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RESULTANT PROCEDURE
A METHOD FOR FINDING THE ZEROS OF REAL POLYNOMIALS

by

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ABSTRACT

The resultant procedure is a method for finding simultaneously all zeros of a polynomial with real coefficients. There are no restrictions on the multiplicity of either the real or complex zeros. The principle of the resultant procedure is as follows.

Assume that the absolute value ρ of a root α of the equation $P(x) = a_0 x^n + a_1 x^{n-1} + \dots + a_n = 0$ is known. Then the resultant $R(p)$ of $P(x)$ and $Q(x) = x^2 + p x + \rho^2$ is computed by a simple recurrence relation. The real roots p_j of the equation $R(p) = 0$ that satisfy

$$-\rho \leq p_j/2 \leq \rho$$

determine the real parts of the roots α of modulus ρ .

The absolute values of α and p are obtained from the pivotal coefficients of a root-squaring algorithm. To be applicable to exceptional polynomials, e.g., polynomials containing cyclotomic factors, and to polynomials with roots of multiplicity up to n , the theory underlying the root squaring method had to be redeveloped, refined, and extended. In particular, criteria have been derived that enable the computer to make clear how to proceed.

The resultant procedure has been coded for the IBM 704 and is in successful operation. The advantage of the code is that it furnishes all roots with predetermined accuracy in a predetermined number of steps. Although the complete theory of the resultant procedure is rather complicated, the resulting formulas are simple. It is believed that this is the first completely automatic code which is based on the root-squaring method.

With small additions, the method can also be used to find the zeros of polynomials with complex coefficients.

1. INTRODUCTION

The numerical determination of the roots of real polynomial equations is of rather great importance in engineering and scientific applications. Although the problem of finding the zeros of a polynomial is a classical one, it is rather difficult to devise a practical numerical method that works for all polynomials equally well.

The root squaring procedures are considered powerful, but it is easy to make arithmetic errors in hand computation. The determination of complex zeros, in particular, of higher multiplicity, required special treatment. Some authors contend that it is impossible to solve cyclotomic polynomials of uneven degree by the Graeffe algorithm. While an experienced hand computer can decide relatively easily where to stop with the iteration of the root squaring operations, and then decide how to proceed further with any specialized method, a general and at the same time practical theory that could be adapted immediately to a high speed computer and furnish an automatic code was missing. While the major objection to the Graeffe algorithm, namely to lead easily to arithmetical errors, is eliminated by the use of electronic computers, many other problems arise, in particular those concerning the control of the flow of computation. Such simple questions as how many significant figures have to be carried along, how many root squaring operations should be performed, even the question, when should a numerically determined solution be accepted as an approximate zero of the polynomial, turn out to be non-trivial for an electronic computer, while a hand computer, in most cases, has a feeling for the right answer immediately. The methods described in this paper rely heavily on the root squaring operations to separate the zeros of polynomials. The concept of "pivotal coefficients" will be important. They determine the approximation of all absolute values of the zeros and their real parts. All arithmetic operations are carried out entirely over the field of real numbers.

The resultant procedure is a method for finding simultaneously all zeros of polynomials with real coefficients. There are no restrictions on the multiplicity of either the real or complex zeros. To obtain the absolute values of the zeros, the method relies on a modification of the Graeffe algorithm. It was necessary to redevelop the root-squaring method and modify it so that polynomial equations with complex roots of arbitrary multiplicity or with cyclotomic polynomials as factors could be handled with equal ease by a high speed computer. The principle of the resultant procedure is given below.

After the absolute value ρ of a root α of the polynomial equation $P(x) = a_0 x^n + a_1 x^{n-1} + \dots + a_n = 0$ has been found, the resultant $R(p)$ of this polynomial $P(x)$ and the quadratic $Q(x) = x^2 + p x + \rho^2$ is determined by a recurrence formula. This resultant is a polynomial in p of degree n . $P(x)$ and $Q(x)$ have a common factor if $R(p) = 0$. Hence the real roots p_j which satisfy the condition $|p_j| \leq 2\rho$ yield the real parts, $-p_j/2$, of the roots α_j of modulus ρ .

We actually compute ρ^* which usually carries a small error. Hence $R(p) = 0$ may yield complex roots p^* which satisfy $|p^*| < 2\rho^*$. The theory shows that the relative error of p^* is of the same order of magnitude as the relative error in ρ^* . Therefore, the resultant procedure is numerically stable. The different steps to be taken to design and/or use the corresponding code are outlined in the next section.

Besides the algorithm for the resultant procedure, quite a few other theoretically interesting results were derived. Some of these are listed below.

One can obtain all absolute values of the roots from the ratios of the regular pivotal coefficients $a_j^{(M)}$.

Contrary to common belief, the Graeffe algorithm does not always yield the moduli of the roots α in descending order.

There exist simple rules for recognizing a large class of exceptional polynomials, but it does not seem worth while to include them in a high speed computer code since these polynomials, which do not occur frequently in technical applications, can also be solved by the resultant procedure.

One could use also the first sub-resultant R' of $P(x)$ and $Q(x)$ to determine p . Although the degree of $R'(p)$ is merely $(n-1)$, the computational work is generally greater in arriving at the final solutions of $P(x)$ than by using $R(p)$.

Although the resultant procedure has been designed to find the zeros of polynomials with real coefficients, it can be extended to polynomials with complex coefficients by solving $P(x) \overline{P(x)} = 0$ of degree $2n$.

2. NOTATION AND OUTLINE OF ALGORITHM FOR THE RESULTANT PROCEDURE

The following symbols listed in alphabetical order for later reference, are used:

H number of significant figures to be carried in root squaring operations

M number of root squaring iterations

$P(x) = \sum_{i=0}^n a_i x^{n-i}$, polynomial with real coefficients a_i and zeros $(\alpha_j)_t$.

$P^{(m)}(x) = \sum_{i=0}^n a_i^{(m)} x^{n-i}$ polynomial with zeros $\left(\alpha_j^{(2^m)}\right)_t$.

$-p = \alpha + \bar{\alpha}$

$q = \alpha \bar{\alpha} = \rho^2$

$Q = x^2 + p x + q$

$R(p)$ resultant of $P(x)$ and $Q(x)$

$T(x) = \sum_{i=0}^n |a_i| x^{n-i}$

t as a subscript indicates the true value of a number instead of its approximation

α_k approximate zero of $P(x)$

δ pivotal bound

ϵ tolerated relative error of α_k with respect to $(\rho_k)_t$.

η separation coefficient

ν multiplicity of zeros of modulus $(\rho)_t$ of $P(x)$

ρ modulus of α

In order to calculate the zeros of $P(x)$, we proceed as follows. The equations listed are sufficient to guarantee good results.

A. Preliminary Considerations

- (1) We decide on the relative accuracy ϵ with which the zeros α_j of $P(x)$ shall be obtained, and on the separation constant η , $\eta > \epsilon$.
 E. g., $\eta = 0.1$ percent means that we expect any two successive but unequal moduli ρ_k and ρ_{k+1} to be considered equal unless, for example,

$$\rho_{k+1} < (1-\eta) \rho_k = .999 \rho_k.$$

- (2) We determine the number M of squaring iterations from the nearest integer of

$$M \geq 3 - (3.3) \log_{10} \eta + (0.1) n. \quad (2.1)$$

The larger we choose η , the faster the routine will be. A more accurate formula is Eq. (7.6) which was used to compute Table I.

- (3) We determine the pivotal bound $\underline{\delta}$.

$$r = (1-\eta)^{M-1}, \quad t = \binom{n}{n/2} - 1 \quad (2.2)$$

$$\underline{\delta} = \frac{(rt + 1)^2}{r^n t + 1} - 1 = \underline{\delta}_a \text{ in Table I,} \quad (\underline{\delta} < 1)$$

A more accurate formula for $\underline{\delta}$ is obtained from Eqs. (6.5) to (6.7). These were used to compute $\underline{\delta}_b$ in Table I.

- (4) The number H of significant figures to be carried in the squaring operations is

$$H = 0.3 M (\bar{\nu} - 1) - \log_{10} \epsilon \quad (2.3)$$

where $\bar{\nu}$ is the highest multiplicity ν of zeros with equal or nearly equal modulus, $1 \leq \nu \leq n$.

Note: Eq. (2.1) and $\eta = 0.01$ always yield $\epsilon < 0.00001$ for $n \leq 60$.

B. Determination of the Moduli of the Roots α

(1) The squaring operations yield

$$a_j^{(m+1)} = (-1)^j \left\{ \left[a_j^{(m)} \right]^2 + 2 \cdot \sum_{\ell=1}^{\min(n-j, j)} (-1)^\ell a_{j-\ell}^{(m)} a_{j+\ell}^{(m)} \right\} \quad (2.4)$$

where $a_j^{(0)} = a_j$, and $j = 0, 1, 2, \dots, n$; $m = 0, 1, 2, \dots, M-1$.

Largest terms should be computed first.

(2) The pivotal coefficients are determined. A coefficient $a_j^{(M)}$ is called pivotal, if it satisfies the condition

$$1 - \underline{\delta} < (-1)^j \frac{\left[a_j^{(M-1)} \right]^2}{a_j^{(M)}} < 1 + \underline{\delta} \quad (2.5)$$

$a_0^{(M)}$ and $a_n^{(M)}$ are always pivotal and are not tested.

(3) The moduli ρ of the roots α and the multiplicity ν of ρ are determined from successive pivotal coefficients by

$$\rho = \left[(-1)^\nu a_{k+\nu}^{(M)} / a_k^{(M)} \right]^{1/(2^M \nu)} \quad (2.6)$$

(4) We test whether $\alpha = \pm \rho$ is a real root. For acceptance test see paragraph D.

C. Determination of the Complex Roots α of Modulus ρ .

(1) The resultant $R(p) = R_n$ is computed by

$$\begin{aligned} B_{k+1} &= p B_k - q B_{k-1}; \quad B_{-1} = 0, \quad B_0 = 1 \\ A_k &= (-1)^k \sum_{j=0}^{n-k} q^{n-j-k} (a_j a_{k+j} - a_{j-1} a_{k+j+1}); \quad a_{-1} = a_{n+1} = 0 \\ R_k &= A_k B_k + R_{k-1}; \quad R_{-1} = 0. \end{aligned} \quad (2.7)$$

(See also Tables III and IV.)

(2) We solve $R(p) = 0$ for $|p| < 2\rho$, using the method of B.

(3) If $x^2 \pm |p|x + \rho^2$ is a quadratic factor of $P(x)$, then

$$\alpha = \mp \frac{1}{2}(|p| \pm i \sqrt{4\rho^2 - p^2}) \quad (2.8)$$

For acceptance test, see paragraph D.

In the rare case where ρ does not lead to a real or complex root α , the polynomial $P(x)$ is exceptional, and $a_{k+\nu}^{(M)}$ is discarded as a pivotal coefficient. When these rules are followed, all roots of $P(x) = 0$ are obtained.

D. Testing of Roots

(1) We compute for $k = 1, 2, \dots, n$

$$b_k = a_k - w b_{k-1} - q b_{k-2} \quad (b_{-1} = 0, b_0 = a_0) \quad (2.9)$$

To test if $\left\{ \begin{matrix} (x-\alpha) \\ (x^2 + px + \rho^2) \end{matrix} \right\}$ is a factor of $P(x)$, set

$$\left\{ \begin{matrix} w = -\alpha, & q = 0 \\ w = p, & q = \rho^2 \end{matrix} \right\} \quad (2.10)$$

(2) The approximation is considered sufficient when

$$T(\rho+2\epsilon\rho) - T(\rho+\epsilon\rho) > \left\{ \begin{matrix} |b_n| & \text{for } (x-\alpha) \\ \left| \frac{1}{2} p b_{n-1} + b_n \right| & \text{for } (x^2 + px + \rho^2) \end{matrix} \right\} \quad (2.11)$$

is satisfied.

The master flow chart for an IBM 704 code and numerical examples are presented in Section 13.

