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APPLICATIONS OF CHIRAL PERTURBATION THEORY: MASS FORMULAS AND THE DECAY $\eta \rightarrow 3\pi$ *

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Abstract

Several new results on the breaking of chiral $SU(3) \times SU(3)$ are presented within the theoretical framework of chiral perturbation theory.

(a) The leading order corrections to the Gell-Mann-Okubo formula for the baryon octet are shown to be of order $\epsilon^{3/2}$, where ϵ is a chiral breaking parameter. An explicit exact expression is given for the leading order corrections which provides a new development in understanding why this formula works so well. Similarly the corrections to the Gell-Mann-Okubo formula for the ground state pseudoscalar octet are shown to be of order $\epsilon^2 \ln \epsilon$ (including η - η' mixing). On the basis of these exact results it is argued that $SU(3) \times SU(3)$ symmetry is as good as $SU(3)$ symmetry $\sim 30\%$ except when one considers electromagnetic interactions. (b) We examine the $\eta \rightarrow 3\pi$ decay on the assumption that it is regulated by a non-electromagnetic isospin violating term of the type $\epsilon_3 u_3$ with u_3 a member of $\bar{3}3 \oplus 3\bar{3}$. The strength ϵ_3 of this term is related to the experimental rate including all leading order chiral symmetry corrections. This estimate of ϵ_3 leads to $\Delta I=1$ hadron level shifts about a factor of 2 or 3 too large, although our estimate of ϵ_3 depends sensitively on the experimental details. (c) Octet enhancement, an exact formalism to describe η - η' mixing and other topics are discussed.

I. Introduction

The ideas of current algebra, partial conservation of axial vector current, and the approximate SU(3) classification of hadronic states were elegantly unified by Gell-Mann, Oakes, and Renner¹ and by Glashow and Weinberg². They suggested that the strong interaction Hamiltonian could be written as

$$H = H_0 + \epsilon_0^0 u_0^0 + \epsilon_8^0 u_8^0, \quad (1.1)$$

where H_0 is invariant under the group SU(3) x SU(3), $\epsilon_0^0 u_0^0$ breaks SU(3)xSU(3) but preserves SU(3), and $\epsilon_8^0 u_8^0$ breaks both. The unrenormalized operators u_0^0 and u_8^0 are assumed to transform as members of the $(\bar{3}, 3) + (3, \bar{3})$ representation (which contains scalar operators u_i^0 , $i=0, \dots, 8$ and pseudoscalar operators ϕ_i^0 , $i=0, \dots, 8$), in analogy with the bare mass terms in a quark model.

It was further suggested that, when the explicit symmetry breaking is turned off, the symmetry is spontaneously broken so that the vacuum (and hence the spectrum of physical states) is only SU(3) invariant. This is implemented by the assumption $\langle u_0^0 \rangle \neq 0$ and $\langle u_8^0 \rangle = 0$ for ϵ_0^0 and $\epsilon_8^0 = 0$.

Associated with this spontaneous symmetry breaking would be an octet of massless Nambu-Goldstone bosons, identified as the π , K, and η mesons.

Gell-Mann, Oakes, and Renner then suggested that the explicit symmetry breaking could be turned on in two steps. In the first step the parameter $c \equiv \epsilon_8^0 / \epsilon_0^0$ is held fixed at $c = -\sqrt{2}$. (This corresponds to giving only the strange quark a bare mass). The term $\epsilon_0^0 (u_0^0 - \sqrt{2} u_8^0)$ breaks the symmetry down to SU(2)xSU(2). The K and η mesons are therefore given small masses, and the degeneracy of the SU(3) multiplets is broken. In the second step,

c is allowed to deviate slightly from $-\sqrt{2}$. This breaks the $SU(2) \times SU(2)$ symmetry, giving the pions a very small mass. These low mass mesons dominate the axial current divergences at low momentum transfer (P.C.A.C.).

The unrenormalized explicit symmetry breaking parameters ϵ_0^0 and ϵ_8^0 are related to the pseudoscalar masses by

$$\epsilon_0^0 = -\frac{1}{\sqrt{6}} \left(\frac{2f_K \mu_K^2}{Z_K^{1/2}} + \frac{f_\pi \mu_\pi^2}{Z_\pi^{1/2}} \right) \quad (1.2)$$

$$\epsilon_8^0 = \frac{2}{\sqrt{3}} \left(\frac{f_K \mu_K^2}{Z_K^{1/2}} - \frac{f_\pi \mu_\pi^2}{Z_\pi^{1/2}} \right),$$

where f_K and f_π are the kaon and pion decay constants ($f_\pi \sim 93$ MeV) and $Z_K^{1/2}$ and $Z_\pi^{1/2}$ are their wave function renormalization constants. If one approximates $Z_K \approx Z_\pi$ and $f_K \approx f_\pi$ (which corresponds to neglecting $\langle u_8^0 \rangle_0$, the $SU(3)$ breaking in the vacuum), then one finds $c \approx -1.25$, close to the $SU(2) \times SU(2)$ value.

For later convenience we define renormalized symmetry breaking parameters ϵ_0 and ϵ_8 by

$$\frac{\epsilon_0}{f_\pi} = -\frac{1}{\sqrt{6}} (2\mu_K^2 + \mu_\pi^2) = -0.21 \text{ GeV}^2 \quad (1.3)$$

$$\frac{\epsilon_8}{f_\pi} = \frac{2}{\sqrt{3}} (\mu_K^2 - \mu_\pi^2) = 0.26 \text{ GeV}^2$$

The formalism of the $(\bar{3}, 3) + (3, \bar{3})$ model is more thoroughly discussed in Ref. (4), which we shall refer to as I.

We take the viewpoint that $SU(2) \times SU(2)$ is accurate⁴ to around 7%, making it by far the best symmetry of the strong Hamiltonian after isospin. This can be seen roughly by the smallness of μ_π^2 compared to other hadronic masses. Other tests of $SU(2) \times SU(2)$, as well as of the $(\bar{3}, 3) + (3, \bar{3})$ model of $SU(3) \times SU(3)$ breaking, are reviewed in Ref. (5).

One of the aims of this paper is to consider how good a symmetry $SU(3) \times SU(3)$ really is. It is not enough to know the values of ϵ_0 and ϵ_8 ; one must also know how to implement the chiral symmetry and how to make quantitative estimates of the dependence of physical quantities on the symmetry breaking terms in (1.1).

There have been two major approaches to this problem. The first approach, pioneered by Glashow and Weinberg,² is based upon the idea of smoothness. One writes down Ward identities for the Green's functions of the theory, saturates the two point functions with meson poles or resonances, and then assumes that the remaining quantities are "smooth" in their momentum dependence. The difficulty with this approach is that there is no way to determine the reliability of the approximations. One cannot argue, for example, that the dominance of the two point function $\langle T(u_i^0 u_j^0) \rangle_0$ by a scalar resonance would become exact in the chiral limit or that the errors are of any definite order in ϵ_0 and ϵ_8 .

The difficulty in controlling the extrapolations explicit in this technique may be seen by the differing results in the literature due to different methods of implementing "smoothness".

The other major approach, stressed by Dashen⁶ and by Dashen and Weinstein⁷, is to do a perturbation expansion in the parameters ϵ_0^0 and

ϵ_8^0 . Dashen and Weinstein showed that pion pole dominance (P.C.A.C.) would become exact in the chiral symmetry limit. They argued that low energy theorems can be interpreted as exact statements concerning the threshold behavior of massless (on shell) Goldstone bosons in an $SU(2) \times SU(2)$ or $SU(3) \times SU(3)$ symmetric world. Higher order terms in ϵ_0^0 and ϵ_8^0 would be corrections to the low energy theorems. Hence PCAC is a consequence of symmetry.

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It was later pointed out by Li and Pagels⁸ that a perturbation expansion in ϵ_0^0 and ϵ_8^0 must contain non-analytic terms. This is due to the fact that Goldstone bosons become massless in the chiral limit, producing a long range component in the strong interactions. Because of the Adler zero the associated infrared singularities are usually finite, but they lead to nonanalytic terms such as $\epsilon \ln \epsilon$ or $\epsilon^{1/2}$ in the expansion. These nonanalytic terms are often the leading corrections to the chiral limit, and because they are due to a finite number of diagrams, they can usually be calculated exactly without assuming any knowledge of the structure of the symmetric part of the Hamiltonian, H_0 .

One can therefore make quantitative statements about the dependence of physical quantities on ϵ_0 and ϵ_8 . For example, one can prove⁹ the model independent result

$$\begin{aligned} \frac{f_K}{f_\pi} - 1 &= \frac{3(\mu_K^2 - \mu_\pi^2)}{64\pi^2 f_\pi^2} \ln \frac{\Lambda^2}{4\mu^2} + O(\epsilon_8) \\ \frac{f_\eta}{f_\pi} - 1 &= \frac{4}{3} \left(\frac{f_K}{f_\pi} - 1 \right), \end{aligned} \quad (1.4)$$

where $\mu^2 \approx 0.17 \text{ GeV}^2$ is the average pseudoscalar mass, and Λ^2 is some cutoff (e.g. $4m_N^2$). Changing Λ merely changes the $O(\epsilon_8)$ part of (1.4). For $\ln \Lambda^2 / 4\mu^2 \approx 2$, (1.4) predicts $f_K/f_\pi \approx 1.22$, as compared with the experimental value 1.26 ± 0.02 . Another result⁴ (which depends on the $(\bar{3}, 3) + (3, \bar{3})$ model) is

$$\frac{Z_K^{1/2}}{Z_\pi^{1/2}} - 1 = \frac{(\mu_K^2 - \mu_\pi^2)}{192\pi^2 f_\pi^2} \ln \frac{\Lambda^2}{4\mu^2} + O(\epsilon_8) \approx 0.025 \quad (1-5)$$

$$\frac{Z_\eta^{1/2}}{Z_\pi^{1/2}} - 1 = \frac{4}{3} \left(\frac{Z_K^{1/2}}{Z_\pi^{1/2}} - 1 \right).$$

One may utilize (1.4) to eliminate the logarithm, yielding

$$\frac{Z_K^{1/2}}{Z_\pi^{1/2}} - 1 = \frac{1}{9} \left(\frac{f_K}{f_\pi} - 1 \right) + O(\epsilon_8). \quad (1.6)$$

In various other applications the singularity is $O(1/\epsilon)$, $O(\epsilon'^3)$, or $O(\epsilon^{3/2})$.

