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Diffraction from Circular and Irregular Apertures

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LOS ALAMOS, NEW MEXICO 87544

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O. P. Judd

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DIFFRACTION FROM CIRCULAR AND IRREGULAR APERTURES

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I. INTRODUCTION

The diffraction patterns originating from uniformly illuminated circular apertures are well known. The far field or Fraunhofer region can be obtained in the closed form Airy pattern solution. In the near field or Fresnel region in which the Fresnel number $N > 1$, the spatial diffraction pattern is obtained by a numerical solution of Maxwell's equations. The results of this calculation in the Fresnel region show the existence of a large intensity peak at the center of the diffraction pattern for even values of N . The detrimental effect of such high power intensity peaks as a source of self-focusing in solid-state lasers has been emphasized previously. It is therefore desirable to reduce the amplitude of these diffraction "spikes" or eliminate them completely.

The large amplitude of the central diffraction spike results from the circular symmetry of the aperture. Deviations from circular symmetry should reduce this intensity peak considerably. This note is an attempt to derive a quantitative analytical estimate of the central peak intensity as a function of aperture irregularity. To do this, the problem has been approached in terms of the Kirchhoff integral formulation of diffraction theory rather than a solution of the paraxial wave equation. Analytic expressions appeared to be more readily derived with the former method.

The basic assumption of this method as applied in this note are the usual ones of scalar versus vector diffraction theory, the aperture radius a is much larger than the wavelength, and that the distance from the aperture r_0 is sufficiently large that $a^2/r_0^2 \ll 1$. These conditions are adequately

satisfied for most practical situations of interest. The Kirchhoff integral formulation therefore appears to be a powerful method for treating diffraction problems where analytic solutions are desired.

The specific problem that is solved here is the on-axis intensity resulting from a (1) uniformly illuminated aperture (phase and amplitude), and (2) Gaussian illumination. The aperture is approximately circular with some small irregularity about an average radius. Extension of the theory to include finite wavefront curvature and spatial beam divergence of the illumination source is straightforward.⁽¹⁾ The theory may also be applicable to tapered apertures which include both phase and amplitude variations.

The results of this analysis indicate that a significant "smoothing" of the on-axis intensity peak can be obtained for $N \geq 20$ by introducing perturbations at the periphery of the aperture with an amplitude of $\approx 1\%$. The greater the degree of irregularity, the greater is the smoothing. Irregular apertures of this type can be easily fabricated and represent a simple method of reducing the spatial variations in intensity introduced by diffraction on apertured optical beams.

II. DIFFRACTION THEORY IN A KIRCHHOFF INTEGRAL FORMULATION

Consider an arbitrary aperture with an intensity distribution $F(\rho, \theta)$. The intensity at the observation point P' with polar coordinates w and β is obtained by summing the contributions from each point ρ and θ in the aperture. The geometry of the problem is shown in Fig. 1. The electric field at P' is just

$$E = \iint F(\rho, \theta) e^{ikr(\rho, \theta)} \rho d\rho d\theta \quad (1)$$

Fig. 1. Configuration and nomenclature for the diffraction problem.

The geometry of the problem gives

$$r^2 = (x-x')^2 + (y-y')^2 + r_0^2 \quad (2)$$

$$r'^2 = x'^2 + y'^2 + r_0^2 \quad (3)$$

where r_0 is the perpendicular distance from the plane of the aperture to the plane of observation. From Fig. 1 and Eqs. (2) and (3), one obtains

$$r = r' \left[1 + \frac{\rho^2}{r'^2} - 2\rho \frac{w \cos(\theta + \beta)}{r'^2} \right]^{1/2} \quad (4)$$

We now assume that ρ^2/r'^2 and $\rho w/r'^2$ are $\ll 1$ so that Eq. (4) becomes to $O(\rho^2/r'^2)$

$$r = r' + \frac{\rho^2}{2r_0} \cos \gamma - \rho \sin \gamma \cos(\theta + \beta) \quad (5)$$

where we have used the relation that $r_0 = r' \cos \gamma$ and $w = r' \sin \gamma$. The point of observation is specified completely by the angles γ and β . For axial symmetry, only γ needs to be specified.