Table I*

Minimum Number of Squaring Operations and Pivotal Bounds

n = 2

η	M	ϵ	$\bar{\delta}$	$\underline{\delta}_a$	$\underline{\delta}_b$
.5	2	.03122	.99799	.4706	.4706
	3	.0097	.99981	.1245	.1245
	4	.000002	1.00000	.0078	.0078
.1	4	.0234	.99888	.7263	.7263
	5	.00212	.99999	.3583	.3583
	6	.000036	1.00000	.0686	.0686
.01	7	.0044	.99996	.8237	.8237
	8	.000596	.99999	.5133	.5133
	9	.00002	1.00000	.1517	.1517
.001	11	.000366	1.00000	.8815	.8815
	12	.000063	1.00000	.6355	.6355
.0001	14	.000058	1.00000	.9215	.9215
	15	.000012	1.00000	.7378	.7378

n = 3

η	M	ϵ	$\bar{\delta}$	$\underline{\delta}_a$	$\underline{\delta}_b$
.5	3	.00195	.99999	.2650	
	4	.000004	1.00000	.0157	
.1	5	.004299	.99996	.8785	
	6	.000073	1.00000	.1420	
.01	9	.000045	1.00000	.3274	
.001	13	.000008	1.00000	.5738	

n = 4

η	M	ϵ	$\bar{\delta}$	$\underline{\delta}_a$	$\underline{\delta}_b$
.5	3	.00488	.99995	.7225	.5481
	4	.00001	1.00000	.0394	.0315
.1	6	.00016	1.00000	.3728	.2902
.01	9	.00011	1.00000	.9087	.6792

n = 5

η	M	ϵ	$\bar{\delta}$	$\underline{\delta}_a$	$\underline{\delta}_b$
.5	4	.00002	1.00000	.0715	
.1	6	.00031	1.00000	.7135	

 $\epsilon \leq 10^{-6}$ $\bar{\delta} = 1.0$

n	η	M	$\underline{\delta}_a$	$\underline{\delta}_b$
2	.5	5	.0001	.0001
	.1	7	.0024	.0025
	.01	10	.0116	.0117
	.001	13	.2530	.2531
	.0001	16	.3740	.3741
4	.5	5	.0001	.0002
	.1	7	.0118	.0095
	.01	10	.0591	.0472
	.001	14	.1722	.1361
	.0001	17	.4119	.3198
6	.5	5	.0006	.0004
	.1	7	.0453	.0214
	.01	10	.2335	.1080
	.001	14	.7271	.3223
	.0001	18	.0546	.0258
8	.5	5	.0021	.0006
	.1	7	.1693	.0382
	.01	10	.9652	.1964
	.001	15	.0381	.0089
	.0001	18	.2052	.0460
10	.5	5	.0077	.0009
	.1	7	.6794	.0561
	.01	11	.0171	.0018
	.001	15	.1420	.0138
	.0001	18	.8382	.0724
12	.5	5	.0284	.0012
	.1	8	.0026	.0011
	.01	11	.0636	.0025
	.001	15	.5685	.0199
	.0001	19	.0037	.0002
14	.5	5	.1075	.0016
	.1	8	.0096	.0002
	.01	11	.2991	.0336
	.001	16	.0005	.0001
	.0001	19	.0138	.0003
16	.5	5	.4313	.0021
	.1	8	.0361	.0003
	.01	12	.0001	.0001
	.001	16	.0019	.0001
	.0001	19	.0524	.0004
20	.5	6	.0001	.0001
	.1	8	.5795	.0004
	.01	12	.0004	.0001
	.001	16	.0278	.0001
	.0001	19	.8801	.0005
30	.5	6	.0735	.0070
	.1	9	.0033	.0007
	.01	12	.3602	.0154
	.001	17	.0001	.0001
	.0001	20	.0001	.0001

- * n degree of polynomial
 η separation coefficient
 ϵ tolerated relative error in modulus
M number of root-squaring operations (Eq. (7.6) page 31)

 $\bar{\delta}$ upper pivotal bound Eq. (6.9) $\underline{\delta}_a$ lower pivotal bound Eqs. (2.2) or (6.8) $\underline{\delta}_b$ lower pivotal bound Eqs. (6.5) to (6.7)For computation use either $\underline{\delta} = \underline{\delta}_a$ or $\underline{\delta} \geq \underline{\delta}_b$ ($\underline{\delta}_b$ is more accurate).

3. ROOT-SQUARING OPERATIONS AND PIVOTAL COEFFICIENTS

Given is a polynomial of degree n with real coefficients

$$P(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n \quad (3.1)$$

Its zeros are denoted by

$$\alpha_k = \rho_k e^{\phi_k i} \quad (i = \sqrt{-1}) \quad (3.2)$$

The root-squaring algorithm consists in forming a sequence of polynomials

$$P^{(1)}(x), P^{(2)}(x), \cdots, P^{(m)}(x), \cdots$$

whose zeros are

$$\alpha_k^{(m)} = \alpha_k^{z^m} = \rho_k^{z^m} e^{z^m \phi_k i} \quad (k = 1, 2, \cdots, n)$$

The polynomials

$$P^{(m)}(x) = a_0^{(m)} x^n + a_1^{(m)} x^{n-1} + \cdots + a_{n-1}^{(m)} x + a_n^{(m)} \quad (m = 0, 1, 2, \cdots) \quad (3.3)$$

are obtained from the relation

$$P^{(0)}(x) = P(x)$$

$$P^{(m+1)}(x) = (-1)^n P^{(m)}(z) P^{(m)}(-z) \quad (m = 0, 1, 2, \cdots)$$

where $z^2 = x$ is substituted on the right hand side. Equating the coefficients on both sides of this equation yields the recurrence formula for the coefficients $a_j^{(m+1)}$,

$$a_j^{(m+1)} = (-1)^j \left\{ \left[a_j^{(m)} \right]^2 + 2 \sum_{\ell=1}^{\min(n-j, j)} (-1)^\ell a_{j-\ell}^{(m)} a_{j+\ell}^{(m)} \right\} \quad (3.4)$$

The sum has to be extended until the dummy index ℓ reaches either j or $(n-j)$ whichever is smaller. The coefficients $a_j^{(m)}$ are Newton's elementary symmetric functions of the zeros $\alpha_k^{(m)}$ multiplied by $\pm a_0^{(m)}$,

$$a_j^{(m)} = (-1)^j a_0^{(m)} \left\{ \left(\alpha_1^{(m)} \alpha_2^{(m)} \dots \alpha_j^{(m)} \right) + \dots \right. \\ \left. + \left(\alpha_{n-j+1}^{(m)} \alpha_{n-j+2}^{(m)} \dots \alpha_n^{(m)} \right) \right\}$$

Using polar coordinates, the general coefficient is

$$a_j^{(m)} = (-1)^j a_0^{(m)} \sum_{\binom{n}{j}} \left(\rho_{k_1} \rho_{k_2} \dots \rho_{k_j} \right)^2 e^{z^m (\phi_{k_1} + \phi_{k_2} + \dots + \phi_{k_j}) i} \quad (3.5)$$

where the sum contains the $\binom{n}{j}$ terms corresponding to the number of possible combinations of j zeros α_k .

The following two lemmas will be used to prove the existence theorem.

Lemma 1. The product of all roots $\alpha_k^{(m)}$ of $P^{(m)}(x) = 0$ which have equal absolute value, $|\alpha_k^{(m)}| = \rho$, ($k = k_1, k_2, \dots, k_\nu$), is

$$\alpha_{k_1}^{(m)} \cdot \alpha_{k_2}^{(m)} \dots \alpha_{k_\nu}^{(m)} = \rho^{2^{m \cdot \nu}} \quad (m = 1, 2, \dots)$$

where ν is the multiplicity of the absolute value ρ .

Proof: Since all roots of $P^{(m)}(x) = 0$ are either real, or occur in pairs of conjugate complex numbers, such that for each argument ϕ_k there exists an argument $\phi_\ell = -\phi_k$, it follows that

$$\alpha_{k_1}^{(m)} \alpha_{k_2}^{(m)} \dots \alpha_{k_\nu}^{(m)} = \left(\rho^{2^m} e^{z^m \phi_{k_1} i} \right) \left(\rho^{2^m} e^{z^m \phi_{k_2} i} \right) \dots \left(\rho^{2^m} e^{z^m \phi_{k_\nu} i} \right) \\ = \rho^{2^{m \cdot \nu}} e^{z^m (\phi_{k_1} + \phi_{k_2} + \dots + \phi_{k_\nu}) i} \\ = \rho^{2^{m \cdot \nu}} e^{z^{m \cdot 0}}$$

Lemma 2. Of the n roots of $P(x) = 0$, let ν_1 roots have the same absolute value ρ_1 , let ν_2 roots have the absolute value ρ_2 , and so on, with

$$\nu_1 + \nu_2 + \nu_3 + \cdots + \nu_N = n \quad 1 \leq \nu_i \leq n$$

and

$$\rho_1 > \rho_2 > \rho_3 > \cdots > \rho_N \geq 0$$

With this arrangement, the term of largest absolute value in the $(\nu_1 + \nu_2 + \cdots + \nu_k)$ th elementary symmetric function of the zeros of $P^{(m)}(x)$ has the form

$$\left(\rho_1^{\nu_1} \rho_2^{\nu_2} \cdots \rho_k^{\nu_k} \right)^{2^m}$$

and is positive unless $k = N$ and $\rho_N = 0$,

for $m = 1, 2, \cdots$ and $k \leq N$.

Proof: Each term of the $(\nu_1 + \nu_2 + \cdots + \nu_k)$ th elementary symmetric function contains $(\nu_1 + \nu_2 + \cdots + \nu_k)$ factors. Because of the arrangement according to magnitude, and because of Lemma 1, the term of largest absolute value is

$$\rho_1^{2^m \nu_1} \rho_2^{2^m \nu_2} \cdots \rho_k^{2^m \nu_k},$$

which is real and positive for $m \geq 1$.

Corollary: When $\kappa = \nu_1 + \nu_2 + \cdots + \nu_k$, the coefficient $a_{\kappa}^{(m)}$ is asymptotically

$$(-1)^{\kappa} a_0^{(m)} \left(\rho_1^{\nu_1} \rho_2^{\nu_2} \cdots \rho_k^{\nu_k} \right)^{2^m}$$

for $m \rightarrow \infty$

Proof: The coefficient $a_{\kappa}^{(m)}$ can be represented in polar coordinates by

$$a_{\kappa}^{(m)} = (-1)^{\kappa} a_0^{(m)} \left(\rho_1^{\nu_1} \rho_2^{\nu_2} \cdots \rho_k^{\nu_k} \right)^{2^m} \left\{ 1 + \delta_{\kappa}^{(m)} \right\} \quad (3.6)$$

where $\kappa = \nu_1 + \nu_2 + \dots + \nu_k$. The expression $\left(\rho_1^{\nu_1} \rho_2^{\nu_2} \dots \rho_k^{\nu_k} \right)^{2^m} \delta_\kappa^{(m)}$ contains the remaining $\binom{n}{\kappa} - 1$ terms of the κ^{th} elementary symmetric function. The largest term in $\delta_\kappa^{(m)}$ is bounded by $\rho_{k+1}^{2^m} / \rho_k^{2^m}$, and therefore

$$\left| \delta_\kappa^{(m)} \right| \leq \left[\binom{n}{\kappa} - 1 \right] \left(\rho_{k+1} / \rho_k \right)^{2^m} < \frac{2^{n+1}}{\sqrt{2\pi n}} \left(\frac{\rho_{k+1}}{\rho_k} \right)^{2^m} \quad (3.7)$$

Since $\left| \rho_{k+1} / \rho_k \right| < 1$, $\delta_\kappa^{(m)}$ can be made arbitrarily small, say $\left| \delta_\kappa^{(m)} \right| \leq \delta_\kappa$, by taking the number of root squarings m sufficiently large. We are now ready to prove the existence theorem. The quantitative evaluation of this formula is postponed to Sections 6 and 7.

Existence Theorem: For the zeros α of the polynomial $P(x)$ of absolute value $|\alpha| = \rho$, there exist coefficients $a_j^{(m)}$ which satisfy for sufficiently large m the relation

$$a_{j+\nu}^{(m)} = (-1)^\nu a_j^{(m)} \rho^{2^m \cdot \nu}$$

within a given accuracy, where ν is the multiplicity of all zeros of absolute value ρ .

This relationship will be used in the following sections in the case of non-exceptional polynomials to determine ρ and ν from the coefficients $a_j^{(m)}$. The proof of the theorem follows from the corollary and the polar coordinate representation of $a_\kappa^{(m)}$, Eq. (3.6). Assume ρ immediately follows ρ_k . Then

$$\begin{aligned} a_{\kappa+\nu}^{(m)} &= (-1)^{\kappa+\nu} a_0^{(m)} \left(\rho_1^{\nu_1} \rho_2^{\nu_2} \dots \rho_k^{\nu_k} \cdot \rho^\nu \right)^{2^m} \left\{ 1 + \delta_{\kappa+\nu}^{(m)} \right\} \\ &= (-1)^\nu a_\kappa^{(m)} \rho^{2^m \cdot \nu} \left\{ \left(1 + \delta_{\kappa+\nu}^{(m)} \right) / \left(1 + \delta_\kappa^{(m)} \right) \right\} \\ &= (-1)^\nu a_\kappa^{(m)} \rho^{2^m \cdot \nu} \left\{ 1 + \delta \right\} \end{aligned} \quad (3.8)$$

where

$$\delta' = \left\{ \left(1 + \delta_{\kappa+\nu}^{(m)} \right) / \left(1 + \delta_{\kappa}^{(m)} \right) \right\} - 1.$$

δ' can be made arbitrarily absolutely small.

For the purpose of convenient reference, the concept of "pivotal coefficient" is introduced.

Definition: Pivotal coefficients $a_j^{(m)}$ satisfy for sufficiently large m the relation

$$\frac{a_j^{(m)^2}}{a_j^{(m+1)}} = (-1)^j$$

within a given accuracy, say δ . The coefficient $a_n^{(m)}$ is pivotal by definition.

Polynomials with $a_n = 0$ will not need special treatment. Since $a_0 \neq 0$, a_0 is always pivotal.

For sufficiently large m , the coefficients $a_{\kappa}^{(m)} = a_{\nu_1+\nu_2+\dots+\nu_k}^{(m)}$, i.e.,

$$a_0^{(m)}, a_{\nu_1}^{(m)}, a_{\nu_1+\nu_2}^{(m)}, \dots, a_{\kappa}^{(m)}, \dots, a_{\nu_1+\nu_2+\dots+\nu_N}^{(m)} = a_n^{(m)} \quad (3.9)$$

satisfy the relation of the existence theorem and the condition for pivotal coefficients. Using again representation in polar coordinates,

$$\frac{a_{\kappa}^{(m)^2}}{a_{\kappa}^{(m+1)}} = \frac{(-1)^{2\kappa} \left(\rho_1^{\nu_1} \dots \rho_k^{\nu_k} \right)^{2^m} \cdot 2 \left(1 + \delta_{\kappa}^{(m)} \right)^2}{(-1)^{\kappa} \left(\rho_1^{\nu_1} \dots \rho_k^{\nu_k} \right)^{2^{m+1}} \left(1 + \delta_{\kappa}^{(m+1)} \right)} = (-1)^{\kappa} (1 + \delta''),$$

where $\delta'' = \left(1 + \delta_{\kappa}^{(m)} \right)^2 / \left(1 + \delta_{\kappa}^{(m+1)} \right) - 1$ can be made arbitrarily absolutely small, say $|\delta''| < \delta$.

