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SUBJECT: An Extension of the Esslinger Shell Relationships to Include Effects of Axial Load and Temperature

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NOMENCLATURE

b = Radial distance from axis of rotation to center of curvature of gradation section

E = Modulus of elasticity

e = Dimension normal or parallel to rotation axis

F = Cross-sectional area of section

F_i, G_i, H_i, K_i = Constants ($i = 1,2,3 \ldots$)

J_o = Moment of inertia of area about axis y-y perpendicular to axis of revolution

J = $s^3/12$ = Cross sectional moment of inertia for rectangular beam of unit width

L = Force in axial direction per unit circumference

M = Meridional bending moment per unit circumference

N = Force in meridional direction per unit circumference

p = Pressure

r = Radius of curvature in meridional direction

S = Radial force per unit circumference

s = Thickness of shell

T = Tangential force per unit length

w' = Radial displacement

ϵ = Unit elongation

ζ = Vertical displacement

ν = Poisson's ratio

ρ = Radial distance from axis of rotation

τ = Temperature excess above reference temperature

ϕ = Meridional angle

χ = Rotation of tangent to meridian

Subscripts

a	=	Reference edge of gradation section
e	=	Terminal intersecting edge of gradation section
i	=	Inside surface of shell
m	=	Mean value
N	=	Normal direction
T	=	Tangential direction

Equations for Shells of Revolution

Maria Esslinger has presented equations for analyzing shells of revolution by numerical methods[1]. The given expressions allow the prediction of shell behavior under edge force and pressure loadings. In order to broaden the range of applicability of this method of analysis, the Esslinger equations have been extended to include the effects of, (a) an external, axial force, and (b), temperature. In addition, expressions have been developed for predicting the axial displacement for all loading conditions. The resulting relationships are presented here. The terminology and the methods of developing equations are those used by Esslinger. Only those expressions necessary for extending the analytical method are presented.

Figure I shows a typical shell element which is referred to as a gradation section. Any gradation section with double curvature can be classified under one of the four categories shown in Figure II. Thus, the equations describing the behavior of a shell element are derived for each category.

I. Forces and Moments at the Terminal Intersecting Point

$$L_e = \frac{\rho_a}{\rho_e} L_a + \frac{p}{2\rho_e} (\rho_{ie}^2 - \rho_{ia}^2)$$

$$S_e = \frac{\rho_a}{\rho_e} S_a + \frac{EF}{\rho_m \rho_e} w + \frac{\nu FN}{\rho_e s} + \frac{\rho_{im}}{\rho_e} \int_{\phi_a}^{\phi_e} p \sin \phi \, r_i \, d\phi$$

$$M_e = -\frac{\rho_a}{\rho_e} S_a e_3 + \frac{\rho_a}{\rho_e} L_a e_4 + \frac{\rho_a}{\rho_e} M_a - \left(\frac{EF}{\rho_m \rho_e} w + \frac{\nu FN}{\rho_e s}\right) e_2$$

$$+ \frac{EJ_o}{\rho_m^2} \chi + \int_{\phi_a}^{\phi_e} pr \sin (\phi_e - \phi) r_i \, d\phi$$

II. Deformations at the Terminal Intersecting Point

The effects of a temperature distribution upon the deformations will be considered. The temperature is assumed to be constant or to vary linearly with the angle ϕ.

The change in inclination is

$$\Delta \chi_e = \frac{\alpha(\rho_e \tau_e - \rho_a \tau_a)}{2r \sin \Delta\phi/2}$$

Thus

$$\chi_e = \chi_a + \int_{\phi_a}^{\phi_e} \frac{M}{EJ} r d\phi + \frac{\alpha(\rho_e \tau_e - \rho_a \tau_a)}{2r \sin (\Delta\phi/2)}$$

The displacement changes

$$\Delta w_e = \alpha(\rho_e \tau_e - \rho_a \tau_a),$$

and the total displacement becomes

$$\Delta w_e = w_a + \chi_a e_3 + \int_{\phi_a}^{\phi_e} \frac{M}{EJ} r(\cos \phi - \cos \phi_e) r d\phi$$

$$+ \left(\frac{N(1 - \nu^2)}{Es} - \nu \frac{w}{\rho} \right) e_4 + \alpha(\rho_e \tau_e - \rho_a \tau_a)$$