The field as given by Eq. (1) can now be written in the form

$$E = \frac{e^{ikr'}}{r'} \int \int F(\rho, \theta) e^{if(\rho, \theta)} \rho d\rho d\theta \quad (6)$$

where

$$f(\rho, \theta) = \frac{k\rho^2}{2r_0} \cos \gamma - k\rho \sin \gamma \cos(\theta + \beta) \quad (7)$$

Eqs. (6) and (7) are the working equations of this section and have been derived elsewhere⁽¹⁾ by a similar method. For the time being, we will assume uniform illumination of the aperture so that $F(\rho, \theta)$ is unity. For convenience, we will also drop the propagation factor $e^{ikr'}/r'$ and reinsert it when needed.

Using the Bessel function identity,

$$e^{if \sin \theta} = \sum_n J_n(r) e^{in\theta} \quad (8)$$

Eqs. (6) and (7) can also be written in the form

$$E = \sum_n \int_0^{2\pi} \int_0^a e^{i(k\rho^2/2r_0)} \cos \gamma J_n(k\rho \sin \gamma) e^{in\theta} \rho d\rho d\theta \quad (9)$$

where a is the aperture radius which in general is a function of θ .

III. FRAUNHOFER DIFFRACTION

In order to check the validity of Eq. (9), we look for the solution in the far field region for a uniformly illuminated circular aperture. The integral over θ can be performed immediately with a resulting contraction of the sum over Bessel functions to the single term $n=0$. In the limit that $r_0 \rightarrow \infty$, one is left with the integral

$$E = \int_0^a J_0(k\rho \sin \gamma) \rho d\rho \quad (10)$$

Using the relation that

$$\frac{d}{dx} [x^{n+1} J_{n+1}(x)] = x^{n+1} J_n(x) \quad (11)$$

one obtains, after dropping some constants,

$$E = 2 \frac{J_1(ka \sin \gamma)}{ka \sin \gamma} \quad (12)$$

The intensity, defined as $|E|^2$ is given as

$$I = I_0 4 \left[\frac{J_1(ka \sin \gamma)}{ka \sin \gamma} \right]^2 \quad (12)$$

where I_0 is the on-axis intensity ($\gamma=0$). This is just the familiar Airy pattern in the far field.

IV. FRESNEL DIFFRACTION

The diffraction pattern is now considered for finite r_0 . This requires the general evaluation of Eq. (9). We now restrict the remaining discussion to the intensity on the axis r_0 . In this case, $\gamma=0$ and the only Bessel function that remains is $n=0$. At this point, we remove the assumption that the radius of the aperture is a constant and consider it a function of θ . Under these conditions, Eq. (9) may be written in the form

$$E = \int_0^{2\pi} d\theta \int_0^{\frac{ka(\theta)}{2r_0}} e^{ix} dx \quad (13)$$

$$= \int_0^{2\pi} [e^{ika(\theta)/2r_0} - 1] d\theta \quad (14)$$

V. NONSYMMETRIC APERTURES

We now assume an arbitrary perturbation on the radius of the aperture. The perturbation can be expressed in the general Fourier form

$$a(\theta) = a \left[1 + \sum_m \epsilon_m \sin(m\theta) \right] \quad (15)$$

where a is the average radius of the aperture. Defining the Fresnel number of the aperture as

$$N = \frac{a^2}{\lambda r_0} ,$$

Eq. (14) may be written in the form

$$E = \int_0^{2\pi} \left[e^{i\pi N (1 + \sum_m \epsilon_m \sin(m\theta))^2} - 1 \right] d\theta \quad (16)$$

We now assume that all $\epsilon_m < 1$ and expand the bracket in the exponent. Using the Bessel function identity in Eq. (8), we obtain

$$E = \int_0^{2\pi} \left[e^{i\pi N \sum_m \pi J_n(2\pi N \epsilon_m) e^{in\theta}} - 1 \right] d\theta \quad (17)$$

where the product is taken over all Fourier components. Performing the integration yields the result that the electric field is given by

$$E = [e^{i\pi N} \prod_m J_0(2\pi N \epsilon_m) - 1] \quad (18)$$

Reinserting the appropriate constants and defining $\bar{I}$ to be the average beam intensity across the aperture, the intensity on axis is given by

$$I = \bar{I} \left[\left(\prod_m J_0(2\pi N \epsilon_m) \right)^2 + 1 - 2 \prod_m J_0(2\pi N \epsilon_m) \cos N\pi \right] \quad (19)$$

which is the desired result.