The set of coefficients (3.9) is called the regular set of pivotal coefficients. We shall see in the following section that

- (a) there may exist other pivotal coefficients, called irregular pivotal coefficients,
- (b) the regular pivotal coefficients furnish the moduli of the zeros of $P(x)$ in descending order.

4. THE EXCEPTIONAL CASES IN THE GRAEFFE ALGORITHM

The existence theorem assured us that we can always find the absolute value of the zeros of a polynomial $P(x)$ within predetermined accuracy from Eq. (3.8), i.e.,

$$\rho z^m \cdot \nu \approx (-1)^\nu a_{j+\nu}^{(m)} / a_j^{(m)} \quad (4.1)$$

The coefficients $a_j^{(m)}$, $a_{j+\nu}^{(m)}$ are successive coefficients of the regular pivotal set. We investigate now the condition under which a coefficient is pivotal. We recall again the representation of $a_j^{(m)}$ by the Newton's elementary symmetric functions in polar coordinates (3.5)

$$a_j^{(m)} = (-1)^j a_0^{(m)} \sum_{\binom{n}{j}} (\rho_{k_1} \rho_{k_2} \dots \rho_{k_j})^2 e^{z^m (\phi_{k_1} + \phi_{k_2} + \dots + \phi_{k_j}) i}$$

For sufficiently large m , and considering only the J absolute largest zeros,

$$\frac{a_j^{(m)^2}}{a_j^{(m+1)}} = (-1)^j \frac{\left\{ \sum_{\binom{J}{j}} (\rho_{k_1} \rho_{k_2} \dots \rho_{k_j})^2 e^{z^m (\phi_{k_1} + \phi_{k_2} + \dots + \phi_{k_j}) i} \right\}^2 \left[1 + \delta_{j,J}^{(m)} \right]^2}{\left\{ \sum_{\binom{J}{j}} (\rho_{k_1} \rho_{k_2} \dots \rho_{k_j})^{2^{m+1}} e^{z^{m+1} (\phi_{k_1} + \phi_{k_2} + \dots + \phi_{k_j}) i} \right\} \left[1 + \delta_{j,J}^{(m+1)} \right]}$$

If $J = j$, only one term appears under the summation signs, and the condition for a pivotal point is satisfied.

If $J > j$, $\binom{J}{j}$ possible significant terms are under the summation signs. If the moduli $(\rho_{k_1} \rho_{k_2} \dots \rho_{k_j})$ are all equal, the condition for pivotal coefficients is

$$\left\{ \sum_{\binom{J}{j}} e^{z^m (\phi_{k_1} + \phi_{k_2} + \dots + \phi_{k_j}) i} \right\}^2 = \sum_{\binom{J}{j}} e^{z^{m+1} (\phi_{k_1} + \phi_{k_2} + \dots + \phi_{k_j}) i}$$

Such a condition cannot exist if all ϕ_k 's are zero since $\binom{J}{j}^2 = \binom{J}{j}$. However, if the ϕ_k 's are of the form π/s ($s = 1, 2, 3, \dots$), many combinations can be found for which the condition is satisfied (or approximately satisfied). In particular, some or all terms may cancel each other; it is even possible to obtain coefficients $a_j^{(m)}$ which become zero. If the moduli $(\rho_{k_1} \rho_{k_2} \dots \rho_{k_j})$ are not all equal, it is still possible that some terms cancel each other. These terms may contain the absolute largest ρ_k 's. As a consequence, one may not obtain the absolute values of the roots α of the polynomial equation $P(x) = 0$ in descending order by the Graeffe algorithm. Examples are given in Sections 5 and 13.

Theorem: The Graeffe algorithm does not necessarily yield the moduli of the roots of $P(x) = 0$ in descending order.

But assume such an irregular pivotal coefficient has been used to determine μ of the ν roots α with absolute value ρ , ($\mu \leq \nu$). We can use this pivotal coefficient to determine the next roots as if it were a regular pivotal coefficient. It can happen that the next pivotal coefficient yields no root, the same ρ as before, or the ρ to some new roots α . To prove this statement, we use again the representation of the coefficients by the elementary symmetric functions in polar coordinates, Eq. (3.5). Since all following coefficients contain the terms of this irregular coefficient $a_\ell^{(m)}$ we have a new sequence of coefficients

$$a_{\ell+j}^{(m)} = (-1)^j a_\ell^{(m)} \left\{ \sum (\rho_{k_1} \rho_{k_2} \dots \rho_{k_j})^2 e^{2^m(\phi_{k_1} + \phi_{k_2} + \dots + \phi_{k_j})i} \right\} \left(1 + \delta_{\ell|j}^{(m)} \right)$$

and these have the same sequence of terms in the brackets () as we had by deriving the set of regular pivotal coefficients $a_0^{(m)}, a_{\nu_1}^{(m)}, \dots, a_\kappa^{(m)}$, with the exception that the factors ρ contained in $a_\ell^{(m)}$ will be missing. We state our findings in the following theorem.

Theorem: Once a pivotal coefficient has yielded the absolute value of a zero of $P(x)$, it may be treated like a regular pivotal coefficient.

Examples are given in Sections 5 and 13.

5. EXAMPLES OF POLYNOMIALS WITH EXCEPTIONAL PIVOTAL COEFFICIENTS

A class of polynomials which demonstrate such extraneous behavior are the polynomials that contain at least one cyclotomic polynomial of odd degree as a factor. We denote the product of all cyclotomic polynomials of equal degree ν contained in $P(x)$ by

$$C_\nu(x) = (x^\nu - c_1)^\alpha (x^\nu - c_2)^\beta \cdots (x^\nu - c_\ell)^\lambda \quad (5.1)$$

and the remaining factor which represents a regular polynomial by $P_0(x)$. Then, a general polynomial can be represented in the form

$$P^{(m)}(x) = P_0^{(m)}(x) \cdot C_3^{(m)}(x) C_5^{(m)} \cdots \quad (5.2)$$

We represent $P^{(m)}(x)$ as

$$P^{(m)}(x) = (x^\nu - c)^k \cdot p_{n-\nu k}^{(m)}(x) = \sum_{j=0}^n a_j^{(m)} x^{n-j} \quad (5.3)$$

By comparing the coefficients of the middle term with the corresponding coefficients of the right hand side of Eq. (5.3), we obtain for each cyclotomic factor $(x^\nu - c)^k$ the following appropriate alternatives:

(a) If $(n - \nu k) < \nu$ ($\nu = 3, 5, \dots$)

$$\frac{a_s^{(m)}}{a_{s-1}^{(m)}} = \frac{a_{s+\nu\kappa}^{(m)}}{a_{s+\nu\kappa-1}^{(m)}} \quad \text{for } s = 1, 2, \dots, (n - \nu\kappa) \quad (5.4)$$

($\kappa = 1, 2, 3, \dots, k$)

$$a_{n-\nu\kappa+1}^{(m)} = \cdots = a_{\nu\kappa-1}^{(m)} = 0 \quad \text{for the remaining coefficients if any.}$$

(b) If $(n - \nu k) \geq \nu$, the coefficients of the cyclotomic polynomial are overlapped completely by the coefficients of $P_{n-\nu k}^{(m)}(x)$, and the first of the above relations (a) holds only approximately, i.e., for sufficiently large m , and in general not for all s mentioned above.

$$\frac{a_s^{(m)}}{a_{s-1}^{(m)}} = \frac{a_{s+\nu\kappa}^{(m)}}{a_{s+\nu\kappa-1}^{(m)}} (1 + \delta''') \quad (5.5)$$

where $\delta^{(m)}$ decreases with increasing m .

In both cases, the absolute value of the roots of the cyclotomic equation can be found as in (4.1) from

$$(-1)^\nu \frac{a_s^{(m)}}{a_s} = \rho^{2m \cdot \nu}$$

in the first case (a) exactly, in the second case (b) approximately. Similar conditions hold for the general equation (5.2). Two simple examples may demonstrate these cases.

Example 1: Let $\nu = 3$, $k = 1$, $n = 4$. Then, Eq. (5.3) is, with $c = r_m^3$,

$$\begin{aligned} P^{(m)}(x) &= (x^3 - r_m^3)(x - \alpha_m) \\ &= x^4 - \alpha_m x^3 - r_m^3 x + \alpha_m r_m^3 \\ &= a_0^{(m)} x^4 + a_1^{(m)} x^3 + a_2^{(m)} x^2 + a_3^{(m)} x + a_4^{(m)} \end{aligned}$$

The ratios are: $\frac{a_1^{(m)}}{a_0^{(m)}} = -\alpha_m$; $\frac{a_2^{(m)}}{a_1^{(m)}} = 0$; $\frac{a_3^{(m)}}{a_2^{(m)}}$ is nonexistent; $\frac{a_4^{(m)}}{a_3^{(m)}} = -\alpha_m$

The pivotal coefficients are $a_0^{(m)}$, $a_1^{(m)}$, $a_3^{(m)}$, $a_4^{(m)}$. The first ratio

$(-1) \frac{a_1^{(m)}}{a_0^{(m)}} = \alpha_m$ always yields a root even if $\alpha_m < r_m^3$. The second

ratio $(-1)^2 \frac{a_3^{(m)}}{a_1^{(m)}} = r_m^3 / \alpha_m$ does not lead to a root, however, according to condition (a),

$$(-1)^3 \frac{a_4^{(m)}}{a_3^{(m)}} = (-1)^3 \frac{a_4^{(m)}}{a_1^{(m)}} = r_m^3.$$

Example 2: We take next $\nu = 3$, $k = 1$, $n = 7$.

$$\begin{aligned} P^{(m)}(x) &= (x^3 - r_m^3)(x^4 + a^{(m)} x^3 + b^{(m)} x^2 + c^{(m)} x + d^{(m)}) \\ &= x^7 + a^{(m)} x^6 + b^{(m)} x^5 + (c^{(m)} - r_m^3) x^4 + (d^{(m)} - a^{(m)} r_m^3) x^3 \\ &\quad - b^{(m)} r_m^3 x^2 - c^{(m)} r_m^3 x - d^{(m)} r_m^3 \end{aligned}$$

Assuming $\rho_1 > \rho_2 > \rho_3 > \rho_4$, but with no restriction on r , the coefficients $a_j^{(m)}$ are, for sufficiently large m , approximately given as follows:

$a_0^{(m)}$	$a_1^{(m)}$	$a_2^{(m)}$	$a_3^{(m)}$	$a_4^{(m)}$
1	$-\rho_1^{2m}$	$(\rho_1 \rho_2)^{2m}$	$-\left[(\rho_1 \rho_2 \rho_3)^{2m} + r_m^3\right]$	$\left[(\rho_1 \rho_2 \rho_3 \rho_4)^{2m} + \rho_1^{2m} r_m^3\right]$
$a_5^{(m)}$	$a_6^{(m)}$	$a_7^{(m)}$		
$-(\rho_1 \rho_2)^{2m} r_m^3$	$(\rho_1 \rho_2 \rho_3)^{2m} r_m^3$	$-(\rho_1 \rho_2 \rho_3 \rho_4)^{2m} r_m^3$		

The pivotal elements are $a_0^{(m)}, a_1^{(m)}, a_2^{(m)}, a_5^{(m)}, a_6^{(m)}, a_7^{(m)}$ which yield $\rho_1, \rho_2, r, \rho_3, \rho_4$ in this order without regard to the respective magnitude of r . Consider, e.g., $a_5^{(m)}$. We can restrict ourselves to list the terms which contain $\rho_1 \rho_2$. Hence

$$\begin{aligned}
 a_5^{(m)} &= -a_0^{(m)} \left\{ \alpha_1^{(m)} \alpha_2^{(m)} r_m^3 + \alpha_1^{(m)} \alpha_2^{(m)} \alpha_3^{(m)} r_m^2 (1 + e^{2\pi i/3} + e^{4\pi i/3}) \right. \\
 &\quad \left. + \alpha_1^{(m)} \alpha_2^{(m)} \alpha_3^{(m)} \alpha_4^{(m)} r_m (1 + e^{2\pi i/3} + e^{4\pi i/3}) + \dots \right\} \\
 &= -a_0^{(m)} \left\{ \alpha_1^{(m)} \alpha_2^{(m)} r_m^3 + 0 + 0 + \dots \right\}
 \end{aligned}$$

The ratios $a_s^{(m)}/a_{s-1}^{(m)}$ are, approximately, for $r < \rho_4$:

$$\begin{array}{ccccccc}
 -\rho_1^{2m} & -\rho_2^{2m} & -\rho_3^{2m} & -\rho_4^{2m} & -r_m^3/(\rho_3 \rho_4)^{2m} & -\rho_3^{2m} & -\rho_4^{2m} \\
 & & \uparrow & \uparrow & & \uparrow & \uparrow \\
 & & \underbrace{\hspace{10em}}_{\text{---}} & & & &
 \end{array}$$

Corresponding ratios are three units apart according to the cyclotomic polynomial $(x^3 - r_m^3)^1$. Hence, again

$$r_m^3 = (-1)^3 \frac{a_5^{(m)}}{a_2^{(m)}} = (-1)^3 \frac{a_6^{(m)}}{a_3^{(m)}} = (-1)^3 \frac{a_7^{(m)}}{a_4^{(m)}} .$$

For $r > \rho_1$ the ratios are

$$\begin{array}{ccccccc}
 -\rho_1^{2m} & -\rho_2^{2m} & -r_m^3/(\rho_1\rho_2)^{2m} & -\rho_1^{2m} & -\rho_2^{2m} & -\rho_3^{2m} & -\rho_4^{2m} \\
 \uparrow & \uparrow & & \uparrow & \uparrow & & \\
 \hline
 & & \text{---} & & & &
 \end{array}$$

and therefore

$$r_m^{(3)} = (-1)^3 \frac{a_3^{(m)}}{a_0^{(m)}} = (-1)^3 \frac{a_4^{(m)}}{a_1^{(m)}} = (-1)^3 \frac{a_5^{(m)}}{a_2^{(m)}}.$$

Similar expressions are obtained when $\rho_1 > r > \rho_4$.