Positive axial displacement corresponds to an increase in the axial dimension of a gradation section. Axial displacement is produced by a change in ζ_a,

$$\Delta \zeta_e = \zeta_a ,$$

by a change in inclination χ_a,

$$\Delta \zeta_e = - \chi_a e_4 ,$$

by deformation of the gradation section,

$$\Delta \zeta_e = - \int_{\phi_a}^{\phi_e} \frac{M}{EJ} r(\sin \phi_e - \sin \phi) r d\phi ,$$

due to a change in length of the gradation section,

$$\Delta\zeta_e = \epsilon_N e_3 = \left(\frac{N(1-\nu^2)}{Es} - \nu\frac{w}{\rho}\right)e_3 \ ,$$

and by the temperature,

$$\Delta\zeta_e = \alpha\tau_m e_3 \ .$$

The total axial displacement at the terminal end is

$$\Delta\zeta_e = \zeta_a - \chi_a e_4 - \int_{\phi_a}^{\phi_e} \frac{M}{EJ} r(\sin\phi_e - \sin\phi) r d\phi$$

$$+ \left[\frac{N(1-\nu^2)}{Es} - \nu\frac{w}{\rho}\right]e_3 + \alpha\tau_m e_3 \ .$$

III. Internal Forces

The internal forces are given in Reference 1 as follows:

$$N = S\cos\phi + L\sin\phi$$

$$T = \frac{Ew}{\rho} + \nu N$$

$$M = M \ .$$

IV. Complete Equations

The axial displacement equation for a convex circular ring with $\phi_e > \phi_a$ (Fig. 11a) will be written. The contributions of the displacement and inclination at $\phi = \phi_a$ are

$$\Delta\zeta_e = \zeta_a - \chi_a \Delta\phi \cos\phi_m .$$

From the deformation of the gradation section

$$\Delta\zeta_e = -\int_{\phi_a}^{\phi_e} \frac{M}{EJ} r(\sin\phi_e - \sin\phi) r d\phi$$

$$= -\frac{r^2}{EJ} \int_{\phi_a}^{\phi_e} \left(\frac{M_a\phi_e - M_e\phi_a}{\Delta\phi} + \frac{M_e - M_a}{\Delta\phi}\phi \right)(\sin\phi_e - \sin\phi) d\phi$$

$$= -\frac{r^2}{Es^3} 2\Delta\phi^2 \left[M_a\left(2\cos\phi_m - \frac{\Delta\phi}{4}\sin\phi_m\right) + M_e\left(\cos\phi_m - \frac{\Delta\phi}{4}\sin\phi_m\right) \right]$$

A change in length of the gradation section gives

$$\Delta\zeta_e = \left(\frac{N(1-\nu^2)}{Es} - \nu\frac{w}{\rho} \right) e_3$$

$$= \left[\frac{1-\nu^2}{Es}\left(\frac{S_a + S_e}{2}\cos\phi_m + \frac{L_a + L_e}{2}\sin\phi_m \right) - \frac{\nu w_a}{\rho_a} \right] r\Delta\phi \sin\phi_m$$

The effect of temperature is

$$\Delta \zeta_e = \alpha \tau_m r \Delta \phi \sin \phi_m$$

Thus, the displacement at the terminal intersection point of the gradation section becomes

$$\frac{\zeta_e}{r} = \frac{\zeta_a}{r} - \chi_a \Delta\phi \cos \phi_m - \frac{2r^2}{s^2} \Delta\phi^2 (2 \cos \phi_m - \frac{\Delta\phi}{4} \sin \phi_m) \frac{M_a}{Esr}$$

$$- \frac{2r^2}{s^2} \Delta\phi^2 (\cos \phi_m - \frac{\Delta\phi}{4} \sin \phi_m) \frac{M_e}{Esr}$$

$$+ \frac{1 - \nu^2}{2} \Delta\phi \sin \phi_m \cos \phi_m \frac{S_a + S_e}{Es}$$

$$+ (1 - \nu^2) \frac{\rho_m}{\rho_e} \Delta\phi \sin^2 \phi_m \frac{L_a}{Es}$$

$$+ (1 - \nu^2) \frac{\rho_{im} r_i}{\rho_e s} \Delta\phi^2 \sin^2 \phi_m \cos \phi_m \frac{p}{E} - \frac{\nu r}{\rho_a} \phi \sin \phi_m \frac{w_a}{r}$$

$$+ \Delta\phi \sin \phi_m \alpha \tau_m$$

The equations for the force, moment and deformations at the terminal intersecting point of a given gradation section will now be fully written. A separate set of equations exists for each of the 4 cases identified earlier. The equations given here are extensions of Esslinger's expressions, and they include the trigonemetric approximations used by her.

