'_{yz}/D & -\sigma'_{xx} \sigma'_{xz}/D \\ A - (\sigma'_{yy})^2/D & B - \sigma'_{yy} \sigma'_{yy}/D & B - \sigma'_{yy} \sigma'_{zz}/D & -\sigma'_{yy} \sigma'_{xy}/D & -\sigma'_{yy} \sigma'_{yz}/D & -\sigma'_{yy} \sigma'_{xz}/D \\ A - (\sigma'_{zz})^2/D & B - \sigma'_{yy} \sigma'_{zz}/D & A - (\sigma'_{zz})^2/D & -\sigma'_{zz} \sigma'_{xy}/D & -\sigma'_{zz} \sigma'_{yz}/D & -\sigma'_{zz} \sigma'_{xz}/D \\ 1/2 - (\sigma'_{xy})^2/D & -\sigma'_{yy} \sigma'_{xy}/D & -\sigma'_{zz} \sigma'_{xy}/D & 1/2 - (\sigma'_{xy})^2/D & -\sigma'_{yz} \sigma'_{xy}/D & -\sigma'_{xz} \sigma'_{xy}/D \\ 1/2 - (\sigma'_{yz})^2/D & -\sigma'_{yy} \sigma'_{yz}/D & -\sigma'_{zz} \sigma'_{yz}/D & -\sigma'_{zz} \sigma'_{yz}/D & 1/2 - (\sigma'_{yz})^2/D & -\sigma'_{xz} \sigma'_{yz}/D \\ 1/2 - (\sigma'_{zx})^2/D & -\sigma'_{yy} \sigma'_{xz}/D & -\sigma'_{zz} \sigma'_{xz}/D & -\sigma'_{xy} \sigma'_{xz}/D & -\sigma'_{yz} \sigma'_{xz}/D & 1/2 - (\sigma'_{zx})^2/D \end{bmatrix}$$

The corresponding shear strains are the engineering shear strains.

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where

$$A = \frac{1-2\nu}{1-2\nu}, \quad B = \frac{2}{1-2\nu}, \quad D = \frac{2}{3} s^2 [1 + H'/3G]$$

$$\{\sigma_{11} \sigma_{22} \sigma_{33} \sigma_{12} \sigma_{13} \sigma_{23}\} = \{\sigma_{xx} \sigma_{yy} \sigma_{zz} \sigma_{xy} \sigma_{xz} \sigma_{yz}\}$$

This is the same matrix as shown by Yamada, except that the shear strains are defined as the tensor components.

Likewise, the coefficients B_{ij} may be expressed in matrix form as follows:

$$[B_{ij}] = \begin{Bmatrix} c - \frac{2}{3} s F' \sigma'_{xx}/D \\ c - \frac{2}{3} s F' \sigma'_{yy}/D \\ c - \frac{2}{3} s F' \sigma'_{zz}/D \\ - \frac{2}{3} s F' \sigma'_{xy}/D \\ - \frac{2}{3} s F' \sigma'_{yz}/D \\ - \frac{2}{3} s F' \sigma'_{xz}/D \end{Bmatrix}$$

where

$$c = \frac{E\alpha}{1-2\nu}, \quad F' = \partial s / \partial T$$

The formulation described above can also be applied to problems where large displacements occur, provided they are interpreted correctly. For large displacements the geometry must be indicated with respect to which the stresses and strains are specified.

In EPSOLA the stress-strain relations are based on the undeformed geometry.

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In terms of quantities referred to the undeformed geometry, we can express Equations (16) in the following form:

$$\bar{s}_{ij} = \bar{C}_{ijkl} \xi_{kl} + \bar{B}_{ij} \theta$$

where

$$\bar{C}_{ijkl} = G (\delta_{ik} \delta_{jl} + \delta_{jk} \delta_{il}) + \lambda (\delta_{ij} \delta_{kl}) - 2G\beta \frac{l_{s'ij} l_{s'kl}}{\frac{2}{3} s^2 [1 + \frac{H'}{3G}]}$$

$$\bar{B}_{ij} = \frac{E\alpha}{1-2\nu} \delta_{ij} - \beta \frac{\partial s / \partial T}{s [1 + H'/3G]} l_{s'ij}$$

These equations are based on

$$f = \frac{1}{2} (l_{s'ij} l_{s'ij}) - \frac{s^2}{3}$$

$$H' = \partial s / \partial l \epsilon_{ij}^P$$

$$\epsilon^P = \frac{l_{s'ij} \xi_{ij}}{s [1 + H'/3G]}$$

and

$$l_{s'ij} = l_{sij} - \frac{1}{3} \delta_{ij} s_{kk}$$

where

l_{sij} = stresses referred to the undeformed configuration.

ξ_{ij} = Lagrangian strain difference referred to the undeformed configuration.

$\bar{s}_{ij}$ = incremental stresses referred to the undeformed configuration.

θ = temperature difference.

These equations are still valid only in cartesian coordinates. In curvilinear coordinates the corresponding equations are given by equations (19) of [1].

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APPENDIX

To show $H' = \frac{EE^*}{E-E^*} :$

In a uniaxial test, $\sigma_{ij} = 0$ except σ_{11} . From Equations (5) we get

$$\dot{\epsilon}_{11} = \frac{\dot{\sigma}_{11}}{E} + \dot{\epsilon}_{11}^P$$

$$\dot{\epsilon}_{22} = \dot{\epsilon}_{33} = - \left(\frac{\nu}{E} \dot{\sigma}_{11} + \frac{1}{2} \dot{\epsilon}_{11}^P \right)$$

where the condition of plastic incompressibility $\dot{\epsilon}_{kk}^P = 0$ has been assumed.

If we take

$$\dot{\sigma}_{11} = E^* \dot{\epsilon}_{11} \quad \sigma_{11} \geq \sigma_y \text{ (yield stress)}$$

we get

$$\dot{\sigma}_{11} \left(\frac{1}{E^*} - \frac{1}{E} \right) = \dot{\epsilon}_{11}^P$$

whence

$$\dot{\sigma}_{11} = H' \dot{\epsilon}_{11}^P$$

where

$$H' = \frac{EE^*}{E-E^*}$$

E^* = current slope of the stress-strain curve.

From the definitions of s and $\dot{\epsilon}^P$, we get from the one-dimensional case:

$$\dot{s} = \frac{3}{2} \dot{\sigma}_{ij} \dot{\epsilon}_{ij} = \dot{\sigma}_{11} \dot{\epsilon}_{11}^P, \quad \dot{\epsilon}^P = \frac{\sigma_{ij} \dot{\epsilon}_{ij}^P}{s} = \dot{\epsilon}_{11}^P$$

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Hence,

$$\dot{s} = H' \dot{\epsilon}^P$$

where

$$H' = \frac{\partial s}{\partial \epsilon^P}$$

In addition, the current value of s can be obtained from the following difference relation:

$$s_n = s_{n-1} + H' \dot{\epsilon}^P$$

In practice the quantity H' is obtained from the total relation

$$\epsilon_{11}^P = \epsilon_{11} - \frac{\sigma_{11}}{E}, \quad \sigma_{11} \geq \sigma_y$$

This results in a s - ϵ^P curve. If the curve is approximated by piecewise continuous segments, then H' will vary depending on the current value of ϵ_n^P . This value is obtained from the difference relation

$$\epsilon_n^P = \epsilon_{n-1}^P + \dot{\epsilon}^P,$$

valid under the assumption of plastic incompressibility.

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