Further complications arise when ν or more zeros of the regular factor of the polynomial have the absolute values $c^{1/2m}$. If, e.g., in the last example $\rho_1 = \rho_2 = \rho_3 = r$, the criterion (b) Eq. (5.5) is not reliable any more. However, the regular pivotal coefficients $a_0^{(m)}$, $a_6^{(m)}$ and $a_7^{(m)}$ yield r and ρ_4 .

Conclusions: It does not seem advantageous to include the criteria of this section in the code for a high speed computer, since the exceptional cases are not frequent in engineering applications, and the selection of pivotal coefficients will always lead to the complete solution of $P(x) = 0$ if the roots are tested to satisfy $P(\alpha) \approx 0$ (Section 12). However, the numerical analyst and the hand computer may find it worthwhile to remember these criteria.

6. CRITERIA FOR THE ACCEPTANCE OR REJECTION OF COEFFICIENTS AS PIVOTAL

We shall derive some rules to recognize the pivotal coefficients after M root squarings when the computation is actually carried out on a high speed computer, or by hand. A pivotal coefficient $a_j^{(M)}$ must satisfy the condition

$$\frac{a_j^{(M-1)^2}}{a_j^{(M)}} = (-1)^j \left(1 + \delta_j^{(M)} \right) \quad (6.1)$$

where the absolute bounds for $\delta_j^{(M)}$ are predetermined. In the following, rules for the determination of these bounds are derived. We first recall that the above condition for the pivotal coefficients can be replaced, for theoretical purposes, by the following ratio of corresponding Newton's elementary symmetric functions

$$\frac{s_j^{(M-1)^2}}{s_j^{(M)}} = 1 + \delta_j^{(M)} .$$

In order to arrive at the bounds for $\delta_j^{(M)}$ we consider some extreme sets of possible zeros of the polynomial $P(x)$. We first consider the coefficient $a_1^{(M)}$ of $P(x)$. Assume we have one root $\alpha_1 = \rho$ and $(n-1)$ roots $\alpha_2 = \dots = \alpha_n = \rho(1-\eta)$.* The first elementary function is then

$$s_1^{(M)} = \rho^{2^M} \left[1 + (n-1)(1-\eta)^{2^M} \right] .$$

The general expression for $\delta_j^{(M)}$ is

$$\delta_j^{(M)} = \frac{s_j^{(M-1)^2} - s_j^{(M)}}{s_j^{(M)}} \quad (6.2)$$

*The symbol η is called "separation coefficient."

Hence, inserting $s_1^{(M-1)}$ and $s_1^{(M)}$ in this formula, and rearranging, yields

$$\delta_1^{(M)} = 2(n-1)(1-\eta)^{2^{M-1}} \frac{1 + \frac{1}{2} (n-2)(1-\eta)^{2^{M-1}}}{1 + (n-1)(1-\eta)^{2^M}}. \quad (6.3)$$

For $\eta = 0$, $\delta_1^{(M)} = n-1$; for $\eta = 1$, $\delta_1^{(M)} = 0$.

In order for the roots ρ and $\rho(1-\eta)$ to be recognized as being different, i.e., in order for $a_1^{(M)}$ to be a pivotal coefficient, the least upper bound $\overline{\delta_1^{(M)}}$ for $|\delta_1^{(M)}|$ must satisfy the inequality

$$\overline{\delta_1^{(M)}} \geq \delta_1^{(M)}.$$

On the other hand, if two roots ρ and $\rho(1-\eta)$ should be considered equal for practical purposes, the $a_1^{(M)}$ should not be recognized as a pivotal coefficient. The most unfavorable condition for multiple real roots exists when there are only these two roots which differ from zero, and therefore

$$s_1^{(M)} = \rho^{2^M} [1 + (1-\eta)^{2^M}].$$

Inserting $s_1^{(M)}$ into the equation (6.2) for $\varepsilon_j^{(M)}$ yields, after some algebra,

$$\delta_1^{(M)} = 2 \left[(1-\eta)^{2^{M-1}} + (1-\eta)^{-2^{M-1}} \right]^{-1}.$$

The greatest lower bound of $|\delta_1^{(M)}|$ to consider two roots ρ and $\rho(1-\eta)$ as equal, i.e., to assure that $a_1^{(M)}$ is not pivotal, is given by

$$\overline{\delta_1^{(M)}} < 2 \left[(1-\eta)^{2^{M-1}} + (1-\eta)^{-2^{M-1}} \right]^{-1} \leq 1. \quad (6.4)$$

Note that this equation does not depend on n . The magnitude of η is usually very small in the cases under consideration.* Hence these inequalities can

*The separation coefficient η in Eq. (6.4) will be identified with the relative error ϵ of ρ in Section 7.

be troublesome only when $\overline{M}$ is unduly large and n very small so that $\delta_1^{(M)} \approx \delta_1^{(M)}$, instead of $\delta_1^{(M)} < \delta_1^M$.

Assume now that half of the roots are ρ , and half of them are $\rho(1-\eta)$, i.e.,

$$\alpha_1 = \dots = \alpha_{n/2} = \rho; \alpha_{(n/2)+1} = \dots = \alpha_n = \rho(1-\eta) .$$

For the sake of simplicity, assume n to be even. The corresponding elementary symmetric function is

$$s_{n/2}^{(M)} = \left(\rho^{2^M \cdot n/2} \right) \sum_{k=0}^{n/2} \binom{n/2}{k}^2 (1-\eta)^{2^M \cdot k} \quad (6.5)$$

and

$$\delta_{n/2}^{(M)} = \frac{s_{n/2}^{(M-1)^2}}{s_{n/2}^{(M)}} - 1 \quad (6.6)$$

The coefficient $a_{n/2}^{(M)}$ has to be pivotal if ρ is considered different from $\rho(1-\eta)$. To assure $a_{n/2}^{(M)}$ to be pivotal, the lower pivotal bound must satisfy

$$\underline{\delta}^{(M)} = \underline{\delta}_{n/2}^{(M)} \geq \delta_{n/2}^{(M)} . \quad (6.7)$$

Eqs. (6.5) to (6.7) have been used to compute Table I, columns $\underline{\delta}_b$. To simplify the determination of $\underline{\delta}_{n/2}^{(M)}$, we make use of the following inequalities:

$$\left[\binom{n}{n/2} - 1 \right] (1-\eta)^{2^M} + 1 \geq \frac{s_{n/2}^{(M)}}{\rho^{2^M \cdot n/2}} \geq \left[\binom{n}{n/2} - 1 \right] (1-\eta)^{2^M \cdot n/2} + 1 .$$

Then, for any η between 0 and 1, we have

$$\underline{\delta} = \underline{\delta}^{(M)} \geq \frac{\left\{ \left[\binom{n}{n/2} - 1 \right] (1-\eta)^{2^{M-1}} + 1 \right\}^2}{\left[\binom{n}{n/2} - 1 \right] (1-\eta)^{2^{M-1} n} + 1} \geq \delta_{n/2}^{(M)} \quad (6.8)$$

as a sufficient condition that ρ and $\rho(1-\eta)$ be recognized as being different. This equation has been used to compute Table I, columns $\underline{\delta}_a$.

Next, we show that $\underline{\delta}_{n/2}^{(M)} \geq \underline{\delta}_1^{(M)}$. In both cases, we have the general formula

$$\underline{\delta}_j^{(M)} \geq \frac{[(x-1)b+1]^2}{(x-1)by+1} - 1 = f(x, y).$$

The derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are both non-negative for the range of the coefficients under consideration; i.e.,

$$0 < n \leq x \leq \binom{n}{n/2} < \infty,$$

$$0 < 2 \leq y \leq n < \infty,$$

$$0 \leq b = (1-\eta)^{2^{M-1}} \leq 1.$$

Therefore

$$\underline{\delta}_{n/2}^{(M)} \geq \underline{\delta}_1^{(M)}.$$

We now turn to the opposite problem. What are the upper bounds $\underline{\delta}_{n/2}^{(M)}$ for $\underline{\delta}_{n/2}^{(M)}$ so that any two roots ρ and $\rho(1-\eta)$ will not be considered as distinct by the test for pivotal coefficients. The most unfavorable case for real roots occurs when $\rho_{(n/2)+1} = \rho_{n/2}(1-\eta)$, all following roots vanish, and the ratio of $\left(\rho_{n/2}/\rho_{(n/2)-1}\right)^{2^{M-1}}$ is negligible in comparison to unity. Then

$$s_{n/2}^{(M-1)} = \left(\rho_1 \rho_2 \cdots \rho_{(n/2)-1} \rho_{n/2}\right)^{2^{M-1}} \left[1 + (1-\eta)^{2^{M-1}}\right].$$

For this case, the criterion that $a_{n/2}^{(M)}$ shall not be pivotal, is

$$\boxed{\bar{\delta} = \overline{\delta^{(M)}} \leq \delta_{n/2}^{(M)} = 2 \left[(1-\eta)^{2^{M-1}} + (1-\eta)^{-2^{M-1}} \right]^{-1}} \quad (6.9)$$

as for $\delta_1^{(M)}$. It can be seen that this greatest lower bound holds for all coefficients, and therefore is independent of j and n . Again $\delta_{n/2}^{(M)} \leq \delta_{n/2}^{(M)}$ for the same η , the equality being approached with increasing M . Eq. (6.9) has been used to compute Table I, column $\bar{\delta}$.

Conclusion: (i) In order that all roots whose absolute values differ by a factor of $1:(1-\eta)$ or more can be separated, one determines a $\bar{\delta}^{(M)}$ from equations (6.8) or (6.7).

(ii) In order that roots whose absolute values differ by a factor of $1:(1-\eta)$ or less be treated as of equal modulus, one determines a $\bar{\delta}^{(M)}$ from equation (6.9).

7. DETERMINATION OF THE MINIMUM NUMBER OF ROOT-SQUARING OPERATIONS

In all previous equations we have assumed that the number M of root squarings is known. In the following, we derive formulas for the determination of M . Preceding the formulation of the existence theorem, we demonstrated, Eq. (3.7), that each pivotal coefficient can be represented as

$$a_{\kappa}^{(m)} = (-1)^{\kappa} a_0^{(m)} \left(\rho_1^{\nu_1} \rho_2^{\nu_2} \cdots \rho_k^{\nu_k} \right) 2^m \quad (\kappa = \nu_1 + \nu_2 + \cdots + \nu_k)$$

within a relative error which is bounded by

$$\left| \delta_{\kappa}^{(m)} \right| \leq \left[\binom{n}{\kappa} - 1 \right] \left(\rho_{k+1} / \rho_k \right) 2^m.$$

If the largest ratio of two consecutive roots is

$$\rho_{k+1} / \rho_k \leq 1 - \eta$$

as previously noted, all $\delta_{\kappa}^{(m)}$ are bounded by

$$\delta_* = \left[\binom{n}{n/2} - 1 \right] (1 - \eta) 2^m < \frac{2^{n+1}}{\sqrt{2\pi n}} (1 - \eta) 2^m \quad (7.1)$$

It follows immediately that for $\rho^{2^m \cdot \nu} \doteq (-1)^{\nu} a_{j+\nu}^{(m)} / a_j^{(m)}$, the relative

error is bounded by $(2 \delta_*) / (1 - \delta_*)$. Therefore, ρ can be determined* from

$$\rho = \exp \left\{ \left[\log (-1)^{\nu} a_{j+\nu}^{(m)} / a_j^{(m)} \right] / (2^{m \cdot \nu}) \right\} \quad (7.2)$$

with a relative error ϵ_t smaller than

$$\left| \epsilon_t \right| \leq \delta_* / (2^{m-1} \nu) \quad (7.3)$$

in second order approximation. This formula is obtained as follows. In the equation for ρ above, the factor containing the error has been omitted. This factor is bounded by

* It should be mentioned that the base is arbitrary, e.g., either 2, e, or 10.

$$\boxed{1 + \epsilon = \exp \left[\log \frac{1 + \delta_*}{1 - \delta_*} \right] / 2^m \cdot \nu} \quad (7.4)$$

where $\epsilon > |\epsilon_t|$. This equation (7.4) has been used with $\nu = 1$ to compute ϵ in Table I. Now,

$$\log \left(\frac{1 + \delta_*}{1 - \delta_*} \right) = 2 \delta_* \left(1 + \frac{\delta_*^2}{3} + \frac{\delta_*^4}{5} + \dots \right) \quad (7.5)$$

and

$$\exp 2 \delta_* / (2^m \nu) + \dots = 1 + 2 \delta_* / 2^m \nu + \frac{2}{3} \delta_*^3 / 2^m \nu + \dots$$

yield the above formula (7.3) for ϵ in second order approximation. Conversely, to obtain ρ within a given relative accuracy ϵ , it is sufficient to satisfy the condition

$$\boxed{\log \frac{1 + \delta_*}{1 - \delta_*} \leq 2^m \nu \log (1 + \epsilon)} \quad (7.6)$$

which follows from Eq. (7.4), or for small δ_* and ϵ

$$2 \delta_* \leq 2^m \nu \epsilon \quad (7.7)$$

which follows from Eq. (7.6). Substituting Eq. (7.1) for δ_* in Eq. (7.6) or (7.7) yields transcendental equations for 2^m . These can be used to determine the number M of root squarings that are sufficient to obtain the required relative accuracy ϵ of ρ . In general, the multiplicity ν is assumed to be unity. E.g., Eq. (7.7) becomes

$$2 \left[\left(\frac{n}{n/2} \right) - 1 \right] (1 - \eta)^{2^m} = 2^m \nu \epsilon$$

or

$$\sqrt[2]{2 \pi n} (1 - \eta)^{2^m} = 2^m \nu \epsilon.$$

It is desirable to minimize M in order to minimize the propagated round-off errors in the squaring operations. Eq. (7.6) and the middle part of (7.1) were used to determine M in Table I. The other equations are less accurate, but simpler to solve.