The arbitrary cutoff Λ that appears for the logarithmic singularities is absent in these cases so the scaling ambiguity is not present.

In I we developed a set of techniques to enable one to determine the leading nonanalytic terms in S-matrix elements and other quantities, including such technical difficulties as renormalization and mass shell constraints. A fairly complete list of applications of these ideas may be found there.⁽¹⁰⁾

These nonanalytic terms are frequently the formally leading terms. Whether they are numerically dominant over the higher order analytic corrections (which cannot be calculated without a detailed knowledge of H_0) is an open question. We make the optimistic assumption that they do in fact dominate in most cases.⁽¹¹⁾ However, our attitude is to use these calculations more as an estimate of how good the zero order (chiral symmetric) terms are rather than as a numerically

accurate estimate of the perturbations. These remarks apply particularly to those cases for which the leading correction has a logarithm. For those cases for which the leading term is $O(\epsilon^{1/2})$ or $O(\epsilon^{3/2})$ we have more confidence.

The nonanalytic terms are usually due to the divergence of a dispersion integral at the lower limit of some two-meson cut. In those applications in which the leading terms are analytic, one cannot prove any exact theorems, but one might hope that the dispersion integrals are still approximately dominated by the thresholds of the two-meson channels. Li and Pagels applied this idea to the meson and baryon mass differences ⁽¹²⁾ and found that octet enhancement for matrix elements that vanish if SU(3) is exact emerged naturally. They reproduced all of the tadpole model results of Coleman and Glashow, ⁽¹³⁾ plus an additional result, and gave a possible solution to the puzzle raised by Dashen ⁽⁶⁾ of how to understand octet enhancement in the chiral symmetry context.

In Chapter II of this paper we will present some new results on nonanalytic terms including the leading corrections to the meson Gell-Mann Okubo (G-M-O) formula. For this we discuss η - η' mixing and the existence of a mixing angle. Also considered are matrix elements of the u_3^0 operators between single meson states, and the $\langle 3\pi | u_3 | n \rangle$ matrix element. When these results are combined with old results a rather definite picture emerges: SU(3)xSU(3) is reliable to within 30% except when computing matrix elements of the effective electromagnetic Hamiltonian. Its accuracy is comparable to the accuracy of SU(3). ⁽¹⁾

We then generalize the Goldstone boson pair mechanism of Ref.(12). We find that from $m_N - m_\Sigma$ and the meson masses, all of the other baryon medium strong and $I=1$ mass differences can be roughly predicted, supporting the idea of threshold dominance.

It is well known that in the current algebra approach to symmetry breaking the corrections to the G-M-O formula for baryons can be represented as a continuum integral. However one can not achieve quantitative control on the integral. From the

framework of chiral perturbation theory, however, the corrections can be shown to be of order $\epsilon^{3/2}$ and hence one may calculate the leading order term exactly. We obtain the exact result

$$\frac{3}{4}m_\Lambda + \frac{1}{4}m_\Sigma - \frac{1}{2}(m_N + m_E) = \frac{g_A^2(\mu_K^2 - \mu_\pi^2)(\mu_\eta - \mu_\pi)}{192\pi f_\pi^2} (3 - 6\alpha + 2\alpha^2) \quad (1.7)$$

where $(f/d)_A = (1-\alpha)/\alpha$ is the (f/d) ratio for axial-vector baryon 8 coupling. The right hand side is very small for the experimental value of $\alpha \approx 2/3$ but could be large for other values, suggestive that the observed octet enhancement is a consequence of the dynamics of the symmetric world or a higher symmetry which determines α .

In Chapter III we consider the $\eta \rightarrow 3\pi$ decay. It has long been known that the decay width is much too large to be electromagnetic in origin. This is still true in spite of the new Cornell experiment⁽¹⁴⁾ which has reduced the width by a factor of three. The theoretical problem can be summarized by Sutherland's theorem⁽¹⁵⁾, which states that the electromagnetic amplitude vanishes in the $SU(2) \times SU(2)$ limit.

It has often been suggested^(16,17) that the decay might be explained by adding a small isospin violating term $\epsilon_3^0 u_3^0$ to the strong Hamiltonian (1.1). The relevant matrix element of u_3^0 does not vanish in the $SU(2) \times SU(2)$ limit. In Chapter III we make a careful study of whether such a scheme can, in fact, work (in the $(\bar{3},3) + (3,\bar{3})$ model). The matrix element $\langle 3\pi | u_3^0 | n \rangle$ is calculated in an $SU(3) \times SU(3)$ expansion up to $O(\epsilon \ln \epsilon)$ and is found to be reliable to $\sim 30\%$. The value of ϵ_3 needed to explain the decay is determined. This value implies an $\epsilon_3 u_3$ contribution to the kaon and baryon $I=1$ mass shifts that is a factor two or three too large to be easily believable. However, the result is very sensitive to the experimental η decay parameters (the combined experimental and theoretical uncertainty in our determination of ϵ_3 is $\gtrsim 50\%$), so the u_3 explanation (perhaps with a different representation for u_3) cannot be definitely ruled out.

In our conclusion we suggest some other possible applications of chiral perturbation theory, and in a technical appendix we describe an exact formalism to treat η - η' mixing.

II. Mass Formulas and the Accuracy of $SU(3) \times SU(3)$.

(A) Nonanalytic Terms

In this section we will present a few new applications of chiral perturbation theory and mention some old ones. Our main goal is to get an idea of the accuracy of $SU(3) \times SU(3)$. Methods of derivation may be found in I and in Appendices A and B. We list results in decreasing order of singularity.

a) $[O(1/\epsilon_0)]$ We have shown⁽¹⁸⁾ that the leading renormalization of the (zero momentum transfer) form factors of the K^+ vector currents is of order ϵ_8^2/ϵ_0 and exactly computable. The result for meson form factors (including a generalization due to Wada⁽¹⁸⁾) is

$$f_+(0) = 1 - \frac{\mu_K^2}{64\pi^2 f_\pi^2} \left[\frac{5}{2} - 6 \ln \frac{4}{3} \right] + O(\epsilon^{3/2}) \approx 0.97 \quad (2.1)$$

where we have set the pion mass equal to zero for simplicity. For baryons, the renormalization is less than 10%.

b) $[O(1/\epsilon_0^{1/2})]$ The leading corrections to the baryon Gell-Mann Okubo formula are discussed later in this chapter. They are of order $\epsilon_8^2/\epsilon_0^{1/2}$ and represent a 3 MeV correction to the G-M-O formula.

c) $[O(\epsilon \ln \epsilon)]$ The meson decay constants and renormalization constants are renormalized⁽⁴⁾ from their $SU(3) \times SU(3)$ values by 10-35% (see Appendix A). The ratios f_K/f_π and $Z_K^{1/2}/Z_\pi^{1/2}$ (eqns. (1.4), (1.5)) are 1.22 and 1.025.

The parameter $c = \epsilon_8^0/\epsilon_0^0$ (see Eqn. (1.2)) is often estimated to be -1.25. Cornwall⁽¹⁹⁾ has computed the leading corrections, yielding $c = -1.29$, a 3% effect. This result can be easily rederived from (1.4) and (1.5).

$$C = \frac{\varepsilon_8^0}{\varepsilon_0^0} = -\sqrt{2} \left[\frac{1 - a\mu_\pi^2/\mu_K^2}{1 + a\mu_\pi^2/2\mu_K^2} \right] \quad (2.2)$$

$$a = \frac{\int_\pi z_K^{1/2}}{\int_K z_\pi^{1/2}} = 1 - \frac{8}{9} \left(\frac{f_K}{f_\pi} - 1 \right) + O(\varepsilon_8) \simeq 0.805$$

Now, consider the Gell-Mann Okubo (G-M-O) formula⁽²⁰⁾ for the pseudoscalar octet. In addition to the defining equations (1.2) there is a third relation between $\varepsilon_0^0, \varepsilon_8^0$ and physical quantities (Appendix A):

$$\sqrt{\frac{2}{3}} \varepsilon_0^0 - \frac{1}{\sqrt{3}} \varepsilon_8^0 = -\frac{\mu_\eta^2 f_\eta}{z_{8\eta}^{1/2}} - \sqrt{\frac{2}{3}} \varepsilon_8^0 \frac{z_{0\eta}^{1/2}}{z_{8\eta}^{1/2}} \quad (2.3)$$

where $z_{i\eta}^{1/2} \equiv \langle 0 | \phi_i^0 | \eta \rangle$, $i=0,8$. (The quantity $z_\eta^{1/2}$ in (1.5) is the same as $z_{8\eta}^{1/2}$).

Combining (1.2) and (2.3) one can prove the following relation, which is exact in the Gell-Mann, Oakes, Renner Model:

$$\frac{3}{4} \left[\frac{f_\eta \mu_\eta^2}{z_{8\eta}^{1/2}} + \sqrt{\frac{2}{3}} \frac{\varepsilon_8^0 z_{0\eta}^{1/2}}{z_{8\eta}^{1/2}} \right] + \frac{f_\pi \mu_\pi^2}{4 z_\pi^{1/2}} - \frac{f_K \mu_K^2}{z_K^{1/2}} = 0 \quad (2.4)$$

The term involving $\varepsilon_8^0 z_{0\eta}^{1/2}$ represents the η - η' mixing effect. In Appendix A, $\varepsilon_8^0 z_{0\eta}^{1/2}$ is calculated and shown to be of $O(\varepsilon_8^2 \ln \varepsilon_0)$. If one sets $f_K = f_\pi = f_\eta$, etc, in (2.4) one obtains the G-M-O formula:

$$\frac{3}{4} \mu_\eta^2 + \frac{1}{4} \mu_\pi^2 - \mu_K^2 = 0 \quad (2.5)$$

This relation works quite well. The left hand side is -0.017 GeV^2 , as compared with the average pseudoscalar mass, $\mu^2 = 0.17 \text{ GeV}^2$.