a. Convex circular ring with $\phi_e > \phi_a$ (Fig. IIa)

$$\frac{S_e}{Es} = (F_1 + F_2)\frac{S_a}{Es} + F_3\frac{w_a}{r} + F_4\chi_a - F_5\frac{p}{E} + F_7\frac{L_a}{Es}$$

$$\frac{M_e}{Esr} = G_o\frac{M_a}{Esr} - (G_1 + G_2)\frac{S_a}{Es} - G_3\frac{w_a}{r} - G_4\chi_a$$

$$+ G_6\frac{p}{E} - (G_7 - G_8)\frac{L_a}{Es}$$

$$\chi_e = \chi_a + H_1\frac{M_a + M_e}{Esr} + H_8\,\alpha\tau_e - H_9\,\alpha\tau_a$$

$$\frac{w_e}{r} = H_2\frac{w_a}{r} + H_3\chi_a + H_4\frac{M_a}{Esr} + H_5\frac{M_e}{Esr} + H_6\frac{S_a + S_e}{Es}$$

$$+ H_{10}\frac{L_a}{Es} + H_{11}\frac{p}{E} + H_{12}\,\alpha\tau_e - H_{13}\,\alpha\tau_a$$

$$\frac{\zeta_e}{r} = \frac{\zeta_a}{r} - K_1\chi_a - K_2\frac{M_a}{Esr} - K_3\frac{M_e}{Esr} + K_4\frac{S_a + S_e}{Es}$$

$$+ K_5\frac{L_a}{Es} + K_6\frac{p}{E} - K_7\frac{w_a}{r} + K_8\,\alpha\tau_m$$

b. Concave circular ring with $\phi_e < \phi_a$

$$\frac{S_e}{Es} = (F_1 + F_2)\frac{S_a}{Es} + F_3\frac{w_a}{r} - F_4\chi_a + F_5\frac{p}{E} - F_7\frac{L_a}{Es}$$

$$\frac{M_e}{Esr} = G_o\frac{M_a}{Esr} + (G_1 + G_2)\frac{S_a}{Es} + G_3\frac{w_a}{r} - G_4\chi_a$$

$$+ G_6\frac{p}{E} - (G_7 - G_8)\frac{L_a}{Es}$$

$$\chi_e = \chi_a + H_1\frac{M_a + M_e}{Esr} - H_8\,\alpha\tau_e + H_9\,\alpha\tau_a$$

$$\frac{w_e}{r} = H_2\frac{w_a}{r} - H_3\chi_a - H_4\frac{M_a}{Esr} - H_5\frac{M_e}{Esr} + H_6\frac{S_a + S_e}{Es}$$

$$- H_{10}\frac{L_a}{Es} - H_{11}\frac{p}{E} + H_{12}\,\alpha\tau_e - H_{13}\,\alpha\tau_a$$

$$\frac{\zeta_e}{r} = \frac{\zeta_a}{r} + K_1 \chi_a + K_2 \frac{M_a}{Esr} + K_3 \frac{M_e}{Esr} + K_4 \frac{S_a + S_e}{Es}$$

$$- K_5 \frac{L_a}{Es} - K_6 \frac{p}{E} - K_7 \frac{w_a}{r} + K_8 \alpha \tau_m$$

c. Convex circular ring with $\phi_e < \phi_a$

$$\frac{S_e}{Es} = (F_1 - F_2) \frac{S_a}{Es} - F_3 \frac{w_a}{r} + F_4 \chi_a + F_5 \frac{p}{E} - F_7 \frac{L_a}{Es}$$

$$\frac{M_e}{Esr} = G_o \frac{M_a}{Esr} + (G_1 - G_2) \frac{S_a}{Es} - G_3 \frac{w_a}{r} + G_4 \chi_a$$

$$+ G_6 \frac{p}{E} - (G_7 + G_8) \frac{L_a}{Es} \quad .$$

$$\chi_e = \chi_a - H_1 \frac{M_a + M_e}{Esr} - H_8 \alpha \tau_e + H_9 \alpha \tau_a$$

$$* \quad \frac{w_e}{r} = H_2 \frac{w_a}{r} - H_3 \chi_a + H_4 \frac{M_a}{Esr} + H_5 \frac{M_e}{Esr} - H_6 \frac{S_a + S_e}{Es}$$

$$- H_{10} \frac{L_a}{Es} - H_{11} \frac{p}{E} + H_{12} \alpha \tau_e - H_{13} \alpha \tau_a$$