When η is assumed to be rather small, so that $(1 - \eta)^{2^M} \approx 1 - 2^M \eta$, and M is so large that δ_* is small, the above equations yield

$$\left\{ \eta + \nu \epsilon / 2 \left[\binom{n}{n/2} - 1 \right] \right\}^{-1} = 2^M$$

or

$$\left\{ \eta + \frac{\nu \epsilon \sqrt{2 \pi n}}{2^{n+2}} \right\}^{-1} = 2^M$$

In most cases, the fractional term in the denominator of the left hand side is small in comparison to η .* Therefore, M is given by

$$M = - \frac{\log \left\{ \eta + \frac{\nu \epsilon \sqrt{2 \pi n}}{2^{n+2}} \right\}}{\log 2}$$

Approximately

$$M \geq 3 - \log_2 \eta + (0.1) n \quad (\text{base } 2) \quad (7.8)$$

or

$$M \geq 3 - (3.3) \log_{10} \eta + (0.1) n \quad (\text{base } 10) \quad (7.9)$$

The interpretation of this result is that the relative accuracy of the moduli ρ depends largely on the separation of the zeros of $P(x)$. Once M , η and ν are determined, ϵ can be estimated from

$$\epsilon < \frac{2^{n-M+2}}{\nu \sqrt{2 \pi n}} (1 - \eta)^{2^M} \quad (7.10)$$

Theoretically this is possible only after the polynomial has been at least partially and approximately solved.

* It does not make sense to separate roots which are distinct only by a factor corresponding to the computational accuracy, i.e., for which $\epsilon \approx \eta$.

Conclusion: (i) First one selects the ratio of the moduli which should be separated, i.e., one estimates η .

(ii) With this η , one determines M from any of the equations (7.6) to (7.9) or Table I.

(iii) With this M , one determines $\underline{\delta}^{(M)}$ of η from Eq. (6.8) or (6.6), $\overline{\delta}^M$ with $\eta = \epsilon$ from Eq. (6.9), and checks the consistency $\overline{\delta}^M(\epsilon) > \underline{\delta}^M(\eta)$.

(iv) Next, a δ is chosen so as to provide some room for rounding errors, i.e., $\delta > \underline{\delta}$.

Numerical Example.

A numerical example may illustrate the use of these formulas even better. Given a polynomial of degree $n = 12$, determine the real roots and the moduli of the complex roots to 6 significant figures ($\epsilon = 10^{-6}$). Determine the minimum number of squaring operations M that insures having all roots separated that are (a) $\eta = 10\%$, (b) $\eta = 1\%$, (c) $\eta = 0.1\%$ different in absolute value. Give the conditions for the coefficients $a_j^{(M)}$ to be pivotal. The solution is given in the table below.

Table II

Numerical Example for the Determination of the
Number of Squaring Operations and Pivotal Bounds

$$n = 12, \epsilon = 10^{-6}$$

Equation	Symbol	$\eta = 0.1$	$\eta = 0.01$	$\eta = 0.001$
(7.9)	$M(\eta, 10^{-6})$	8	11	15
(6.8)	$\frac{\underline{\delta}^{(M)}(\eta)}{\overline{\delta}^M(10^{-6})}$	.00253	.0644	.568
(6.9)	$\underline{\delta}^{(M)}(10^{-6})$	< 1	.999 999 5	.999 966 5
(6.1) Necessary Conditions for pivotal coefficient $a_j^{(M)}$	$\left \frac{(-1)^j \left[\frac{a_j^{(M-1)}}{a_j^{(M)}} \right]^2}{a_j^{(M)}} - 1 \right $	< 0.003	< 0.065	< 0.569
	$\frac{(-1)^j \left[\frac{a_j^{(M-1)}}{a_j^{(M)}} \right]^2}{a_j^{(M)}}$	> 0	> 0	> 0

8. ROUNDING ERRORS

We have also to consider the propagation of the rounding errors due to the arithmetic operations which are truncated to a certain number of significant figures. The initial coefficients $a_j = a_j^{(0)}$ are considered accurate.

Generally, the coefficients $a_j^{(m+1)}$ will have the same relative accuracy as the largest term of the right hand side of

$$a_j^{(m+1)} = (-1)^j \left\{ a_j^{(m)^2} + 2 \sum_{\ell} (-1)^\ell a_{j-\ell}^{(m)} a_{j+\ell}^{(m)} \right\}.$$

A loss in relative accuracy may occur when some of the largest terms cancel each other. Such a loss can be prevented in particular at the beginning, if the products $a_j^{(m)} a_k^{(m)}$ ($j, k = 0, 1, 2, \dots, n$) are retained in double precision; also if the sequence of computation is so ordered that the largest product is first computed, then the next largest, and so on. In general, there will be a loss in relative accuracy if $a_j^{(m+1)}$ is smaller than $a_j^{(m)^2}$. A particularly unfavorable situation occurs, when all, or many roots of $P(x) = 0$ are almost equal. When the roots are equal, then

$$P(x) = a_0 (x - \alpha)^n = a_0 \sum_{k=0}^n (-1)^k \binom{n}{k} \alpha^k x^{n-k}$$

and the coefficients of $P^{(m)}(x)$ are

$$a_k^{(m)} = (-1)^k a_0^{(m)} \binom{n}{k} \rho^{2^m}$$

where $\rho = |\alpha|$. The ratio of successive coefficients becomes

$$(-1)^k a_k^{(m)^2} / a_k^{(m+1)} = \binom{n}{k}.$$

If the roots are almost equal, the ratio is of the same order, and the loss of significant figures is given approximately by

$$h \approx \log_{10} a_k^{(m)^2} / a_k^{(m+1)} \approx \log_{10} \binom{n}{k} \quad (8.1)$$

for each squaring operation.

For M such operations, the number of significant figures lost is Mh . Let ϵ be the accuracy of the zeros. We assume that in most situations we cannot expect a higher accuracy in the solutions than we have in the precision of the arithmetic operations, and that we may lose up to Mh significant figures. Hence, these operations should be carried out with

$$H = -\log_{10} \epsilon + Mh$$

significant figures. Approximating $\binom{n}{k}$ by the bound $2^{n+1}/\sqrt{2\pi n}$, we have

$$H \leq -\log_{10} \epsilon + M \left\{ (n+1) \log_{10} 2 - \frac{1}{2} \left[\log_{10} n + \log_{10} 2\pi \right] \right\} \quad (8.2)$$

Therefore, to obtain the modulus ρ with a relative error ϵ , the total number of significant figures to be carried in the arithmetic operation is bounded by

$$H \leq 0.3 M (n-1) = \log_{10} \epsilon \quad (8.3)$$

This equation is an approximation to Eq. (8.2).

If the highest multiplicity of any modulus ρ is $\bar{\nu}$, similar considerations as before yield the equation for the number of significant figures to be carried in the squaring operation,

$$H \approx 0.3 M (\bar{\nu} - 1) - \log_{10} \epsilon \quad (8.4)$$

9. RESULTANTS

A necessary and sufficient condition that two polynomials

$$P(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n$$

and

$$G(x) = b_0 x^m + b_1 x^{m-1} + \cdots + b_m$$

have at least one common divisor is the vanishing of the determinant of order $(m+n)$, the resultant R ,

$$R = \begin{vmatrix} a_0 & a_1 & a_2 & \dots & 0 \\ 0 & a_0 & a_1 & \dots & 0 \\ 0 & 0 & a_0 & \dots & 0 \\ & & & \dots & \\ 0 & 0 & 0 & \dots & a_n \\ b_0 & b_1 & b_2 & \dots & 0 \\ 0 & b_0 & b_1 & \dots & 0 \\ & & & \dots & \\ 0 & 0 & 0 & \dots & b_m \end{vmatrix}$$

A necessary and sufficient condition that these polynomials have at least a common divisor of second degree is the vanishing of the resultant R and the first sub-resultant R' . This sub-resultant is obtained from R by deleting the n^{th} and $(m+n)^{\text{th}}$ rows in R and the last two columns [columns $(n+m-1)$ and $(n+m)$].

In a similar way one can obtain conditions for common divisors of higher degree.

The numerical evaluation of these resultants seems to be rather tedious; however, if the polynomial $G(x)$ is of second degree or less, it turns out that the resultants can be computed quite easily by simple recurrence formulas. In order to obtain these formulas we make a Laplace expansion of the first row for R' and of the first two rows for R . We find that the following determinants B_k will appear as minors:*

*Expressions giving B_k in closed form are Eqs. (10.8).

$$B_k = \begin{vmatrix} p & q & & & & 0 \\ 1 & p & q & & & \\ & 1 & p & q & & \\ & & \cdot & \cdot & \cdot & \cdot \\ & & & & 1 & p & q \\ 0 & & & & & 1 & p \end{vmatrix} \quad (9.1)$$

This determinant is of triple-diagonal form and of order k . It satisfies the recurrence formula

$$B_j = p B_{j-1} - q B_{j-2} \quad (j = 1, 2, \dots, k) \quad (9.2)$$

where, for convenience, we set

$$B_{-1} = 0, \quad B_0 = 1. \quad (9.3)$$

Then,

$$B_1 = p, \quad B_2 = p^2 - q, \quad B_3 = p^3 - 2 p q, \quad \dots, \text{etc.}$$

Now, we replace $G(x)$ by $Q(x) = x^2 + p x + q$. If we expand R' in the first row, the co-factor of a_k is seen to be the product of a triangular minor of value 1 and B_{n-k-1} , i.e.,

$$R' = \begin{vmatrix} a_0 & a_1 & a_2 & \boxed{a_3} & a_4 & a_5 & a_6 & \cdots & a_{n-3} & a_{n-2} & a_{n-1} \\ \boxed{\begin{matrix} 1 & p & q \\ & 1 & p \\ & & 1 \end{matrix}} & q & p & q \\ & & \boxed{\begin{matrix} p & q & \cdot & \cdot & \cdot \\ 1 & p & q & \cdot & \cdot & \cdot \\ & 1 & p & \cdot & \cdot & \cdot \\ & & & p & q \\ & & & 1 & p & q \\ B_{n-k-1} & & & 1 & p \end{matrix}} \end{vmatrix}$$

or

$$R' = \sum_{k=0}^{n-1} (-1)^k a_k B_{n-k-1} \quad (9.4)$$

If we expand the resultant R of $P(x)$ and $Q(x)$ in the first two rows, the corresponding co-factors are seen to be the product of three minors of the form:

$$\begin{vmatrix} 1 & p & q & & & \\ & 1 & p & q & & \\ & & 1 & p & \dots & \\ & & & \dots & \dots & \\ & & & & & 1 \end{vmatrix} = 1^i \quad (9.5)$$

$$\begin{vmatrix} p & q & & \dots & \\ 1 & p & q & \dots & \\ & 1 & p & \dots & \\ & \dots & & \dots & \\ & & & & 1 & p \end{vmatrix} = B_k \quad (9.6)$$

where $k = (j - i)$,

$$\begin{vmatrix} q & & & & 0 \\ p & q & & & 0 \\ 1 & p & q & & 0 \\ & 1 & p & q & 0 \\ & & \dots & \dots & \\ 0 & & & & q \end{vmatrix} = q^\ell \quad (9.7)$$

where $\ell = (n-j)$. The resultant R is therefore:

$$R = \sum_{\binom{n+2}{n}} (-1)^{i+j} \begin{vmatrix} a_i & a_j \end{vmatrix} q^{n-j} B_{(j-i)} \quad (9.8)$$

where $\begin{vmatrix} a_i & a_j \end{vmatrix}$ stands for the determinant $\begin{vmatrix} a_i & a_{j+1} \\ a_{i-1} & a_j \end{vmatrix}$.

10. APPLICATIONS OF RESULTANTS TO ROOT FINDING ALGORITHM FOR DESK AND ELECTRONIC COMPUTERS

From the investigation of Graeffe's algorithm and its modifications, we have seen that no great difficulties are encountered in obtaining the absolute values of either simple or multiple roots. If, however, the roots are complex, in particular with multiple equal absolute values, the methods so far published were incomplete (e.g., restricted to two equal pairs of conjugate complex roots) or were not at all suited for machine computation. The basic idea of the resultant procedure is as follows:

Once an absolute value ρ of a root is established, we may compute the resultant R of

$$P(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n \quad (10.1)$$

and

$$Q(x) = x^2 + px + q \quad (10.2)$$

where $q = \rho^2$ is known. This resultant R is again a polynomial of degree n , Eqs. (9.8) and (11.4) below, but its unknown is the parameter p . Solving this equation

$$R(p) = 0 \quad (10.3)$$

will yield all linear and quadratic factors with absolute value ρ , since $R(p) = 0$ is a vanishing resultant. Moreover, we know theoretically, that $R(p) = 0$ can have only real admissible roots since complex roots of $P(x)$ appear in conjugate pairs, and $-p = \alpha + \bar{\alpha} = 2\rho \cos \phi$ is a real number. We know also that we need consider only those roots p which satisfy the condition

$$p^2 \leq 4q \quad (10.4)$$

The equality sign occurs only if there exists a real root α with absolute value ρ . The multiplicity of ρ equals the total number of real solutions p counting their multiplicity, and satisfying (10.4). (The proof follows immediately upon considering multiple roots p as limits of distinct roots.)