The leading corrections to (2.5) can be determined using (1.4), (1.5) and the results on $\varepsilon_8^0 z_{0\eta}^{1/2}$:

$$\frac{3}{4} \mu_\eta^2 + \frac{1}{4} \mu_\pi^2 - \mu_K^2 - \frac{\varepsilon_8 \ln(\Lambda^2/4\mu^2)}{96\pi^2 f_\pi} \left[\frac{5\varepsilon_8}{2f_\pi} - \sqrt{3} \mu_\eta^2 + \frac{\mu_\pi^2}{\sqrt{3}} + \frac{2\mu_K^2}{\sqrt{3}} \right] = 0 + O(\varepsilon_8^2) \quad (2.6)$$

The first term in the brackets is due to η - η' mixing; the other terms are from the ratios of the f 's and Z 's (the vacuum symmetry breaking). Putting in numbers, we find

$$(-0.017 - 0.038 + 0.013) \text{GeV}^2 = -0.042 \text{GeV}^2 = O(\epsilon_8^2) \quad (2.7)$$

where -0.017 is the G-M-O combination, -0.038 is from the mixing, and 0.013 is from the vacuum symmetry breaking.

The corrections have made the formula worse, although (2.5) is still small compared to $\mu^2 = 0.17 \text{GeV}^2$. What could be the source of the difficulty? It could be an indication that the $(\bar{3}, 3) + (3, \bar{3})$ model (or the whole chiral symmetry scheme) is not correct. Our suspicion, however, is that in this application the $\epsilon \ln \epsilon$ terms are not correctly estimating $Z_{\eta\eta'}^{1/2}$.

We would like to comment on the standard mixing angle formula

$$\frac{3}{4}(\mu_{\eta'}^2 \cos^2 \theta + \mu_{\eta}^2 \sin^2 \theta) + \frac{1}{4}\mu_{\pi}^2 - \mu_K^2 = O(\epsilon_8^2) \quad (2.8)$$

Can such a formula (which has no predictive power) be theoretically justified? One can certainly define θ as the angle which satisfies (2.8). However, it is shown in Appendix A that θ can only be interpreted as the angle which rotates the η' and η states into "singlet" and "octet" states if one turns off all strong interactions. This interpretation cannot be justified by dominating any dispersion relation by a pole or by keeping the leading symmetry breaking terms. In other words, a mixing angle is appropriate for the bare mass terms in a Lagrangian, but not for physical hadron states.

Even if one does ignore the strong interactions and introduce a mixing angle in terms of a rotation of the η' and η one can only obtain (2.8) by ignoring vacuum symmetry breaking. Finally, G-M-O is supposed to be correct only to order ϵ_8 , so there is no justification for keeping the term $\sin^2 \theta \mu_{\eta'}^2$ in (2.8) which

is of order ϵ_8^2 .

In Chapter III we will need the matrix element of u_3^0 between single meson states. Up to $O(\epsilon \ln \epsilon)$, the perturbation of the matrix elements around $SU(3) \times SU(3)$ are

$$\begin{aligned} \langle K^+ | u_3^0 | K^+ \rangle &= - \langle K^0 | u_3^0 | K^0 \rangle \\ &= - \frac{Z_S^{1/2}(0)}{2f_K(\epsilon)} \left[1 - \frac{\sqrt{3} \epsilon_8 \ln(\Lambda^2/4\mu^2)}{32\pi^2 f_\pi^3} - \frac{7}{12\sqrt{3}} \frac{\epsilon_8 \ln(\Lambda^2/4\mu^2)}{32\pi^2 f_\pi^3} \right] \quad (2.9) \\ &= - \frac{Z_S^{1/2}(0)}{2f_K(\epsilon)} (1.06) = - \frac{Z_S^{1/2}(0)}{2f_\pi(\epsilon)} (0.87) \end{aligned}$$

where $f_K(\epsilon)$ and $f_\pi(\epsilon)$ are the physical values of the decay constants, and $Z_S^{1/2}(0)$ is the value of $Z_\pi^{1/2}$, Z_K , and $Z_{8\eta}$ in the $SU(3) \times SU(3)$ limit. Similarly,

$$\begin{aligned} \langle \pi^0 | u_3^0 | \eta \rangle &= - \frac{Z_S^{1/2}(0)}{\sqrt{3}f_\eta(\epsilon)} \left[1 - \frac{\sqrt{3} \epsilon_8 \ln(\Lambda^2/4\mu^2)}{32\pi^2 f_\pi^3} - \frac{11}{6\sqrt{3}} \frac{\epsilon_8 \ln(\Lambda^2/4\mu^2)}{32\pi^2 f_\pi^3} \right] \quad (2.10) \\ &= - \frac{Z_S^{1/2}(0)}{\sqrt{3}f_\eta(\epsilon)} (0.94) = - \frac{Z_S^{1/2}(0)}{\sqrt{3}f_\pi(\epsilon)} (0.73) \end{aligned}$$

We see that these matrix elements are changed by less than 30% by the leading symmetry breaking terms. Formulas (2.9) and (2.10) are exact to leading order. They incorporate all effects, including internal dynamics, mass shell constraints, renormalization, and vacuum symmetry breaking.

In Chapter III and Appendix B we consider a still more complicated matrix element: $\langle 3\pi | u_3^0 | \eta \rangle$. The leading corrections to its $SU(3) \times SU(3)$ symmetric value are $\sim 34\%$.

Finally, we mention the one known example in which the perturbation expansion completely breaks down. Dashen⁽⁶⁾ has shown that in the $SU(3) \times SU(3)$ limit the electromagnetic contributions to $\mu_K^2 + \mu_K^2$ and $\mu_\pi^2 + \mu_\pi^2$ are equal. That is, the ratio of the octet part of the effective electromagnetic Hamiltonian ($\mu_K^2 + \mu_K^2$)

to the 27-dimensional part. $(\mu_{\pi}^2 + \mu_{\pi^0}^2)$ becomes a Clebsch-Gordan coefficient (unity) in the $SU(3) \times SU(3)$ limit. Hence, the tadpole mechanism⁽¹³⁾ (tadpole is used in its original meaning: a dynamical enhancement of the octet part of H_{em}) cannot occur in the chiral symmetry limit.

Since, in fact, octet enhancement does occur ($\mu_K^2 + \mu_{K^0}^2 = -0.004 \text{ GeV}^2$ while $\mu_{\pi}^2 + \mu_{\pi^0}^2 = +0.0013 \text{ GeV}^2$), Dashen concluded that the perturbation expansion must break down in this application.

This was shown explicitly in Ref.(21): the leading $\epsilon \ln \epsilon$ corrections to the electromagnetic mass shifts were computed and shown to be larger than the leading term (The large corrections were due to the internal meson loops, not the off shell extrapolations of the external particles). Hence the $SU(3) \times SU(3)$ perturbation expansion breaks down in this application⁽²²⁾, and the Dashen theorem does not correctly estimate the electromagnetic mass shifts.

The meson mass shifts therefore provide no evidence for or against the existence of an explicit $\epsilon_3 u_3$ term in the Hamiltonian. The origin of octet enhancement is left unexplained.

The conclusion to be drawn from all of this is that regardless of the order of the leading corrections, $SU(3) \times SU(3)$ is reliable in almost all cases to around 30% or better, making its accuracy comparable to that of $SU(3)$. The only exception seems to be the matrix elements of the time-ordered product of two currents.

(B) Octet Enhancement

We have seen that octet enhancement cannot be explained as a chiral limit theorem. Li and Pagels have shown^(9,12) that octet enhancement for quantities which vanish in the $SU(3)$ symmetry limit emerges naturally from the assumption of the Goldstone boson pair mechanism. This assumption is that the dispersion integrals for $\langle a | \lambda \cdot V | b \rangle$ (V is a vector current) are dominated by the thresholds of the two meson cuts. For mesons, the G-M-O relation (2.5) and the (octet)

Coleman-Glashow formula

$$\sqrt{3} \Delta_{\pi^0 \eta} = \mu_{K^+}^2 - \mu_{K^0}^2 \quad (2.11)$$

follow without any assumption concerning the origin of the symmetry breaking.

Moreover, the 27-dimensional $\mu_{\pi^+}^2 - \mu_{\pi^0}^2$ is predicted to vanish.

When this idea is extended to the baryons, the G-M-O formula, Coleman-Glashow formula, and the tadpole results⁽¹³⁾ all follow. Baryon mass differences are given by formulas like

$$M_a - M_b \approx A(\mu_c^2 - \mu_d^2) \int \frac{4\Lambda^2 dt \sqrt{t}/t}{(\mu_c + \mu_d)^2} \approx 4A(\mu_c^2 - \mu_d^2) \Lambda \quad (2.12)$$

where Λ is a cutoff, A is a constant, and μ_c and μ_d are meson masses. Here we would like to extend these results by retaining the $\epsilon^{3/2}$ terms associated with the lower limit of (2.12). We obtain a one parameter fit to all baryon mass differences. The results are

$$\begin{aligned} M_N - M_\Sigma &= \frac{g_A^2}{192\pi f_\pi^2} (\mu_K^2 - \mu_\pi^2) \left[(2\Lambda - \mu_K - \mu_\pi)(15 - 48\alpha + 28\alpha^2) + (2\Lambda - \mu_K - \mu_\eta)(3 - 8\alpha + 4\alpha^2) \right] \\ M_P - M_n &= \frac{g_A^2}{16\pi f_\pi^2} (\mu_{K^+}^2 - \mu_{K^0}^2) \left[(2\Lambda - 2\mu_K) \frac{1}{6} (3 - 4\alpha^2) + (2\Lambda - \mu_\pi - \mu_\eta) \frac{1}{3} (3 - 4\alpha) \right] \end{aligned} \quad (2.13)$$

and similar

formulas for $m_\Sigma - m_\Xi$, $m_\Lambda - m_\Xi$, $m_\Sigma^+ - m_\Sigma^-$, and $m_\Xi^+ - m_\Xi^0$ which can easily be inferred from Eq.(4.1) of Ref.9. The cutoff Λ is chosen to be 705 MeV, to fit $m_N - m_\Sigma$, g_A is 1.24, and $(f/d)_A = (1-\alpha)/\alpha$ is the f/d ratio for axial-vector-baryon couplings. Experimentally, $\alpha \approx 0.66 \pm 0.02$. Using $\alpha = 2/3$ and the experimental meson masses, the predictions for the baryon mass shifts are given in Table I.