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*The sign of the coefficient H_6 is wrong in the original paper.

$$\frac{\ell_e}{r} = \frac{\ell_a}{r} - K_1\chi_a + K_2\frac{M_a}{Esr} + K_3\frac{M_e}{Esr} + K_4\frac{S_a + S_e}{Es}$$

$$+ K_5\frac{L_a}{Es} + K_6\frac{p}{E} + K_7\frac{w_a}{r} + K_8\,\alpha\tau_m$$

d. Concave circular ring with $\phi_e > \phi_a$

$$\frac{S_e}{Es} = (F_1 - F_2)\frac{S_a}{Es} - F_3\frac{w_a}{r} - F_4\chi_a - F_5\frac{p}{E} + F_7\frac{L_a}{Es}$$

$$\frac{M_e}{Esr} = G_o\frac{M_a}{Esr} - (G_1 - G_2)\frac{S_a}{Es} + G_3\frac{w_a}{r} + G_4\chi_a$$

$$+ G_6\frac{p}{E} - (G_7 + G_8)\frac{L_a}{Es}$$

$$\chi_e = \chi_a - H_1\frac{M_a + M_e}{Esr} + H_8\,\alpha\tau_e - H_9\,\alpha\tau_a$$

$$\frac{w_e}{r} = H_2\frac{w_a}{r} + H_3\chi_a - H_4\frac{M_a}{Esr} - H_5\frac{M_e}{Esr}$$

$$- H_6\frac{S_a + S_e}{Es} + H_{10}\frac{L_a}{Es} + H_{11}\frac{p}{E} + H_{12}\,\alpha\tau_e - H_{13}\,\alpha\tau_a$$

$$\frac{\zeta_e}{r} = \frac{\zeta_a}{r} + K_1\gamma_a - K_2 \frac{M_a}{Esr} - K_3 \frac{M_e}{Esr} + K_4 \frac{S_a + S_e}{Es}$$

$$- K_5 \frac{L_a}{Es} - K_6 \frac{p}{E} + K_7 \frac{w_a}{r} + K_8 \alpha \tau_m$$

In these equations*:

$$F_1 = \frac{\rho_a}{\rho_e} \qquad F_4 = F_3(\cos \phi_a - \cos \phi_m)$$

$$F_2 = \frac{\nu r}{\rho_e} \Delta\phi \cos \phi_a \qquad F_5 = \frac{r_i \rho_{im}}{s\rho_e} \Delta\phi \sin \phi_m$$

$$F_3 = \frac{r^2}{\rho_m \rho_e} \Delta\phi \qquad F_7 = \frac{\nu r}{\rho_e} \Delta\phi \sin \phi_a$$

$$G_o = F_1 \qquad G_4 = F_3 \frac{\Delta\phi^2}{6} \sin^2 \phi_m$$

$$G_1 = F_1 \Delta\phi \sin \phi_m \qquad G_6 = F_5 \frac{\Delta\phi}{2 \sin \phi_m}$$

$$G_2 = F_2(\cos \phi_m - \cos \phi_e) \qquad G_7 = F_7(\cos \phi_m - \cos \phi_e)$$

$$G_3 = F_3(\cos \phi_m - \cos \phi_e) \qquad G_8 = F_1 \Delta\phi \cos \phi_m$$

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*The constants do not have consecutive subscripts so that the equations here more nearly become an integral part of Esslinger's work.

$$H_1 = \frac{r^2}{s^2} 6\Delta\phi^2$$

$$H_2 = 1 - \frac{\nu r}{\rho_a} \Delta\phi \cos \phi_m$$

$$H_3 = \Delta\phi \sin \phi_m$$

$$H_4 = \frac{r^2}{s^2} 2 \Delta\phi^2 (2 \sin \phi_m + \frac{\Delta\phi}{4} \cos \phi_m)$$

$$H_5 = \frac{r^2}{s^2} 2 \Delta\phi^2 (\sin \phi_m + \frac{\Delta\phi}{4} \cos \phi_m)$$

$$H_6 = \frac{1 - \nu^2}{2} \Delta\phi \cos^2 \phi_m$$

$$H_8 = \frac{\rho_e}{r\Delta\phi}$$

$$H_9 = \frac{\rho_a}{r\Delta\phi}$$

$$H_{10} = (1 - \nu^2) \frac{\rho_m}{\rho_e} \Delta\phi \sin \phi_m \cos \phi_m$$