For the actual computation of the resultant R of $P(x)$ and $Q(x)$, we want to obtain R as a polynomial of order n in p

$$R(p) = r_n p^n + r_{n-1} p^{n-1} + \cdots + r_1 p + r_0 \quad (10.5)$$

Since B_k is a polynomial of order k in p , say

$$B_k = b_{k,k} p^k + b_{k,k-1} p^{k-1} + \cdots + b_{k,1} p + b_{k,0} = \sum_{j=0}^k b_{k,j} p^j \quad (10.6)$$

the Laplace expansion (9.8) is ordered according to terms containing $B_k(p)$ as factor. Figuratively speaking, we "slide" the minors B_k along the first subdiagonal. This leads to the following algorithm, starting at step (1) with $k = 0$. See Table III.

Table III

Algorithm for the Computation of the Resultant $R(p)$

Step	Operation	Remarks
(1)	$B_{k+1} = p B_k - q B_{k-1}$	$B_0 = 1; B_{-1} = 0$
(2)	$A_k = (-1)^k \sum_{j=0}^{n-k} \left(a_j a_{j+k} - a_{j-1} a_{j+k+1} \right) q^{n-j-k}$	$a_{-1} = a_{n+1} = 0$
(3)	$R_k = R_{k-1} + A_k B_k$	$R_{-1} = 0$
(4)	Repeat for $k = 1, 2, \dots, n$	$R_n = R(p)$

We see that $R(p)$ is obtained by a recurrence formula which is derived from

$$R = \sum_{j=0}^n A_j B_j \quad R_k = \sum_{j=0}^k A_j B_j = R_{k-1} + A_k B_k$$

$$R(p) = R_n \quad (10.7)$$

Table III can be written in more detailed form in order to show the inner and outer loops of one complete cycle of the recurrence formula. See Table IV.

Table IV

Recurrence Formulas of $R(p)$ for High Speed Computer

Step	Operation	Remarks
Preset	$b_{-1,0} = 0, b_{0,0} = 1, a_{-1} = a_{n+1} = 0, k = 0$	$b_{k-1} = 0$ for all k
$b_{k+1,j} = b_{k,j-1} - q b_{k-1,j}$	$j = 0, 1, \dots, k$ $b_{k+1,k+1} = 0$
	$A_k = (-1)^k \sum_{j=n-k}^0 (a_j a_{k+j} - a_{j-1} a_{k+j+1}) q^{n-j-k}$	
	$r_{k,j} = r_{k-1,j} + A_k \cdot b_{k,j}$	$r_{0,j} = 0, r_{kk} = A_k$ $j = 0, 1, \dots, k-1$
(4)	Repeat steps (1) to (3) for $k = 1, 2, \dots, n$	Coefficients of $R(p)$ $r_j = r_{n,j}$

The coefficients of the resultant (10.5) are given by $r_{n,j} = r_j$. The sequence of computation for economical coding and use of the high speed memory can be understood from the figure below.

	(j)	(j-1)		(i)	
(k-1)	-qb ..	0		0	(b)
				r ..	(r)
(k)	0	b ..	0	+A _k b ..	(b)
				= r ..	(r)
(k+1)	= b ..	0			(b)

Figure 1. Geometrical Representation of Computation Steps (1) and (3) of Tables III or IV

It is seen that alternate $b_{k,j}$ are zero and therefore require neither computation nor storage space. Also, only alternate $r_{k,j}$ require computation.

This point becomes even clearer when we consider a scheme which can be used for hand computation in the course of code checking.

To compute the coefficients r_j of $R(p)$ on a desk calculator, one determines first the coefficients

$$b_{k,j}^* \text{ of } p^j q^{\left[\frac{k-j}{2}\right]} \text{ of } B_k = B_k(p,q). \text{ Explicitly}$$

$$B_k = \sum_{j=0}^{\left[\frac{k}{2}\right]} (-1)^j \binom{k-j}{j} p^{k-2j} q^j$$

$$= \sum_{j=0}^k b_{k,j}^* p^j q^{\left[\frac{k-j}{2}\right]} \quad (10.8)$$

A practical arrangement for the recursive calculation of the coefficients $b_{k,j}^*$ is given in Table V which also gives the computing scheme. It may be noted that the sub-diagonals are the binomial coefficients. With these coefficients, the polynomials B_k are easily constructed. To exemplify, B_0 to B_6 are listed in Table VI. The resultant $R(p,q)$ is then obtained as previously from equation (10.7),

$$R_n = \sum_{j=0}^n A_j B_j$$

Table V

Coefficients of $B_k = B_k(p,q)$

$$B_k = \sum_{j=0}^{\left[\frac{k}{2}\right]} (-1)^j \binom{k-j}{j} p^{k-2j} q^j = \sum_{j=0}^k b_{k,j}^* p^j q^{\left[\frac{k-j}{2}\right]}$$

Coefficients $b_{k,j}^*$

j =	10	9	8	7	6	5	4	3	2	1	0
k = 0											1
1										1	
2									1		-1
3								1		-2	
4							1		-3		1
5						1		-4		3	
6					1		-5		6		-1
7				1		-6		10		-4	
8			1		-7		15		-10		1
9		1		-8		21		-20		5	
10	1		-9		28		-35		15		-1
	etc										

Computing Scheme

Table VI
Algebraic Functions B_0 to B_6

$B_0 =$					1	A_0
$B_1 =$				p		A_1
$B_2 =$			p^2		$-q$	A_2
$B_3 =$		p^3		$-2pq$		A_3
$B_4 =$		p^4		$-3p^2q$	$+q^2$	A_4
$B_5 =$	p^5		$-4p^3q$		$+3pq^2$	A_5
$B_6 =$	p^6	$-5p^4q$		$+6p^2q^2$	$-q^3$	A_6

From Step 2 in the computation scheme for the A_k , it is seen that we have no loss of accuracy in evaluating the determinants $|a_j a_{j+k}|$ if we use double precision arithmetic since we assumed the coefficients a_j to be exact. To compute $R(p,q)$, Table V indicates that we must expect some minor loss in significant figures through cancellation. However, since $q = \rho^2$ is usually not accurate to one full word length, there is little gain in computing B_k and therefore R_k with double precision, except for the A_k 's.

The question arises if it would be more advantageous to normalize $P(x)$ so that $q = 1$. This would simplify the recurrence formulas somewhat, but introduce the error of ρ into the normalized coefficients $a_j \rho^{n-j}$. Only if these coefficients are carried along in double precision, the normalized A_k 's would be at most of accuracy comparable to those of the unnormalized A_k 's, as can be shown easily by substituting $a_j \rho^{n-j}$ for a_j into the equation for A_k with $q = 1$.

Finally, we do not use the sub-resultant $R'(p) = 0$ to determine the real part of the complex zeros of $P(x)$ since the corresponding roots p are not of multiplicity two or higher. Therefore the numerical work to determine the complex zeros of $P(x)$ would be almost doubled.

11. NUMERICAL STABILITY OF RESULTANT PROCEDURE

We investigate the influence of the error in ρ on the solution p of the resultant. The resultant of the polynomial

$$P(x) = a_0 x^n + a_1 x^{n-1} + \dots + a_n$$

and the quadratic

$$Q(x) = x^2 + px + q$$

is a polynomial of degree n , both in p and q , say $R(p, q)$. In order that $P(x)$ and $Q(x)$ have at least one common factor, it is necessary that $R(p, q) = 0$. However, in practical computation $q = \rho^2$ will only be known within a certain accuracy. Hence, $Q(x) = x^2 + px + q$ cannot be a factor of $P(x)$, and hence $R(p, q) \neq 0$. Therefore, we have to determine the bounds of $p - p_t$ if p is the nearest solution to the theoretically accurate p_t of the equation

$$R(p, q) = 0 \quad . \quad (11.1)$$

When a distinction between theoretically accurate quantities and approximations is necessary, symbols with subscript t refer to the true values as opposed to approximated values. It may be noted that the roots p can be real or complex. We shall prove that the resultant procedure is numerically stable. To do this, we need the following theorems.

Theorem: The number of zeros α of $P(x)$ (with their proper multiplicity considered) that satisfy $|\alpha| = \rho$, and $\alpha + \bar{\alpha} = -p$ equals the multiplicity of the roots p of $R(z, \rho^2) = 0$.

The polynomial $R(z, \rho^2)$ has n zeros. The roots of $P(x) = 0$ with $|\alpha| = \rho$ shall be varied so that no two of them are equal, and that for each of them a different p must follow. Now, the coefficients of $P(x)$ are brought continuously to the original values, and the total multiplicity of p cannot be decreased. In particular, to a single real root α corresponds a zero $p = -2\alpha$, of multiplicity one; to a pair of conjugate complex roots α and $\bar{\alpha}$ corresponds a zero $-p = \alpha + \bar{\alpha}$, of multiplicity two.

Theorem: The multiplicity of a particular root p of $R(z, q) = 0$ equals the multiplicity of the root q of the particular equation $R(p, z) = 0$, where $R(p, q)$ is the resultant of $P(x)$ and $x^2 + px + q$.

The proof is obtained again by perturbation. Assume that p is a zero of $R(z, q)$ of multiplicity μ . Then q is a root of $R(p, z) = 0$. We now vary the roots α of $P(x) = 0$ that correspond to p so that the real parts remain

constant, but the modulus changes. For each of these different moduli exists a q , and by proper perturbation of the α 's, μ different values for q will be obtained. Hence, q has at least the multiplicity μ of p . Repeating a similar reasoning for a given multiplicity μ of q , and fixing $|\alpha|^2 = q$, yields for p a multiplicity of at least μ . Hence the multiplicities of p and q are equal.

Theorem: If the ℓ^{th} partial derivative of $R(p,q)$ with respect to p vanishes, then the ℓ^{th} derivative of $R(p,q)$ with respect to q vanishes also, provided p and q correspond to a root α of $P(x) = 0$.

Since p and q are roots of the algebraic equation $R(p,q) = 0$, both of equal multiplicity, the theorem follows immediately. Furthermore, by the same reasoning, if $\partial^\ell R / \partial p^\ell = \partial^\ell R / \partial q^\ell = 0$, it follows that $\partial^\ell R / \partial p^k \partial q^{\ell-k} = 0$ also. This is seen from $R = (q - \alpha(p))^\ell \cdots = (p - \beta(q))^\ell \cdots$

We study next the influence of a perturbation of q on p . To do this, we expand the resultant into a Taylor series of not more than $(n+1)$ terms.

$$R(x + \xi, z + \zeta) = R(x, z) + \left(\xi \frac{\partial}{\partial x} + \zeta \frac{\partial}{\partial z} \right) R(x, z) + \cdots + \left(\xi \frac{\partial}{\partial x} + \zeta \frac{\partial}{\partial z} \right)^n R(x, z)$$

Since $R(p, q) = 0$ by definition, and $\frac{\partial R}{\partial x} = 0$ for $x = p$ corresponding to complex roots, we have in the general case for multiple p

$$\left(\xi \frac{\partial}{\partial x} + \zeta \frac{\partial}{\partial z} \right)^\ell R(p, q) + \cdots + \left(\xi \frac{\partial}{\partial x} + \zeta \frac{\partial}{\partial z} \right)^n R(p, q) = 0 \quad .$$

In practical applications, the disturbances ξ and ζ must always be small to be meaningful. Hence we can neglect all terms which contain factors of the form $\xi^{\ell+1}$, $\xi^\ell \zeta$, $\dots$, $\xi^{\ell-k} \zeta^{k+1}$, $\dots$, $\zeta^{\ell+1}$. The influence of ζ on ξ can then be determined from the homogeneous form

$$\left(\xi \frac{\partial}{\partial x} + \zeta \frac{\partial}{\partial z} \right)^\ell R(p, q) = 0 \quad . \quad (11.2)$$

We now introduce the amplification factor

$$\kappa = \xi / \zeta$$

and substitute $\xi = \kappa \zeta$ in the equation (11.2) above. Since $\zeta \neq 0$, we have replacing $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ by $\left(\frac{\partial}{\partial p} \right)_t$, $\left(\frac{\partial}{\partial q} \right)_t$,

$$\xi^\ell \left(\kappa \frac{\partial}{\partial p} + \frac{\partial}{\partial q} \right)_t^\ell R(p, q) = 0$$

or

$$\sum_{k=0}^{\ell} \binom{\ell}{k} \left[\frac{\partial^\ell R(p, q)}{\partial^{\ell-k} p \partial^k q} \right]_t \kappa^{\ell-k} = 0 \quad (11.3)$$

This is a polynomial of order ℓ in κ , and independent of ξ . Therefore, the roots κ_j ($j=1, 2, \dots, \ell$) are constants, and we have proved the following theorem.

Theorem: The Resultant Procedure is numerically stable.

The ℓ roots of the polynomial equation correspond to the multiplicity ℓ of p_t , and $p^* = p_t + \kappa_j \xi$ approximates one of the ℓ roots of

$$R(z, q + \xi) = 0$$

in which p_t branches out when q is perturbed. For practical application, we need to be sure only that at least one κ_j is sufficiently small, such that $|p^* - (\alpha + \bar{\alpha})_t|$ satisfies the accuracy condition. Therefore we need only an upper bound for the smallest root κ_j . This bound is given by the ℓ^{th} root of the constant coefficient in the polynomial for κ .

$$\kappa_{\min} \leq \left| \frac{\partial^\ell R(p, q) / \partial q^\ell}{\partial^\ell R(p, q) / \partial p^\ell} \right|_t^{1/\ell} \quad (11.3)$$

In explicit form

$$R(p, q) = \sum_{j=0}^n p^{n-j} \sum_{h=0}^{[j/2]} (-1)^{j+h+n} \binom{n+h-j}{h} \sum_{i=0}^{j-2h} \left(a_i a_{n+i+2h-j} - a_{i-1} a_{n+i+2h-j+1} \right) q^{j-i-h} \quad (11.4)$$

If one investigates this formula further, one sees that

$$\kappa_{\min} = 1 \quad \text{for } \ell = 0 \text{ and } n$$

and

$$\kappa_{\min} < \left[\frac{\max \left| \begin{array}{cc} a_k & a_{h+1} \\ a_{k-1} & a_h \end{array} \right| \left(1 + q^{\frac{n}{2}} \right)}{\min \left| \begin{array}{cc} a_k & a_{h+1} \\ a_{k-1} & a_h \end{array} \right| \left(1 + |p|^{\frac{n}{2}} \right)} \right]^{\frac{2}{n}}$$

for $\ell = n/2$, with $k < n/2$ and $h > n/2$. A more careful estimation of $\kappa_{\min}$ reveals that $\kappa_{\min}$ is generally of order one, since the extreme values of the formula are never attained.