These numbers are in rough agreement with experiment, supporting the view that the two meson thresholds are dominating the dispersion integrals.⁽²³⁾ Several comments are in order: a) These predictions are not exact theorems of the type discussed earlier. b) The results are independent of the origin of SU(3) and SU(2) breaking. c) The numbers are rather sensitive to the value of α used and to the validity of the meson Coleman-Glashow formula (2.11). d) The I=1 predictions probably do not properly include the electromagnetic Born terms (for $m_p - m_n$ the Born term is 0.8 MeV.)

Although the order ϵ_8 contributions to (2.13) are only approximate, the order $\epsilon_8^2/\epsilon_0^{1/2}$ pieces are in fact exact (and independent of the $(\bar{3},3)+(3,\bar{3})$ model). Forming the G-M-O combination we find the leading corrections (which can also be found from the methods of I):

$$\frac{3}{4} M_\Lambda + \frac{1}{4} M_\Sigma - \frac{1}{2} (M_N + M_\Xi) = \frac{g_A^2 (\mu_K^2 - \mu_\pi^2) (\mu_\eta - \mu_\pi)}{192 \pi f_\pi^2} (3 - 6\alpha + 2\alpha^2) + O(\epsilon_8^2) \quad (2.14)$$

The left-hand side is experimentally 7.5 ± 5.0 MeV, while the right hand is $28 \text{ MeV} \times (3 - 6\alpha + 2\alpha^2) = -3.1$ MeV for $\alpha = 2/3$. The correction is tiny and of the wrong sign. It is interesting that $3 - 6\alpha + 2\alpha^2$ vanishes for $\alpha = 0.63$. It has always been puzzling that the G-M-O formula works so well. The leading correction (2.14) is a large number times a function of α which nearly vanishes due to an apparent accident⁽²⁴⁾ of the closeness of α to 0.63. It appears that the success of the octet formula is being governed by the value of α , which can only be understood on the basis of the dynamics of the SU(3) symmetric interactions or a higher symmetry, such as SU(6).

The $\epsilon_8^{3/2}$ terms in (2.13) are almost purely octet and they work fairly well (Table I). It is only the tiny 27-dimensional piece in (2.14) that is of the

wrong sign⁽²⁵⁾ if $\alpha > 0.63$.

The Coleman-Glashow formula

$$M_p - M_n + M_{\Sigma^-} - M_{\Sigma^+} + M_{\Xi^0} - M_{\Xi^-} = 0 \quad (2.15)$$

which is satisfied to within experimental errors, does not have any $\epsilon_8/\epsilon_0^{1/2}$ corrections even though the individual mass differences in (2.13) do. The corrections are of order $\alpha \epsilon_8 \ln \epsilon_0$, (here, α is the fine structure constant).

In summary, $SU(3) \times SU(3)$ seems to work well in most cases. Octet enhancement is predicted from the Goldstone boson pair mechanism but its extreme success in the baryon G-M-O formula seems to be an accident of the value of α . The leading corrections to the G-M-O formulas for both the mesons and the baryons are predicted to be small, but of the wrong sign⁽²⁵⁾.

III. The $\eta \rightarrow 3\pi$ decay

The large experimental widths for the principal decay modes of the η have long been a puzzle. The $\eta \rightarrow 2\gamma$ width was a factor of five greater than expected⁽²⁶⁾ from SU(3) and the π^0 decay rate. Similarly, the $\eta \rightarrow 3\pi$ rates seemed far too large to be compatible with electromagnetism.

A recent Cornell experiment⁽¹⁴⁾ has reduced the 2γ rate by a welcome factor of three.⁽²⁷⁾ Since the 2γ decay is used to determine the total width, the $\eta \rightarrow 3\pi$ rates are also reduced by a factor of three. Nevertheless, the 3π widths are still two orders of magnitude larger than expected from electromagnetism.^(15,17) The basic problem is summarized by Sutherland's theorem:⁽¹⁵⁾ the electromagnetic amplitude vanishes in the $SU(2) \times SU(2)$ limit.

Various authors^(16,17) have suggested that the decay could be explained by adding a small isospin violating term $\epsilon_3^0 u_3^0$ to the strong Hamiltonian (1.1). The decay amplitude due to this term need not vanish in the $SU(2) \times SU(2)$ limit. The u_3^0 term can be thought of as a difference between the bare masses of the nonstrange quarks or as a Coleman-Glashow tadpole.⁽¹³⁾ It could also be the effect of new interactions, such as the neutral gauge bosons used in some gauge theories to cancel the divergences associated with electromagnetism.

In order to predict the η decay rate one must know ϵ_3^0 and must compute the matrix element $\langle 3\pi | u_3^0 | \eta \rangle$.

A typical approach⁽¹⁷⁾ has been to estimate ϵ_3^0 from the observed mass difference $\mu_{K^+}^2 - \mu_{K^0}^2$, assuming that the purely electromagnetic part of the splitting is given by Dashen's theorem⁽⁶⁾ (see Chapter II). As we have shown⁽²¹⁾ that Dashen's theorem does not correctly estimate the electromagnetic mass difference, this approach must be abandoned.

The calculation of $\langle 3\pi | u_3^0 | \eta \rangle$ involves going to the $SU(3) \times SU(3)$ limit and possesses some subtleties.⁽²⁸⁾ In Section B we will compute this matrix

element (in the $(\bar{3},3) + (3,\bar{3})$ model) in an $SU(3) \times SU(3)$ expansion correct up to $O(\epsilon \ln \epsilon)$. These leading corrections are reasonably small (34%), so the chiral calculation is fairly reliable. We then determine ϵ_3 from the observed decay parameters. Unfortunately, the number obtained depends very sensitively on the experimental numbers. In Section C we use this value to determine the $\epsilon_3 u_3$ contribution to the proton-neutron and other mass differences. They turn out to be uncomfortably large (for example $m_p - m_n|_{u_3} \sim -7.9$ MeV.), although the combined experimental and theoretical uncertainty in this number is at least 50%. The implications of this result are then discussed.

(A) Experimental Results and Electromagnetic Contribution

The branching ratios for the $3\pi^0$ and the $\pi^+\pi^-\pi^0$ decays are $\Gamma_{000}/\Gamma_{\text{tot}} = 30.0 \pm 1.1$ and $\Gamma_{+-0}/\Gamma_{\text{tot}} = 23.9 \pm 0.6$, where⁽¹⁴⁾ $\Gamma_{\text{tot}} = 0.85 \pm 0.12$ KeV.

Experiment is consistent with a linear form for the $\pi^+\pi^-\pi^0$ matrix element:

$$T_{+-0} = A + BE_0, \quad (3.1)$$

where E_0 is the energy of the π^0 . Experimentally,⁽²⁹⁾ the slope parameter B/A is around $-2/\mu_\eta$.

It is usually assumed that the final state has isospin one. This is partly a theoretical prejudice: If the decay is electromagnetic, then only the $I=1$ part can contribute because of G parity. Of course, an $\epsilon_3 u_3$ term is also $I=1$.

If the final state is $I=1$, then from Bose statistics

$$\begin{aligned} T_{000} &= T_{+-0} + T_{-0+} + T_{0+-} \\ &= 3A + B\mu_\eta \end{aligned} \quad (3.2)$$

The widths for the $3\pi^0$ and $\pi^+\pi^-\pi^0$ decays are then given by⁽³⁰⁾

$$\begin{aligned}\Gamma_{000} &= 827 |3A + B\mu_\pi|^2 \text{ eV}, \\ \Gamma_{+-0} &= 489 |3A + B\mu_\pi|^2 (1 + 0.02y(1+y)) \text{ eV},\end{aligned}\tag{3.3}$$

where $y = (\mu_\eta - 3\mu_\pi) / (\mu_\eta + 3A/B)$.

Using the experimental value $B/A \approx -2/\mu_\eta$ one can then predict that $\Gamma_{+-0}/\Gamma_{000} = 0.59$. (The prediction would be $2/3$ if one neglected the $\pi^+ - \pi^0$ mass difference and set $B=0$. It is not very sensitive to B/A .) Experimentally, $\Gamma_{+-0}/\Gamma_{000} = 0.80 \pm 0.05$. This is in crude agreement with the $I=1$ assumption, but an $I \geq 3$ final state clearly cannot be ruled out.⁽³¹⁾

Consider the possibility that the decay is electromagnetic. Then the amplitude is

$$T_{ijk} = -\langle ijk | H_{em} | \eta \rangle\tag{3.4}$$

and the quantities A and B of (3.1) and (3.2) are functions of the chiral symmetry breaking parameters ϵ_0 and ϵ_8 . In the $SU(2) \times SU(2)$ limit both T_{+-0} and T_{000} must vanish when the four-momentum of the (on-shell) π^0 goes to zero, because^(6,7)

$$T_{ij0} \xrightarrow[p_{\pi^0 \rightarrow 0}]{\frac{1}{f_\pi}} \langle ij | [F_3^5, H_{em}] | \eta \rangle = 0\tag{3.5}$$

But for $p_{\pi^0} = 0$, $T_{+-0} = A$ and $T_{000} = 3A + B\mu_\eta$. Hence, in the $SU(2) \times SU(2)$ limit

$$A = B = 0\tag{3.6}$$

so the electromagnetic decay is suppressed by a factor of μ_π^2 (or $\mu_\pi^2 \ln \mu_\pi^2$).

This result is independent of any detailed model and depends only on the assumption that neutral axial charges commute with H^{em} (up to anomalous terms which are higher order in $\alpha \sim 1/137$) and that $SU(2) \times SU(2)$ is a good hadron symmetry. This is Sutherland's theorem⁽¹⁵⁾ stated as a chiral limit theorem. (The original statement was that T_{+-0} and T_{000} must vanish at the off mass-shell point $p_{\pi 0} = 0$.)

Various authors^(15,17) have tried to estimate these μ_{π}^2 terms by assuming linear extrapolation formulas in the masses of the three pions, which they vary independently (via PCAC). In this approach the slope $B/A \sim -2/\mu_{\eta}$ is predicted correctly but the magnitude of A always comes out far too small. Dittner, Dondi, and Eliezer,⁽¹⁷⁾ for example, find $\Gamma_{+-0} = 0.6 \text{ eV}$ in the $(\bar{3}, 3) + (3, \bar{3})$ model and similar results in other models (one must make a model dependent $SU(3) \times SU(3)$ calculation to compute A).

Therefore, the vanishing of A in the $SU(2) \times SU(2)$ limit seems to rule out the possibility that the decay is electromagnetic (unless, perhaps, there is some unexpected very strong final state enhancement of the amplitude).