In the limit of perfectly circular aperture, $\epsilon_m \rightarrow 0$ and Eq. (19) becomes

$$I = 4\bar{I} \sin^2 \frac{N\pi}{2} \quad (20)$$

Eq. (20) yields the familiar result that the intensity along the axis in the Fresnel region of diffraction exhibits maxima and minima at odd and even Fresnel numbers, respectively. This is the characteristic behavior of the central diffraction "spike" resulting from a uniformly illuminated circular aperture.

In order to investigate the effects of a non-circular aperture, we assume $2\pi N \epsilon_m \gg 1$. Using the asymptotic form for $J_0(x)$, Eq. (19) becomes

$$I = \bar{I} \left[1 - \pi \sqrt{\frac{4}{\pi N \epsilon_m}} \cos(2\pi N \epsilon_m - \pi/4) \cos N\pi \right] \quad (21)$$

One observes in Eq. (21) that as $\pi N \epsilon_m \rightarrow \infty$, the oscillatory behavior of the on-axis intensity vanishes. From the standpoint of self-focusing, one wants to minimize any peaks in the intensity above the average. For perfect circular symmetry, Eq. (20) indicates that this peak value can be four times the average value. The amount of reduction, resulting from any one Fourier component, can be estimated from Eq. (21) and is indicated in Fig. 2.

From the figure, it is obvious that a significant reduction in the intensity variation can be achieved with an irregularity of an amount $\epsilon_m = \frac{10}{\pi N} \approx \frac{3}{N}$. At a Fresnel number of 20, this amounts to a variation in radius of 15 percent, a value easily achieved in practice. In order to reduce the variation of intensity from 66 percent to 20 percent, one must increase $\pi N \epsilon_m$ by an order of magnitude. These results are summarized in Fig. 2 where we have plotted the value of $I_{\max}/\bar{I}$ versus normalized Fresnel number $\pi N \epsilon_m$. Since the effect of many finite amplitude Fourier components on the oscillatory behavior of the intensity enters as a product of reduction factors, it is obvious that the more irregular the perturbation, the more the oscillatory intensity behavior is diminished. It should be noted that each Fourier component of the irregularity provides the same reduction factor.

Fig. 2. On-axis intensity dependence as a function of Fresnel number and perturbation amplitude.

The oscillatory variation of the on-axis intensity is largest for a uniformly illuminated aperture. One case of a nonuniform aperture illumination is now considered.

VI. GAUSSIAN ILLUMINATION

For this case, the source distribution is assumed to be of the form

$$F(\rho, \theta) = e^{-\frac{\rho^2}{b^2}}. \quad (22)$$

Inserting this form into Eq. (9) yields the previous results with N , replaced by

$$N \rightarrow N + \frac{1}{\pi} \frac{a^2}{b^2}. \quad (23)$$

In the limit of the circular aperture, Eq. (20) becomes

$$I \propto \left\{ \left[1 - e^{-a^2/b^2} \right]^2 + 4e^{-a^2/b^2} \sin^2 \frac{\pi \eta}{2} \right\}. \quad (24)$$

As $a^2/b^2 \rightarrow 0$, the uniform illumination result is recovered. In the opposite limit, $a^2/b^2 \rightarrow \infty$, the beam

doesn't "see" the aperture and the dependence on N vanishes.

The limit of a noncircular aperture is obtained by the substitution

$$J_0(x) \rightarrow e^{-a^2/b^2} J_0(x) \quad (25)$$

in Eqs. (19) and (21). The result is that the value of $\pi N \epsilon$ is replaced by

$$\pi N \epsilon \rightarrow \pi N \epsilon e^{2a^2/b^2} \quad (26)$$

in the expressions derived previously.

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1. Principles of Optics, M. Born and E. Wolf, p. 382 ff, Pergamon Press (1970).