Conclusion: The relative error in the part of α or $p/2$ is $\kappa_{\min} \epsilon$, where ϵ is the relative error in ρ , and $\rho^2 = q$. $\kappa_{\min}$ is of order one.

We note that the approximate values p^* which correspond to the minimum κ_j may be complex, although p_t is theoretically real. But the absolute value of p^* can differ from the absolute value of p_t only by $2 \kappa_{\min} \epsilon$. However, the algorithm as described above, in Sections 3 through 8, yields just the desired modulus of p^* which in turn approximates the magnitude of p_t with prescribed accuracy.

12. TESTING OF ROOTS

We derive a criterion that enables us to decide whether a linear or quadratic polynomial approximates a factor of $P(x)$.

The numerical value of $P(\alpha)$ can be determined by making use of the algorithm of synthetic division. We may write the following identities

$$P(x) = (x - \alpha)(b_0 x^{n-1} + b_1 x^{n-2} + \dots + b_{n-1}) + R \quad (12.1)$$

or

$$P(x) = (x^2 + px + q) (b_0 x^{n-2} + b_1 x^{n-3} + \dots + b_{n-2}) + xR' + R \quad (12.2)$$

In the first identity, $P(\alpha) = R$. In the second identity, if $\alpha^2 + p\alpha + q = 0$, $P(\alpha) = \alpha R' + R$. The recurrence relation for the synthetic division is given by

$$\begin{aligned} b_{-1} &= 0, & b_0 &= a_0 \\ b_k &= a_k - pb_{k-1} - qb_{k-2} \quad (k = 1, 2, \dots, n) \end{aligned} \quad (12.3)$$

If one sets $p = -\alpha$, $q = 0$, the algorithm is identical to Horner's scheme. Hence, we have

$$\text{for Eq. (12.1): } R = b_n \quad (12.4)$$

$$\text{for Eq. (12.2): } R' = b_{n-1}, \quad R = b_n + pb_{n-1} \quad (12.5)$$

and need only one computing routine for both cases.

In order that $P(\alpha) = 0$, R or R' and R respectively must be zero. However, α is only an approximate zero of $P(x)$, and therefore $P(\alpha) \neq 0$. If α_t is a true zero of multiplicity ℓ of $P(x)$, expansion in a Taylor series yields

$$P(\alpha) = \frac{(\alpha - \alpha_t)^\ell}{\ell!} P^{(\ell)}(\alpha_t) + \dots + \frac{(\alpha - \alpha_t)^n}{n!} P^{(n)}(\alpha_t) \quad (12.6)$$

We introduce the polynomial

$$T(x) = \sum_{i=0}^n |a_i| x^{n-i} \quad (12.7)$$

which has the property that for all derivatives

$$T^{(k)}(\rho) \geq |P^{(k)}(\alpha)| \quad (k = 0, 1, 2, \dots, n) \quad (12.8)$$

where ρ is the modulus of α . If ϵ is, as previously, the tolerated relative error of ρ ,

$$|\alpha - \alpha_t| \leq \epsilon \rho, \quad (12.9)$$

we obtain from Eqs. (12.6) to (12.9)

$$\begin{aligned} T(\rho + 2\epsilon\rho) - T(\rho + \epsilon\rho) &= \epsilon\rho T'(\rho + \epsilon\rho) + \dots + \frac{(\epsilon\rho)^\ell}{\ell!} T^{(\ell)}(\rho + \epsilon\rho) + \dots + \frac{(\epsilon\rho)^n}{n!} T^{(n)}(\rho + \epsilon\rho) \\ &\geq \left| \frac{(\alpha - \alpha_0)^\ell}{\ell!} P^{(\ell)}(\alpha_t) + \dots + \frac{(\alpha - \alpha_0)^n}{n!} P^{(n)}(\alpha_t) \right| = |P(\alpha)| \end{aligned} \quad (12.10)$$

Hence, a necessary condition that $(x - \alpha)$ approximates a linear factor of $P(x)$ is, from (12.1) and (12.9)

$$T(\rho + 2\epsilon\rho) - T(\rho + \epsilon\rho) \geq |R| \quad (12.11)$$

Similarly, necessary conditions that $x^2 + px + q$ with $\alpha + \bar{\alpha} = -p$ and $\alpha\bar{\alpha} = q$ approximates a quadratic factor of $P(x)$ are, from (12.2) and (12.9)

$$T(\rho + 2\epsilon\rho) - T(\rho + \epsilon\rho) \geq \begin{cases} |\alpha R' + R| \\ |\bar{\alpha} R' + R| \end{cases}$$

or combining both conditions

$$T(\rho + 2\epsilon\rho) - T(\rho + \epsilon\rho) \geq \frac{1}{2} |-p R' + 2 R| \quad (12.12)$$

Substituting equations (12.4) and (12.5) into (12.11) and (12.12) yields the desired criterion in the final form.

Theorem: Using the notation of this paragraph, the following criteria hold.

A necessary condition that $(x - \alpha)$ approximates a linear factor of $P(x)$ is

$$T(\rho + 2\epsilon\rho) - T(\rho + \epsilon\rho) \geq |b_n| \quad (12.13)$$

A necessary condition that $x^2 + px + q$ approximates a quadratic factor is

$$T(\rho + 2\epsilon\rho) - T(\rho + \epsilon\rho) \geq |1/2 p b_{n-1} + b_n| \quad (12.14)$$

13. DESCRIPTION OF IBM 704 CODE AND NUMERICAL EXAMPLES

In writing a code for a high speed computer, the whole power of a theory is seldom utilized. Concessions are made in order to obtain a simpler code. For example, the root-squaring operation in the code for the IBM 704 to be described is performed in floating point double precision, but the double products, $2 a_{j+\ell}^{(m)} a_{j-\ell}^{(m)}$, are not computed according to descending magnitude, and therefore the retention of significant figures is not optimized. However, the squared term, $\left(a_j^M\right)^2$, is added last to preserve some accuracy in many cases. Because of the restriction to at most 20 significant figures in the arithmetic operations, a loss in accuracy for zeros of higher multiplicity must be anticipated, although the numerical examples prove that, in many cases, Eq. (2.3) is very conservative. An error in the original coefficients may be introduced at the beginning through the conversion from the decimal to the binary number system.

In programming the resultant procedure for the IBM 704, paragraphs B through D of Section 2 were followed closely. The master flow chart on the following pages should provide a good picture of the flow of computation. As can be seen, the root-squaring algorithm is a subroutine for the determination of the pivotal coefficients as well as for the actual resultant procedure. It should also be evident that the synthetic division subroutine can be used (a) to divide by linear and (b) by quadratic factors, and (c) to calculate the numerical values of $T(x)$ and $T(x')$. Also, there is but one subroutine for the acceptance test of both (a) linear, and (b) quadratic factors. This makes the code rather concise. The $(2^m \cdot \nu)^{\text{th}}$ root is obtained by a standard logarithmic procedure. Quite extensive studies by the author have shown that this is the quickest way to obtain the moduli. The present subroutine to take the $(2^m \cdot \nu)^{\text{th}}$ root restricts the expected accuracy of the moduli ρ to at most 18 significant figures.

The resultant algorithm yields the real as well as the complex zeros of a polynomial. However, it seems more advantageous to test the moduli ρ and their negative values just after the root-squaring operations to see whether they are real zeros or not. This eliminates the necessity of building an error criterion into the resultant procedure. On the other hand, the real zeros may be determined twice by the code. The advantage is that a comparison of these real zeros gives an indication of the error propagation within the resultant algorithm.

In view of these facts, the output routine lists the following quantities:

- (1) Multiplicity of zero, obtained by factorization.
- (2) Real part of zero.

Figure 2
MASTER FLOW CHART
Root-squaring and Resultant Algorithms

Figure 3
MASTER FLOW CHART
Acceptance Tests for Zeros

Figure 4

MASTER FLOW CHART

Determination of Multiplicities and Output Routine

- (3) Imaginary part of zero.
- (4) Remainder of synthetic division.
- *(5) Modulus of complex zero.
- *(6) Multiplicity of modulus of zero, obtained from pivotal coefficients $a_j^{(M)}$.

For complex zeros, only the positive imaginary part is listed, but the multiplicity is given as 2, 4, etc., counting also the corresponding conjugate zero. The output is optional on line or on tape. Experience with over 800 polynomials solved with the resultant procedure indicated that $M = 10$ and $\epsilon = 10^{-8}$ was always adequate. These polynomials had their origin in scientific and engineering problems.

In the following, numerical examples of exceptional polynomials are given in order to elucidate further the theoretical developments of the previous sections. The numbers given are in semi-logarithmic representation, e.g., -0.000320 is written as -3.20, -4.

Example 1:

$$x^5 - 4x^4 + 3x^3 - 125x^2 + 500x - 375 = 0$$

The solution, as represented by the high speed printer, is for $M = 4$; $\epsilon = 10^{-6}$.

<u>Mult.</u>	<u>Real Part</u>	<u>Imaginary Part</u>	<u>Remainder</u>	<u>Modulus</u>	<u>Mod. Mult.</u>
1	3.000000, 0		-1.165271, -5		1
1	5.000000, 0		-7.145107, -5		3
2	-2.499997, 0	4.330128, 0	-8.992851, -3	5.000000, 0	3
1	1.000000, 0		-0.000000, -1		1

The moduli of the zeros appear in order $\rho_1 = 3$, $\rho_2 = 5$, $\rho_5 = 1$ instead of in descending order as is expected in regular root-squaring procedures. The reason for this behavior is the cyclotomic factor contained in the polynomial which is

$$(x^3 - 5^3)(x - 3)(x - 1) \quad .$$

The coefficients $a_2^{(M)}$ and $a_3^{(M)}$ are irregular pivotal coefficients. If ϵ is chosen large enough, the moduli appear in order $\rho_1 = 3$, $\rho_2 = 1$, $\rho_3 = 5$.

*Optional

Example 2:

$$x^{12} - x^{11} + 6x^{10} - 4x^9 - 29x^8 + 45x^7 - 82x^6 + 64x^5 - 384x^4 + 256x^3 \\ + 1856x^2 - 2880x + 1152 = 0$$

Figure 2

$$(x^6 - 2^6)(x + 2)(x^2 + 3^2)(x - 1)^3$$

Zeros of polynomial with moduli of multiplicity 7, 3, and 2.

The location of the roots are indicated in Figure 2. This polynomial equation was chosen to demonstrate that a sequence of four pivotal coefficients $(a_2^{(M)}; a_3^{(M)}; a_8^{(M)}; a_9^{(M)})$ can yield three moduli of equal magnitude. The real roots indicated by an asterisk (*) are those due to the extraneous pivotal coefficients $a_3^{(M)}$ and $a_8^{(M)}$, and are repeated in the resultant procedure.

The total multiplicity of $\rho = 2$ is given by $\nu = 1 + 5 + 1 = 7$. A further remark concerns the pure imaginary roots $\alpha = \pm 3i$. Since $\rho = 3$ is only accurate to 7 significant figures, the real parts of the approximation for α represents the accumulation of all rounding errors. Since the multiplicity for $\rho = 3$ is 2, the root with two asterisks (**) may be discarded. The complete solution is given below. ($M = 8$).

<u>Mult.</u>	<u>Real Part</u>	<u>Imaginary Part</u>	<u>Remainder</u>	<u>Modulus</u>	<u>Mod. Mult.</u>
**2	- 5.829133, -6	3.000000, 0	-1.164946, 0	3.000000, 0	2
2	5.829133, -6	3.000000, 0	1.164801, 0	3.000000, 0	2
*2	- 2.000000, 0		0.000000, -1		1
*1	2.000000, 0		0.000000, -1		1
2	- 2.000000, 0		1.728535, -6		5
1	2.000000, 0		-1.484752, -4		5
2	-10.000000, -1	1.732051, 0	-9.416938, -4	2.000000, 0	5
2	10.000000, -1	1.732051, 0	-5.922913, -4	2.000000, 0	5
*2	- 2.000000, 0		0.000000, -1		1
*1	2.000000, 0		0.000000, -1		1
3	1.000000, 0		0.000000, -1		3

Example 3:

$$x^4 + x^3 + x^2 + x + 1 = 0.$$

It is seen that the solution of the equation for the primitive fifth roots of the unity represents no difficulty. (Compare Section 14. Historical survey.) The solution is given below for $M = 6$. It is accurate to all seven digits given.

<u>Mult.</u>	<u>Real Part</u>	<u>Imaginary Part</u>	<u>Remainder</u>
2	-8.090170, -1	5.877852, -1	-1.616451, -7
2	3.090170, -1	9.510565, -1	3.566978, -8

Example 4:

The zeros of the Chebyshev polynomial $T_{12}(x)$ can be determined to seven-figure accuracy with $M = 8$. $T_{12}(x) = 2048x^{12} - 6144x^{10} + 6912x^8 - 3584x^6 + 840x^4 - 72x^2 + 1$

<u>Mult.</u>	<u>Real Part</u>	<u>Mult.</u>	<u>Real Part</u>
1	-9.914448, -1	1	-6.087614, -1
1	9.914448, -1	1	6.087614, -1
1	-9.238795, -1	1	-3.826834, -1
1	9.238795, -1	1	3.826834, -1
1	-7.933533, -1	1	-1.305262, -1
1	7.933533, -1	1	1.305262, -1

The smallest separation coefficient is $\eta = 0.0681$. In order to obtain smaller coefficients a_j in the input, we choose for a polynomial of order 20 the transformed Chebyshev polynomial $C_{20}(x) = \frac{1}{2} T_{20}\left(\frac{x}{2}\right)$.

$$C_{20}(x) = x^{20} - 20x^{18} + 170x^{16} - 800x^{14} + 2275x^{12} - 4004x^{10} + 4290x^8 - 2640x^6 + 825x^4 - 100x^2 + 2.$$

The smallest separation coefficient is $\eta = 0.0246$. The zeros of $C_{20}(x)$ are given below for $M = 8$.