(B) Addition of an $\varepsilon_3 u_3$ Term

Due to the failure of electromagnetism to account for the decay, it has been suggested^(16,17) that one should add an $I=1$ term to the strong symmetry breaking Hamiltonian. Then (1.1) becomes

$$H = H_0 + \varepsilon_0^0 u_0^0 + \varepsilon_3^0 u_3^0 + \varepsilon_8^0 u_8^0 \quad (3.7)$$

where, hopefully, $|\varepsilon_3^0| \ll |\varepsilon_0^0|, |\varepsilon_8^0|$. In contrast to the electromagnetic case the u_3 contribution to the $\eta \rightarrow 3\pi$ decay does not vanish in the $SU(2) \times SU(2)$ limit, so it appears to be a good candidate to explain the decay. In this section we will calculate the parameters A and B in an expansion about the

$SU(3) \times SU(3)$ limit. Throughout we will assume that u_0^0 , u_3^0 , and u_8^0 belong to the $(\bar{3}, 3) + (3, \bar{3})$ representation.

We must compute the matrix element $T_{ijk}(p_i, p_j, p_k; \varepsilon_0^0; \varepsilon_8^0) = -\varepsilon_3^0 \langle ijk | u_3^0 | \eta \rangle$, assuming the form (3.1) for T_{+-0} . Now, in the $SU(2) \times SU(2)$ limit, T_{+-0} vanishes when $p_{\pi^+} = 0$ because $[{}^5F_{\pi^+}, u_3^0] = 0$. But $E_0 = \mu_\eta/2$ when $p_{\pi^+} = 0$, so from (3.1) the slope B/A is correctly predicted to be $-2/\mu_\eta$ in the $SU(2) \times SU(2)$ limit. We have argued in the Introduction that $SU(2) \times SU(2)$ results are valid to $\sim 7\%$, so we will work in the $SU(2) \times SU(2)$ limit from now on. From (3.1) and (3.2) we have

$$\begin{aligned} T_{+-0} &= A(\varepsilon_0^0) \left(1 - \frac{2E_0}{\mu_\eta}\right) \\ T_{000} &= A(\varepsilon_0^0) \end{aligned} \quad (3.8)$$

To determine A we must go to the $SU(3) \times SU(3)$ limit (always maintaining the ratio $\varepsilon_8^0/\varepsilon_0^0 = -\sqrt{2}$). The ratio E_0/μ_η is ambiguous in the chiral limit, so rather than work directly⁽³²⁾ with T_{+-0} we will utilize the exact result that for $p_{\pi^0} \rightarrow 0$,

$$\begin{aligned} T_{+-0}(p_{\pi^0}=0, \varepsilon_0^0) &= A(\varepsilon_0^0) \\ &= \frac{1}{4} \frac{\varepsilon_3^0}{\mu_\pi} \langle + - | [{}^5F_3, u_3^0] | \eta \rangle \\ &= \frac{\varepsilon_3^0}{4\mu_\pi} \langle + - | \left(\sqrt{\frac{2}{3}} \phi_0^0 + \frac{1}{\sqrt{3}} \phi_8^0 \right) | \eta \rangle \end{aligned} \quad (3.9)$$

where the ϕ_0^0 and ϕ_8^0 operators carry zero momentum. Notice that if the underlying symmetry group were $U(3) \times U(3)$, as is naively expected in quark models but not present in fact,⁽³³⁾ then ϕ_0^0 and ϕ_8^0 could be written as linear combinations of $\partial \cdot A_8$ and $\partial \cdot A_0$ so that A (and B) would vanish in the $SU(2) \times SU(2)$ limit.

We will evaluate (3.9) exactly in the $SU(3) \times SU(3)$ limit. In Appendix B

we sketch the calculation of the leading $\epsilon \ln \epsilon$ corrections.

$A(0)$ is determined from the following unambiguous⁽³⁴⁾ prescription:

(a) Define the off mass shell Green's function

$$G(p_\eta^2, \epsilon_0^0) = \frac{i\epsilon_3^0}{f_\pi} \int d^4x e^{ip_\eta x} (\square + \mu_\eta^2) \times \\ \langle + - | T(\phi_\eta(x) (\frac{\sqrt{3}}{2}\phi_0^0(0) + \frac{1}{\sqrt{3}}\phi_8^0(0))) | 0 \rangle \quad (3.10)$$

A suitable interpolating field ϕ_η (including the effects of η - η' mixing)

is defined in Appendix A.

(b) Approximate $A(\epsilon_0^0) = G(\mu_\eta^2, \epsilon_0^0)$ by $G(0, \epsilon_0^0)$. The error induced is of $O(\epsilon_0 \ln \epsilon_0)$ and is calculated in Appendix B.

(c) For $\epsilon_0^0 \neq 0$, use the exact Dashen-Weinstein^(6,7) method for $SU(2) \times SU(2)$ to evaluate $G(0, \epsilon_0^0)$.

$$G(0, \epsilon_0^0) = \frac{-i\epsilon_3^0}{f_\pi^3} \mu_\eta^2 \int d^4x \langle 0 | [{}^5F_\pi, [{}^5F_\pi, T(\frac{\sqrt{3}}{2}\phi_\eta(x)\phi_0^0(0) + \frac{1}{\sqrt{3}}\phi_\eta(x)\phi_8^0(0))]] | 0 \rangle \quad (3.11)$$

where $[{}^5F, [{}^5F, T(\)]]$ is a shorthand meaning to keep the various equal time commutators.

(d) Drop all terms which vanish as $\epsilon_0^0 \rightarrow 0$.

The result of all this is

$$G(0, \epsilon_0^0) = \frac{-4i\epsilon_3^0}{3\sqrt{3}f_\pi^3} \frac{\mu_\eta^2}{Z_{8\eta}^{1/2}} \int d^4x \langle T(\phi_8^0(x), \phi_8^0(0)) \rangle_0 + O(\epsilon \ln \epsilon) \quad (3.12)$$

But the propagator in (3.12) is just $-iZ_{8\eta}/\mu_\eta^2$, where $Z_{8\eta}$ is defined after (2.3), so we have

$$A(0) = G(0,0) = \frac{-4\epsilon_3^0 Z_{8\eta}^{1/2}}{3\sqrt{3}f_\pi^3} \equiv \frac{-4\epsilon_3}{3\sqrt{3}f_\pi^3} \quad (3.13)$$

For $\epsilon_0 = 0$, $z_{8\eta}^{1/2}$ is equal to $z_s^{1/2}(0)$ defined in Appendix A, and we have defined the renormalized ϵ_3 as $\epsilon_3^0 z_s^{1/2}(0)$.

In Appendix B we calculate the $\epsilon_0 \ln \epsilon_0$ corrections to $A(0)$ exactly.

The result is

$$A(\epsilon_0) = \frac{-4\epsilon_3}{3\sqrt{3}f_\pi} \left[1 + \sqrt{\frac{2}{3}} \frac{45}{16} \frac{\epsilon_0 \ln(\Lambda^2/4\mu^2)}{32\pi^2 f_\pi^3} + O(\epsilon_0) \right] \quad (3.14)$$

where ϵ_3 is still defined as $\epsilon_3^0 z_s^{1/2}(0)$, and $f_\pi(\epsilon)$ is the physical value of the pion decay constant. For $\ln \frac{\Lambda^2}{4\mu^2} = 2$, the correction term is -34%, which indicates that $SU(3) \times SU(3)$ is reasonably dependable in this application.

From (3.3) and the experimental width Γ_{+-0} , one obtains²⁹ $|A| = 0.703$ for $B/A = -2.08/\mu_\eta$ and $|A| = 0.572$ for $B/A = -1.87/\mu_\eta$. These numbers would be reduced by 15% if Γ_{000} were used instead of Γ_{+-0} . These two values yield

$$\left| \frac{\epsilon_3}{f_\pi} \right| = \begin{cases} 1.19 \times 10^{-2} \text{ GeV}^2 \\ 0.97 \times 10^{-2} \text{ GeV}^2 \end{cases} \quad (3.15)$$

respectively. This is to be compared with the $SU(3) \times SU(3)$ breaking parameters $\epsilon_8/f_\pi = 0.26 \text{ GeV}^2$ and $\epsilon_0/f_\pi = -0.21 \text{ GeV}^2$, and the $SU(2) \times SU(2)$ breaking parameter⁽³⁵⁾ $\epsilon/f_\pi = -\mu_\pi^2 = -1.8 \times 10^{-2} \text{ GeV}^2$.

(C) Implications of the $\epsilon_3 u_3$ Terms

Assuming that the $\eta \rightarrow 3\pi$ decay is due to an $\epsilon_3 u_3$ term, we have given what we believe is a reliable determination of ϵ_3 .

We now consider the other implications of an $\epsilon_3 u_3$ term. Using the matrix element (2.9), and assuming $\epsilon_3 > 0$, we can find the $\epsilon_3 u_3$ contribution to $\mu_K^2 - \mu_{K^0}^2$. The result, listed in Table II for the two values of

ϵ_3 in (3.15), is about twice the experimental mass difference.

Since u_3^0 and u_8^0 are members of the same octet, the $\epsilon_3^0 u_3^0$ baryon mass differences can be predicted from the medium strong differences:

$$\begin{aligned} M_P - M_n \Big|_{u_3} &= E \left[\left(\frac{f}{d} \right)_B + 1 \right] \\ M_{\Sigma^+} - M_{\Sigma^-} \Big|_{u_3} &= E \left[2 \left(\frac{f}{d} \right)_B \right] \\ M_{\Xi^-} - M_{\Xi^0} \Big|_{u_3} &= E \left[- \left(\frac{f}{d} \right)_B + 1 \right] \end{aligned} \quad (3.16)$$

where $(f/d)_B \sim -3.29$ from the medium strong differences, and

$E \equiv \sqrt{3} \epsilon_3 (m_{\Sigma} - m_{\Lambda}) / 2 \epsilon_8 Z_s^{1/2}(0)$. The small deviation of $\epsilon_8 Z_s^{1/2}(0)$ from ϵ_8 can be determined from Appendix A. The numbers predicted from (3.16) are given in Table II.

We see that the u_3 contributions predicted for the mass differences are probably too large⁽³⁶⁾ to be believable, although there is no reliable way to calculate the electromagnetic contributions.