$$H_{11} = F_5 (1 - \nu^2) \Delta\phi \cos \phi_m$$

$$H_{12} = \frac{\rho_e}{r}$$

$$H_{13} = \frac{\rho_a}{r}$$

$$K_1 = \Delta\phi \cos \phi_m$$

$$K_2 = \frac{r^2}{s^2} 2\Delta\phi^2 (2 \cos \phi_m - \frac{\Delta\phi}{4} \sin \phi_m)$$

$$K_3 = \frac{r^2}{s^2} 2\Delta\phi^2 (\cos \phi_m - \frac{\Delta\phi}{4} \sin \phi_m)$$

$$K_4 = H_{10} \frac{\rho_e}{2\rho_m}$$

$$K_5 = H_{10} \tan \phi_m$$

$$K_6 = H_{11} \tan \phi_m$$

$$K_7 = H_3 \frac{\nu r}{\rho_a}$$

$$K_8 = H_3$$

REFERENCES

1. Maria Esslinger, Statische Berechnung von Kesselboden; Julius Springer, Berlin, Germany, 1952, pp 25-35.

English Translation:

Static Calculation of Boiler Bottoms, ORNL CF 56-12-37 (August 22, 1957) pp 31-42.

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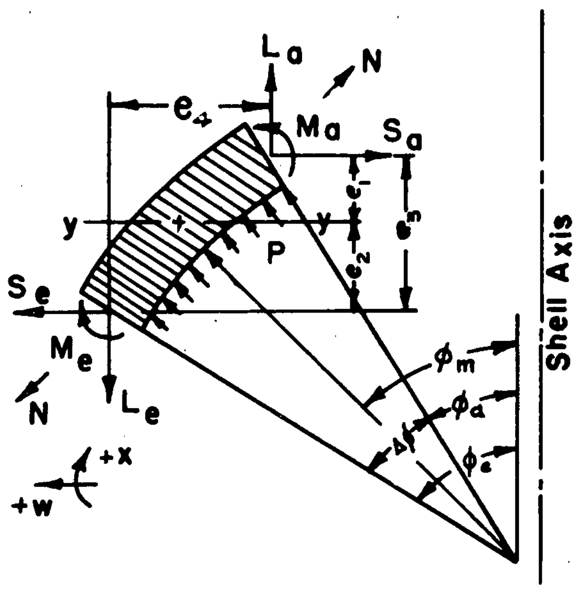
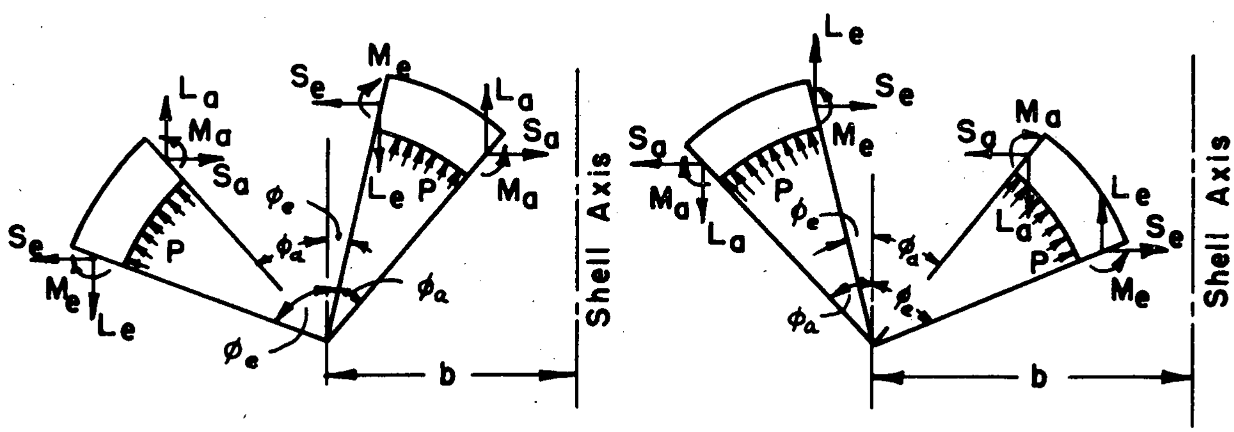


GRADATION SECTION
FIGURE 1



(a) Convex $\phi_e > \phi_a$

(b) Concave $\phi_e < \phi_a$

(c) Convex $\phi_e < \phi_a$

(d) Concave $\phi_e > \phi_a$

GRADATION SECTION CATEGORIES
FIGURE 2