What are the errors in the numbers in Table II? The numbers are uncertain by 20% because of B/A . They would be reduced 15% had we used Γ_{000} rather than Γ_{+-0} to determine ϵ_3 (old values for the widths would increase the numbers by $\sqrt{3}$). There is also a 15% error due to the quoted uncertainties in Γ_{tot} , the branching ratios, and the baryon masses. Finally, if the $\epsilon \ln \epsilon$ corrections in (3.14) had been ignored, the numbers in Table II would be reduced by a factor $2/3$. Hence, the uncertainty in the numbers is at least 50%.

What can be concluded from all this? The decay seems clearly to be non-electromagnetic. Also, an $\epsilon_3 u_3$ term in the $(\bar{3}, 3) + (3, \bar{3})$ model probably cannot account for the decay⁽³⁷⁾ using the present experimental parameters. The problem cannot be blamed on the $SU(3) \times SU(3)$ extrapolation: The corrections are $\sim 30\%$.

The numbers predicted from the $\epsilon_3 u_3$ term are in error only by a factor of 2 or 3, not an order of magnitude. Also, they depend sensitively on the dynamical details. Perhaps u_3^0 is not a member of the $(\bar{3},3) + (3,\bar{3})$ multiplet, or contains a piece which is not. If the u_3 term is the effect of some neutral gauge bosons, for example, it may not have simple $SU(3) \times SU(3)$ transformation properties. This could account for the failure of the prediction on $\Gamma_{+-0}/\Gamma_{000}$ (after (3.3)) based on the $I=1$ assumption. Of course, detailed predictions would be difficult to make in such a case.

One could also abandon the $(\bar{3},3) + (3,\bar{3})$ model for all the terms in (3.7). But the $(8,8)$ model seems no better,⁽¹⁷⁾ and the $(\bar{6},6) + (6,\bar{6})$ model has serious problems of its own.⁽³⁸⁾

Perhaps the decay is basically due to the u_3 term (or even electromagnetism), but is enhanced by some strong final state interaction (such as a heavy pion)⁽³⁹⁾ to which the chiral symmetry arguments are blind.

It would be worthwhile to study (both experimentally and theoretically) the other implications of a u_3 term. Possibilities include violations of $SU(2)$ symmetry in e decays and in meson-baryon coupling constants.⁽⁴⁰⁾ Bačé⁽⁴¹⁾ has considered the effects on nonleptonic K decays, and Osborn and Wallace⁽³⁰⁾ have considered the πN scattering length.

IV Conclusion

We have argued that $SU(2) \times SU(2)$ is the best symmetry of the strong Hamiltonian after isospin. In Chapter II we observed that $SU(3) \times SU(3)$ and $SU(3)$ are comparable and valid to $\sim 30\%$ in most cases. Some applications of chiral perturbation theory were given. We also argued that octet enhancement for matrix elements which vanish in the $SU(3)$ limit could be understood from the Goldstone boson pair mechanism.

In III we showed that an $\epsilon_3 u_3$ term is of the right order of magnitude to explain the $\eta \rightarrow 3\pi$ decay, but the details depend sensitively on the experimental parameters and on the representation content of u_3 .

Other possible applications of these ideas (in addition to those at the end of Chapter III) might include the decays $\eta' \rightarrow \eta\pi\pi$ and $K_S \rightarrow 2\pi$, the effects of symmetry breaking on g_A and the meson-baryon couplings, and the extension of these ideas to other symmetry breaking models.

Appendix A. The η - η' System

In the third chapter of I we presented a formalism to describe the η - η' system which assumed the existence of an η - η' mixing angle. Such a mixing angle exists only if one drops a certain continuum integral. As no such assumption was made anywhere else in I, our treatment was inconsistent. (42) In this Appendix we will correct this inconsistency by giving an exact formalism for the η - η' system.

In the $(\bar{3},3) + (3,\bar{3})$ model there are two unrenormalized fields ϕ_8^0 and ϕ_0^0 and two states $|\eta\rangle$ and $|\eta'\rangle$ with $I=Y=0$. We expect that for $\epsilon_8=0$ there will be no mixing effects and that ϕ_8^0 (ϕ_0^0) will correspond to the η (η') state. Furthermore, as $\epsilon_0 \rightarrow 0$ we expect $\mu_\eta^2 \rightarrow 0$ but $\mu_{\eta'}^2 \neq 0$: the η becomes a Goldstone boson but not the η' .

Let us define the renormalization constant $Z_{ia}^{1/2} \equiv \langle 0 | \phi_i^0 | a \rangle$ for $i=0$ or 8 and $a=\eta$ or η' . They are real by T invariance. In the $\epsilon_8=0$ limit, $Z_{0\eta}^{1/2}$ and $Z_{8\eta'}^{1/2}$ vanish, while $Z_{8\eta}^{1/2} = Z_\pi^{1/2} = Z_K^{1/2} = Z_s^{1/2}(\epsilon_0^0)$; $Z_{0\eta'}^{1/2}$ approaches some different value. Assuming canonical quantization rules for ϕ_0^0 and ϕ_8^0 one can prove formally that

$$\begin{aligned} 0 &= Z_{0\eta}^{1/2} Z_{8\eta}^{1/2} + Z_{0\eta'}^{1/2} Z_{8\eta'}^{1/2} + \int_{s_{08}}^{\infty} ds \rho_{08}^0(s) \\ 1 &= Z_{0\eta}^{1/2} + Z_{0\eta'}^{1/2} + \int_{s_{ii}}^{\infty} ds \rho_{ii}^0(s) \end{aligned} \quad (A.1)$$

for $i=0$ or 8 , where

$$\rho_{ij}^0(s) = \frac{(2\pi)^3}{2} \sum_n \delta^4(q-p_n) \{ \langle 0 | \phi_i^0 | n \rangle \langle n | \phi_j^0 | 0 \rangle + (i \leftrightarrow j) \} \quad (A.2)$$

with $q^2=s$. Of course, in a real field theory these integrals will in general diverge.

The η - η' mixing effects are incorporated in the $Z_{ia}^{1/2}$. An η - η' mixing angle θ could be defined in various ways. For example, it could be defined as the angle which satisfies Eqn. (2.8). A theoretically useful definition, however, would be to define "singlet" and "octet" states $|\bar{0}\rangle$ and $|8\rangle$ as a rotation by θ of the physical states $|\eta'\rangle$ and $|\eta\rangle$ such that

$$\langle 0 | \phi_8^0 | \bar{0} \rangle = \langle 0 | \phi_0^0 | 8 \rangle = 0 \quad (\text{A.3})$$

But (A.3) implies

$$\tan \theta = \frac{Z_{8\eta'}^{1/2}}{Z_{8\eta}^{1/2}} = - \frac{Z_{0\eta}^{1/2}}{Z_{0\eta'}^{1/2}} \quad (\text{A.4})$$

This last equality is true if and only if the ρ_{08}^0 integral in (A.1) can be neglected relative to the renormalization constants. However, we shall see that $Z_{0\eta}^{1/2}$, ρ_{08}^0 and probably $Z_{8\eta'}^{1/2}$ all vanish like $\epsilon_8 \ln \epsilon_0$. Hence, one cannot justify dropping the continuum integral in (A.1) on the basis of a small ϵ_8 . A mixing angle (as defined in A.3) can only be justified if the interactions in the symmetric Hamiltonian H_0 are neglected.

Now define f_η and $f_{\eta'}$ by

$$\langle 0 | \partial \cdot A_8 | a \rangle = \mu_a^2 f_a \quad (\text{A.5})$$

$a = \eta$ or η' . For $\epsilon_8 \rightarrow 0$, $f_{\eta'}$ vanishes and $f_\eta = f_\pi = f_K \equiv f_s(\epsilon_0)$. Then using the result⁽¹⁾

$$\partial \cdot A_8 = -\left(\sqrt{\frac{2}{3}}\epsilon_0^0 - \frac{1}{\sqrt{3}}\epsilon_8^0\right)\phi_8^0 - \sqrt{\frac{2}{3}}\epsilon_8^0\phi_0^0 \quad (\text{A.6})$$

one has

$$\frac{\mu_a^2 f_a}{z_{8a}^{1/2}} = - \left[\sqrt{\frac{2}{3}} \varepsilon_0^0 - \frac{1}{\sqrt{3}} \varepsilon_8^0 \right] - \sqrt{\frac{2}{3}} \varepsilon_8^0 \frac{z_{0a}^{1/2}}{z_{8a}^{1/2}} \quad (\text{A.7})$$

for $a = \eta$ or η' . The Ward identities (3.11) of I become

$$\begin{aligned} \sqrt{\frac{2}{3}} \langle v_0^0 \rangle_0 - \frac{1}{\sqrt{3}} \langle v_8^0 \rangle_0 &= z_{8\eta}^{1/2} f_\eta + z_{8\eta'}^{1/2} f_{\eta'} \\ &\quad - \left(\sqrt{\frac{2}{3}} \varepsilon_0^0 - \frac{1}{\sqrt{3}} \varepsilon_8^0 \right) \int_{s_{88}}^\infty ds \rho_{88}^0(s)/s \\ &\quad - \sqrt{\frac{2}{3}} \varepsilon_8^0 \int_{s_{08}}^\infty ds \rho_{08}^0(s)/s \\ \sqrt{\frac{2}{3}} \langle v_8^0 \rangle_0 &= z_{0\eta}^{1/2} f_\eta + z_{0\eta'}^{1/2} f_{\eta'} \\ &\quad - \left(\sqrt{\frac{2}{3}} \varepsilon_0^0 - \frac{1}{\sqrt{3}} \varepsilon_8^0 \right) \int_{s_{08}}^\infty ds \rho_{08}^0(s)/s \\ &\quad - \sqrt{\frac{2}{3}} \varepsilon_8^0 \int_{s_{00}}^\infty ds \rho_{00}^0(s)/s \end{aligned} \quad (\text{A.8})$$

Now, from (A.7) one has

$$\mu_{\eta'}^2 f_{\eta'} = -\sqrt{\frac{2}{3}} \varepsilon_8^0 z_{0\eta'}^{1/2} + O(\varepsilon^2 \ln \varepsilon) \quad (\text{A.9})$$

which implies that $f_{\eta'}$ vanishes like ε_8 , not like $\varepsilon_8 \ln \varepsilon_0$. Then, using the fact that the continuum integrals in (A.8) have no threshold singularities,⁽⁴³⁾ we have

$$\sqrt{\frac{2}{3}} \langle v_0^0 \rangle_0 - \frac{1}{\sqrt{3}} \langle v_8^0 \rangle_0 = z_{8\eta}^{1/2} f_\eta + O(\varepsilon) \quad (\text{A.10a})$$

$$\sqrt{\frac{\epsilon}{3}} \langle u_8^0 \rangle_0 = z_{0\eta}^{1/2} f_\eta + O(\epsilon_8) \quad (\text{A.10b})$$

Equation (A.10a) will be used to determine $z_{8\eta}^{1/2}$ and f_η ; from (A.10b) and (3.18) of I we can determine the mixing quantity $z_{0\eta}^{1/2}$ which was needed for the corrections to the Gell-Mann Okubo formula in Chapter II:

$$\frac{z_{0\eta}^{1/2}}{z_{8\eta}^{1/2}} = -\sqrt{\frac{2}{3}} \frac{5\epsilon_8 \ln(1^2/4\mu^2)}{96\pi^2 f_\pi^3} + O(\epsilon_8) \quad (\text{A.11})$$

We would now like to define a pair of suitable interpolating fields to define off-shell Green's function for the η and η' . They are

$$\begin{aligned} \phi_\eta &\equiv (z_{0\eta}^{1/2} \phi_8^0 - z_{8\eta'}^{1/2} \phi_0^0) / D \\ \phi_{\eta'} &\equiv (-z_{0\eta}^{1/2} \phi_8^0 + z_{8\eta}^{1/2} \phi_0^0) / D \\ D &\equiv z_{0\eta}^{1/2} z_{8\eta}^{1/2} - z_{8\eta'}^{1/2} z_{0\eta'}^{1/2} \end{aligned} \quad (\text{A.12})$$

In the $\epsilon_8 = 0$ limit, $\phi_\eta = \phi_8^0 / z_{8\eta}^{1/2}$, $\phi_{\eta'} = \phi_0^0 / z_{0\eta'}^{1/2}$. The fields are constructed so that

$$\langle 0 | \phi_a | b \rangle = \delta_{ab}, \quad (\text{A.13})$$

for $(a, b) \in (\eta, \eta')$, and

$$\partial \cdot A_8 = f_\eta \mu_\eta^2 \phi_\eta + f_{\eta'} \mu_{\eta'}^2 \phi_{\eta'} \quad (\text{A.14})$$

One can use PCAC techniques for ϕ_η (or ϕ_8^0) by writing them in terms of $\partial \cdot A_8$ and $\phi_{\eta'}$ (or ϕ_0^0) from A.14 (A.6) and then verifying that the $\phi_{\eta'}$ (ϕ_0^0) pieces vanish to the required order.

Equation (A.12) can be inverted to yield

$$\begin{aligned}\phi_8^0 &= z_{8\eta}^{1/2} \phi_\eta + z_{8\eta'}^{1/2} \phi_{\eta'} \\ \phi_0^0 &= z_{0\eta}^{1/2} \phi_\eta + z_{0\eta'}^{1/2} \phi_{\eta'}\end{aligned}\tag{A.15}$$

Since $z_{0\eta}^{1/2}$ and probably also $z_{8\eta'}^{1/2}$ vanish like $\epsilon_8 \ln \epsilon_0$, we see that ρ_{08}^0 (defined in (A.2)) must also vanish like $\epsilon_8 \ln \epsilon_0$, as was asserted after (A.4).

In analogy to (3.22) of I one can easily derive

$$\langle a | u_\ell^0(0) | a \rangle = \frac{d\mu_a^2}{d\epsilon_\ell^0}\tag{A.16}$$

for $\ell=0$ or 8 , $a=\eta$ or η' , with u_ℓ^0 carrying zero momentum. Also,

$$\langle \eta'(k^2) | u_\ell^0 | \eta(k^2) \rangle = 0\tag{A.17}$$

where u_ℓ^0 carries no momentum and $k^2 = \mu_\eta^2$ or $\mu_{\eta'}^2$.

Finally, as in (3.23) of I one can show

$$\frac{d}{d\epsilon_\ell^0} (z_{ia}^{1/2} z_{ja}^{1/2}) = -\frac{d}{dk^2} [(k^2 - \mu_a^2)^2 \int d^4x d^4y e^{ik \cdot (x-y)} \langle 0 | T(\phi_i^0(x) \phi_j^0(y) u_\ell^0(0)) | 0 \rangle] \Big|_{k^2 = \mu_a^2}\tag{A.18}$$

From (A.18) one can derive various results, such as

$$D_z^\ell = \frac{1}{2} \frac{d}{dk^2} \langle \eta(k^2) | u_\ell^0(0) | \eta(k^2) \rangle \Big|_{k^2 = \mu_\eta^2}\tag{A.19}$$

where

$$D_Z^l = \frac{1}{D} \left[z_{0\eta}^{1/2} \frac{dz_{8\eta}^{1/2}}{d\varepsilon_0} - z_{8\eta}^{1/2} \frac{dz_{0\eta}^{1/2}}{d\varepsilon_0} \right] = \frac{1}{z_{8\eta}^{1/2}} \frac{dz_{8\eta}^{1/2}}{d\varepsilon_0} + O(\varepsilon |h^2 \varepsilon) \quad (\text{A.20})$$

(D is defined in (A.12).)

From Equations (A.10a), (A.19), and (A.20) one can use the techniques described in I to show

$$\begin{aligned} \frac{f_i}{f_i(0)} - 1 &= \frac{-\ln(\Lambda^2/4\mu^2)}{32\pi^2 f_\pi^3} (\sqrt{6}\varepsilon_0 + \frac{3}{2}d_i\varepsilon_8) + O(\varepsilon) \\ \frac{z_i^{1/2}}{z_i^{1/2}(0)} - 1 &= \frac{-\ln(\Lambda^2/4\mu^2)}{192\pi^2 f_\pi^3} (14\sqrt{\frac{2}{3}}\varepsilon_0 + d_i\varepsilon_8) + O(\varepsilon) \end{aligned} \quad (\text{A.21})$$

for $i=1, \dots, 8$. By $f_8(z_8^{1/2})$ we mean $f_\eta(z_{8\eta}^{1/2})$; d_i is the Gell-Mann d symbol d_{ii8} .

If $T(K^2; \varepsilon_0, \varepsilon_8)$ is a Green's function involving an η field carrying momentum K , then (as in 3.30 and 3.31 of I) the derivative of the on-mass shell matrix element $T(\mu_\eta^2; \varepsilon_0, \varepsilon_8)$ is

$$\begin{aligned} \frac{dT(\mu_\eta^2; \varepsilon_0, \varepsilon_8)}{d\varepsilon_0} &= -iT_{u_0}^S(\mu_\eta^2; \varepsilon_0, \varepsilon_8) \\ &= -iT_{u_0}^I(\mu_\eta^2; \varepsilon_0, \varepsilon_8) + D_Z^l T(\mu_\eta^2; \varepsilon_0, \varepsilon_8) \\ &\quad + \frac{d\mu_\eta^2}{d\varepsilon_0} T_\eta(\mu_\eta^2; \varepsilon_0, \varepsilon_8) \end{aligned} \quad (\text{A.22})$$

where D_Z^l is defined in (A.20). $T_{u_0}^S$ and $T_{u_0}^I$ are the matrix element T with a u_0 operator inserted everywhere except on the external legs. The technical difference between the two is defined in I. Both still contain

η' poles in the (η, u_ℓ^0) channel. The quantity T_η is defined as

$$T_\eta(k^2, \varepsilon_0, \varepsilon_8) \equiv \frac{T(k^2, \varepsilon_0, \varepsilon_8) - T(\mu_\eta^2, \varepsilon_0, \varepsilon_8)}{k^2 - \mu_\eta^2} \quad (\text{A.23})$$

At $k^2 = \mu_\eta^2$, T_η is just dT/dk^2 . Of course, one must also add the counterterms associated with other external legs to (A.22).

As in (3.32) of I, one can approximate (A.22) by a formula in which the quantities on the right are evaluated at $k^2 = 0$ (the η' poles in the (η, u_ℓ^0) channel present no difficulty). The result is

$$\begin{aligned} \frac{dT(\mu_\eta^2, \varepsilon_0, \varepsilon_8)}{d\varepsilon_\ell^0} &= -iT_{u_\ell^0}(0, \varepsilon_0, \varepsilon_8) \\ &\quad + \left(\frac{1}{\mu_\eta^2} \frac{d\mu_\eta^2}{d\varepsilon_\ell^0} - D_2^\ell \right) T(0, \varepsilon_0, \varepsilon_8) \\ &\quad + \frac{d\mu_\eta^2}{d\varepsilon_\ell^0} T_\eta(0, \varepsilon_0, \varepsilon_8) + O(\text{CONSTANT}) \end{aligned} \quad (\text{A.24})$$

where $T_{u_\ell^0}$ includes u_ℓ^0 insertions everywhere, even on the external leg.

One may usually⁽⁴⁴⁾ set $\varepsilon_8 = 0$ on the right, inducing an error of $O(\varepsilon_8 \ln \varepsilon_0)$.

The error in (A.24) would be $O(\varepsilon \ln \varepsilon)$ if T_η were evaluated at μ_η^2 . If there are "dependent variable singularities" (see Chapter II-c of I) the error can be $O(1)$ when T_η is evaluated at $k^2 = 0$. This presents no difficulty in practice since we are usually only interested in the $\ln \varepsilon_0$ part of the derivative. The same comments apply to Equations (2.71) and (3.32) of I.

Appendix B: Corrections to the $SU(3) \times SU(3)$

Calculation of the $\eta \rightarrow 3\pi$ Amplitude

In this appendix we will sketch the calculation of the leading ($\epsilon \ln \epsilon$) corrections to the result (3.13) for A. We will stay on the line $\epsilon_8^0 = -\sqrt{2} \epsilon_0^0$ of exact $SU(2) \times SU(2)$ symmetry.

We will work with the expression (3.10) for $A(\epsilon_0^0) = G(\mu_\eta^2, \epsilon_0^0)$. Since f_π is an explicit factor in (3.10) there is no need to expand it around $\epsilon_0 = 0$. From (3.10) and (A.24) we find

$$\begin{aligned} \frac{1}{f_\pi} \frac{d}{d\epsilon_0^0} [f_\pi G(\mu_\eta^2, \epsilon_0^0)] &= -i G(0, \epsilon_0^0)_{u_0^0 = -\sqrt{2} u_8^0} - \frac{2\sqrt{3} z_{8\eta}^{1/2}}{\mu_\eta^2 f_\pi} G(0, \epsilon_0^0) \\ &+ G(0, \epsilon_0^0) \left[\frac{-z_\pi^{1/2}}{z_\pi^{1/2}} \frac{dz_\pi^{1/2}}{d\epsilon_0^0} - \frac{1}{f_\pi} \frac{df_\pi}{d\epsilon_0^0} + \frac{2}{\sqrt{3}\mu_\eta^2} \frac{d}{d\epsilon_0^0} \left(\frac{\epsilon_0^0 z_{0\eta}^{1/2}}{f_\pi} \right) \right] \\ &- \frac{2\sqrt{3} z_{8\eta}^{1/2}}{f_\pi} \frac{d}{dp^2} G(p^2, \epsilon_0^0) \Big|_{p^2 = \mu_\eta^2} + O(\epsilon \ln \epsilon) \end{aligned} \quad (B-1)$$

The interpretation of (B.1) is that $-i G(0, \epsilon_0^0)_{u_0^0 = -\sqrt{2} u_8^0}$ represents the ϵ_0 dependence of the internal dynamics. The G/μ_η^2 comes from differentiating the Klein-Gordon operator: it cancels the $u_0^0 - \sqrt{2} u_8^0$ insertions on the external η leg in the first term. The $G(0, \epsilon_0^0) [\dots]$ term represents renormalization effects, and the last term is due to the variation of the mass of the external η with ϵ .

In (B.1) we have chosen to evaluate $\frac{dG}{dp^2}$ at μ_η^2 rather than zero. We have rewritten the $[G(0, \epsilon_0^0) d\mu_\eta^2/d\epsilon_0^0]/\mu_\eta^2$ term of (A.24) using the expression (A.7) for μ_η^2 . The $dz_\pi^{1/2}/d\epsilon_0^0$ term in (B.1) is the counterterm associated with the external pions (This is clear from (2.46) of I since $d\mu_\pi^2/d\epsilon_0^0 = 0$).

Using (A.11) and (A.21) one easily finds that ϵ_0^0 times the bracketed counter-terms in (B.1) is

$$\epsilon_0^0 G(0, \epsilon_0^0) [\dots] = \frac{11}{2} \sqrt{3} \frac{\epsilon_0 \ln(\Lambda^2/4\mu^2)}{32\pi^2 f_\pi^3} A(0) + O(\epsilon_0) \quad (B.2)$$

To calculate the first term in (B.1) we use the exact $SU(2) \times SU(2)$ expression

$$\begin{aligned}
 -iG(0, \varepsilon_0^0) &= \frac{\varepsilon_3^0 \mu_\pi^2}{f_\pi} \int d^4x d^4z \langle + - | T[(\phi_0^0(z) - \sqrt{2}\phi_8^0(z)) \\
 &\quad \phi_\pi(x) (\frac{\sqrt{2}}{3}\phi_0^0(0) + \frac{1}{\sqrt{3}}\phi_8^0(0))] | 0 \rangle \\
 &= -\frac{\varepsilon_3^0 \mu_\pi^2}{f_\pi} \int d^4x d^4z \langle 0 | [F_{\pi+}^\dagger [F_{\pi-}^\dagger T\{(\phi_0^0 - \sqrt{2}\phi_8^0)\phi_\pi \\
 &\quad (\frac{\sqrt{2}}{3}\phi_0^0 + \frac{1}{\sqrt{3}}\phi_8^0)\}] | 0 \rangle \quad (B.3)
 \end{aligned}$$

where, just as in (3.11), we mean take all the equal time commutators. For the second term in (B.1) we use (3.11). We now compute all the equal time commutators for these two terms (using the expression⁽⁴⁵⁾ (A.12) for ϕ_π), utilize equations⁽⁴⁵⁾ (A.6) and (A.14), and then drop all terms which vanish or go to constants as $\varepsilon_0 \rightarrow 0$. The result is that the first two terms in (B.1) are

$$\begin{aligned}
 \frac{i\varepsilon_3^0}{f_\pi^4} \int d^4z &\left[-\frac{4\sqrt{2}}{3} \langle T(\phi_{\pi+}^0(z)\phi_{\pi-}^0(0)) \rangle_0 \right. \\
 &+ \frac{4\sqrt{2}}{9} \langle T[\phi_0^0(z)(\phi_0^0(0) - \sqrt{2}\phi_8^0(0))] | 0 \rangle_0 \\
 &+ \frac{14}{9} \langle T[\phi_8^0(z)(\phi_0^0(0) - \sqrt{2}\phi_8^0(0))] | 0 \rangle_0 \\
 &\left. + O(\text{CONSTANT}) \right] \quad (B.4)
 \end{aligned}$$

Each of these terms diverges like ε_0 due to the two meson cut (none have a two pion cut)... The divergent part is calculated as in (3.15) of I to give (when multiplied by ε_0^0) the same expression as in (B.2).

Finally, the dG/dp^2 term in (B.1) has singularities due to the three diagrams in Figure 1. In the first diagram the invariant mass of the cut is of the dependent variable type (see I). The contribution of these diagrams (when multiplied by ϵ_0^0) is

$$\begin{aligned} & -\frac{2\sqrt{\frac{2}{3}} \frac{\epsilon_0^0}{f_\pi} \frac{d}{dp^2} G(p^2, \epsilon_0^0) \big|_{p^2=\mu^2}}{f_\pi} \\ & = -\frac{83}{16} \sqrt{\frac{2}{3}} \frac{\epsilon_0 \ln(\Lambda^2/4\mu^2)}{32\pi^2 f_\pi^3} A(0) + O(\epsilon) \end{aligned} \quad (B.5)$$

(The first diagram contributes 75/16 and the last two 1/2).

Putting the various pieces together, we have

$$A(\epsilon_0) = \frac{-4\epsilon_3}{3\sqrt{3} f_\pi^3(\epsilon) f_\pi^2(0)} \left[1 + \frac{83}{16} \sqrt{\frac{2}{3}} \frac{\epsilon_0 \ln(\Lambda^2/4\mu^2)}{32\pi^2 f_\pi^3} + O(\epsilon_0) \right] \quad (B.6)$$

Part of the $\epsilon \ln \epsilon$ correction can be absorbed into the renormalization of $f_\pi(0)$ by the symmetry breaking interaction (see A.21) to give⁽⁴⁶⁾

$$A(\epsilon_0) = \frac{-4\epsilon_3}{3\sqrt{3} f_\pi^3(\epsilon)} \left[1 + \frac{45}{16} \sqrt{\frac{2}{3}} \frac{\epsilon_0 \ln(\Lambda^2/4\mu^2)}{32\pi^2 f_\pi^3} + O(\epsilon_0) \right] \quad (B.7)$$

where $f_\pi(\epsilon)$ is the physical (renormalized) value of f_π . For $\ln \Lambda^2/4\mu^2=2$, the $\epsilon \ln \epsilon$ term is ≈ -0.34 .

We conclude by commenting that the relatively small (34%) corrections in (B.7) are due to the cancellation of much larger factors. We do not consider this to be a lucky accident. The different terms in (B.1) merely single out for separate consideration different aspects of the same Feynman diagrams (e.g. the ϵ dependence of the masses of the internal and external lines, of the vertices,

etc.). Their near cancellation is not accidental: it simply means that the relevant graphs are small.

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- 34) The chiral limit techniques of Dashen and Weinstein^(6,7) would yield 3/4 of (3.13). The problem is that the ϕ_8^0 part of (3.9) is proportional to the $\eta\eta \rightarrow \pi^+ \pi^-$ amplitude in the chiral limit, except one of the Klein-Gordon operators is missing. Hence, it is of the ambiguous form $\infty \times 0$.
- 35) The Hamiltonian (1.1) can be rewritten as $H_1 + \epsilon^0 \sigma^0$, where H_1 is $SU(2) \times SU(2)$ symmetric, $\sigma^0 = \sqrt{\frac{2}{3}} u_o^0 + \frac{1}{\sqrt{3}} u_8^0$, and $\epsilon^0 = \epsilon / Z_\pi^{1/2} = \sqrt{\frac{2}{3}} \epsilon_o^0 + \frac{1}{\sqrt{3}} \epsilon_8^0$.
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- 46) If in Ref. (21) we had written the zero order contribution to $\mu_K^2 + \mu_{K^0}^2$ in terms of the physical $f_K^{-2}(\epsilon)$ instead of $f_s^{-2}(0)$ the remaining corrections would have been larger : 200% rather than 130%.

Figure Caption

- 1) The three diagrams contribution to the singular part of dG/dp^2 in Equation (B.1).

Table Caption

Table I. The baryon mass differences predicted from the threshold dominance relations (2.13).

Table II. The predicted $\epsilon_3 u_3$ contributions to the $\Delta I=1$ mass differences:

- a) The quantity.
- b) Prediction for $B/A = -2.08/\mu_\eta$.
- c) Prediction for $B/A = -1.87/\mu_\eta$.
- d) Experimental mass difference.

Table I

Quantity	Predicted Value (MeV)	Experimental Value
$m_N - m_\Sigma$	-252(input)	-252
$m_\Sigma - m_\Xi$	-188	-123
$m_\Lambda - m_\Xi$	-235	-199
$m_p - m_n$	-2.3	-1.3
$m_\Sigma^+ - m_\Sigma^-$	-5.6	-7.9
$m_\Xi^- - m_{\Xi^0}$	+3.3	+6.6

Table II

a	b	c	d
$\mu_K^2 - \mu_{K^0}^2$	-0.0104 GeV ²	-0.0084 GeV ²	-0.004 GeV ²
$m_p - m_n$	-7.9 MeV	-6.4 MeV	-1.3 MeV
$m_\Sigma^+ - m_\Sigma^-$	-22.7 MeV	-18.5 MeV	-7.9 MeV
$m_\Xi^- - m_{\Xi^0}$	+14.8 MeV	+12.0 MeV	+6.6 MeV