<u>Machine Solution</u>	<u>Accurate</u>	<u>Relative Error</u>
$\pm$	$\pm$	
1.993790	1.993835	$\left. \begin{array}{l} < 3 \times 10^{-5} \\ \approx 10^{-5} \end{array} \right\}$
1.944715	1.944739	
1.847798	1.847759	
1.705300	1.705280	
1.520815	1.520819	$< 4 \times 10^{-6}$
1.298896		$\left. \begin{array}{l} < 10^{-6} \end{array} \right\}$
1.044997		
0.7653669		
0.4668907		
0.1569182		

Example 5:

$$x^4 - 2020x^3 + 1039109x^2 - 19199090x + 90090000 = 0.$$

The roots of this equation are 9, 10, 1000, and 1001. This example serves to demonstrate the use of Table I.

- (a) To obtain $\alpha_1 = 9$, and $\alpha_2 = 10$ to 6 significant figures, the number of squaring operations must be at least $M = 7$.
- (b) To separate the roots $\alpha_3 = 1000$ and $\alpha_4 = 1001$, we read from Table I under $\epsilon = 10^{-6}$, $\eta = 0.001$, that $M \geq 14$.

The computations given below are for $M = 6, 10, 12, 14$. We see that only $M = 14$ (or higher) satisfies all requirements of accuracy and separation.

M = 6: Not sufficient accuracy

<u>Mult.</u>	<u>Real Part</u>	<u>Imaginary Part</u>	<u>Remainder</u>	<u>Modulus</u>	<u>Mod. Mult.</u>
2	1.000500, 3		-2.455132, 5		2
1	1.000018, 1		1.807226, 2		1
1	8.999834, 0		1.630430, 2		1

M = 10: α_3, α_4 not separated.

<u>Mult.</u>	<u>Real Part</u>	<u>Imaginary Part</u>	<u>Remainder</u>	<u>Modulus</u>	<u>Mod. Mult.</u>
2	1.000500, 3		-2.455132, 5		2
1	1.000000, 1		1.132813, -1		1
1	9.000000, 0		1.132813, -1		1

M = 12: Separated, but not sufficient accuracy.

<u>Mult.</u>	<u>Real Part</u>	<u>Imaginary Part</u>	<u>Remainder</u>	<u>Modulus</u>	<u>Mod. Mult.</u>
1	1.001004, 3		4.035211, 3		1
1	9.999960, 2		3.982039, 3		1
1	1.000000, 1		1.132813, -1		1
1	9.000000, 0		1.132813, -1		1

M = 14: Correct solution.

<u>Mult.</u>	<u>Real Part</u>	<u>Imaginary Part</u>	<u>Remainder</u>	<u>Modulus</u>	<u>Mod. Mult.</u>
1	1.001000, 3		3.704297, 1		1
1	1.000000, 3		-7.339844, 0		1
1	1.000000, 1		1.132813, -1		1
1	9.000000, 0		1.132813, -1		1

Another example is found at the end of this report* where a polynomial equation is solved with the standard setting of the Code AN C203. This Code is available through SHARE. The sample print-out includes all possible options. It is seen that the number of computed figures is larger than in the examples given in this paragraph. Also, in the INTERMEDIATE PRINT OUT, the roots are printed in reversed order, i.e., the results obtained last are printed first, and the results computed first, are printed last.

*Description of Code AN C203, page 66.

14. HISTORICAL SURVEY OF ROOT-SQUARING METHODS

It seems almost impossible to trace the true origin of the numerical methods of the approximate solution of algebraic equations. E. T. Bell⁽⁸⁾ states that the problem of approximation is said to have been effectively solved by the Chinese in the thirteenth and fourteenth centuries A.D. This work, if authentic, excelled most of what Europeans accomplished in numerical solutions until almost the same procedures were reinvented by W. G. Horner (1819) and C. H. Graeffe⁽²²⁾ (1837). A. Ostrowski⁽³⁹⁾ contends that the fundamental idea of the method of numerical solution of algebraic equations which is called (with right) the Graeffe method, was formulated very clearly by Dandelin⁽¹⁷⁾ in 1826, and some years later, 1834, by Lobachevski,⁽³²⁾ while E. T. Whittaker and G. Robinson⁽⁴⁷⁾ attribute its real origin to E. Waring⁽⁴⁶⁾ in 1776.

Graeffe's merit consists in having developed practical methods for the determination of the arguments of the complex zeros of polynomials, although the moduli of pairs of zeros were assumed to be different. His methods are essentially reproduced in most of the more recent text books, as those by Hildebrand⁽²³⁾ (1956), Householder⁽²⁷⁾ (1953), Scarborough⁽⁴⁴⁾ (1956), and others.^(45,48) Dandelin's concept of decomposition of the transformed polynomial was neglected by Graeffe, but again considered by E. Carvallo⁽¹³⁾ (1890) and G. Polya^(40,41) (1913), and is treated in many foreign text books as for example in reference 45 (Strubecker, 1956). Graeffe's method was made popular by J. F. Encke⁽¹⁸⁾ (1841) who also proposed the use of Moivre's identities to determine the arguments of complex roots. A clear description of Encke's method was given by C. Rounge⁽⁴²⁾ (1899), and the explicit equations appeared again in a paper by W. T. Olver⁽³⁷⁾ (1952). To apply these equations, the original polynomial must be of even order. This requirement can always be met by multiplying, or dividing, the given polynomial by a linear factor. Several papers (Polya⁽⁴¹⁾ (1915), Ostrowski⁽³⁹⁾ (1940), Bodewig⁽¹⁰⁾ (1946), Hoel and Wall⁽²⁶⁾ (1947), and Brooker⁽¹³⁾ (1952)) considered the evaluation of the error in the root-squaring process. As in most numerical methods, there is a distinction between the a priori and the posteriori evaluation of the error, i.e., before and after the equation has been solved.

A practically useful extension of the Graeffe method was made by Brodetsky and Smeal⁽¹²⁾ (1924) and brought into a convenient form by Lehmer⁽³⁰⁾ (1945). The algorithm permits obtaining directly the real zeros with the proper sign, and also the proper argument and modulus in the case of a pair of complex zeros. Strubecker⁽⁴⁵⁾ and Zurmühl⁽⁴⁸⁾ make the statement that the Brodetsky-Smeal method fails in the case of multiple roots, or several pairs of complex roots of equal modulus. Although the algorithm can be adjusted to such cases, it is rather cumbersome, and requires special consideration for each particular case.

So far, all investigations described were intended for use in hand computation. The first paper that refers to sequence controlled calculators in conjunction with the root squaring algorithm is probably that by Fry (1945), followed by others (Brooker,⁽¹³⁾ Hoel and Wall,⁽²⁶⁾ etc.). The high speed computers were used merely to do the bulk of the work, that is the root squaring transformations of the given polynomial. In spite of the extended literature on the root-squaring algorithm, simple criteria suitable for easy coding for an automatic computer have never been included. Hence, the tendency was to use iterative methods, such as the Aitken-Bernoulli, Newton-Raphson, Bairstow-Hitchcock, Friedman, or Lin procedures. For many practical applications they are adequate. Except for the first process convergence is not proved a priori; and in case of multiple roots or almost equal roots, it may be difficult to obtain the requested accuracy. On the other hand, as late as 1945, Lehmer⁽³⁰⁾ could state with full right: "Accounts of the Graeffe process are nearly always given in examples. To be sure, the 'general ideas' of the process are also set forth, but I have yet to see a writer admit that the Graeffe process fails utterly in such simple cases as the equation

$$x^4 + x^3 + x^2 + x + 1 = 0$$

for the primitive fifth roots of unity. Although the Graeffe process is of considerable practical importance, its theoretical justification, including the estimation of errors, is not at all easy. . . ."

It is from here on where the present paper hopes to make a contribution in several independent directions. First, the emphasis on the decomposition of the polynomials was shifted to the new concept of the pivotal coefficients. This made it possible to easily solve equations of the type given above (see the numerical examples in Section 13). Secondly, for the sake of completeness, the exceptional cases were investigated and, it is felt, given adequate space within this paper. These exceptional cases were ignored in all papers concerned with the root-squaring procedure until the past year. The results may be compared with the recent notes by Hirschleber⁽²⁴⁾ (1956/57). Thirdly, the resultant of the given polynomial and a quadratic factor has been introduced and used to determine the real parts of complex roots. This resultant may seem the most impressive independent part of the method. It should be mentioned that this resultant, $R(p)$, could easily be replaced by another polynomial in p , obtained either by decomposition, or separation of real and imaginary parts, or by the Encke polynomial, etc.; even by a modification of the Lin or Bairstow procedures, which aim also to obtain vanishing resultants R and R' . To be sure, the resultant procedure is very convenient for a high speed computer. Next, the design of an automatic code needs an a priori theory. Finally, once a possible solution to the problem is found, the computer needs a criterion to decide for itself whether to accept this solution or not. Such a general criterion has also been introduced. It turned out that these simple problems are by no means elementary, but that the answers could be found in relatively simple equations.

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Description of Code AN C203

IDENTIFICATION

AN C203, Zeros of a Polynomial in Double Precision.
 Mary Anne Fisherkeller - April 28, 1959.
 Argonne National Laboratory, Lemont, Illinois.

PURPOSE

This routine computes both real and complex zeros of the real polynomial $P(x) = a_0x^n + a_1x^{n-1} + \dots + a_n$. All arithmetic is done in double precision.

In addition to the zeros with their multiplicities and remainder terms, the original coefficients of $P(x)$ are output.

Optional output includes: (1), an intermediate output of the zeros, remainder terms, moduli, and multiplicity of moduli in double precision, and (2), a final output of the coefficients generated by the zeros found, and these coefficients adjusted by a_0 of $P(x)$. Output, other than the intermediate, is in single precision.

RESTRICTIONS

Core storage of 8192 is required.

When assembling the symbolic deck, the following subroutines must be assembled concurrently: BS INTP, BS CONV, BS OUT, BS LNX, BS DPSQ, BS EXP, UA CSH2, UA SPH1, MU RDI2.

METHOD

The method employed in the program is the Resultant Procedure, which is based on the root-squaring process. The accuracy is therefore dependent on M , the number of root-squaring iterations.

With sense switch 2 down, the program automatically sets $M = 10$, $\Delta = .2$, and $\epsilon = 10^{-8}$. This is considered the standard setting.

If one wishes to shorten the running time for a series of similar polynomials, or if some results do not seem satisfactory, the section on "Determination of M , Δ , ϵ " should be consulted.

Reference: "Resultant Procedure, a Method for Finding the Zeros of Real Polynomials" by Erwin Bareiss. Argonne National Laboratory Report. (ANL 5997) (1959).

USAGE

AN C203 is designed to be a main program, not a subroutine, and all of 8K core storage is required for its operation.

Storage allows for an n such that $1 \leq n \leq 99$, but for an $n > 20$, the accuracy of the results should be carefully investigated.

The input deck consists of the absolute binary deck backed by the data and a blank card. The first card is a lower binary loader.

The SHARE No. 2 Printer Panel is used if output is on-line.

Logical Tape 2 is used if output is off-line.

Sense Switches:

- 1 down Always (Indicates card input)
- 2 up M, DELTA, and EPSILON are input
- 2 down M, DELTA, and EPSILON are automatically set by the program
- 3 up On-line record is suppressed
- 3 down Record is made on-line
- 4 up Intermediate output is wanted
- 4 down Intermediate output is not wanted
- 5 up Record is made on Tape 2
- 5 down Tape record is suppressed
- 6 up Polynomial is not generated from zeros found and original coefficients only are output
- 6 down Polynomial is generated from zeros found and original coefficients, generated coefficients, and the generated coefficients multiplied by a_0 are output

Input is from cards read on-line. For each polynomial, the first card contains the decimal integer n , beginning in column 1. This card is followed by the coefficients of $P(x)$, $a_0, a_1, \dots, a_n$, punched on consecutive cards beginning in column 1 and ending in column 72.

The fractional part (F) of a number is always preceded by the character (period), and consists of from one to twenty decimal digits. The first decimal digit must not be zero unless $F = 0$. If F is negative, the character - (minus) precedes the decimal point. A positive F has no indication of sign. If a number has a non-zero decimal exponent Q , the

fraction is followed by the character E, followed by Q, which consists of from one to eleven decimal digits, with a minus character preceding Q if Q is negative. If the character E does not appear in the number, the exponent is assumed to be zero. Consecutive numbers are separated by commas, and the first blank to the right of column 1 indicates that all punching to the right of this blank is non-relevant.

If sense switch 2 is up, the card following the coefficients must contain the decimal integer M, beginning in column 1. The next card must contain DELTA, EPSILON in the floating decimal notation of the coefficients.

There is no separation between cases, but a blank card must follow the last data card to halt the program.

Stops:

3450₈ - HTR 2546₈ Final Stop
1350₈ - HTR 1070₈ Division by Zero Stop

CODING INFORMATION

Fifty-one locations must be reserved for COMMON and all code located below 3500₁₀ in the core.

Card Decks:

- 1) Symbolic (SAP) deck: 1170 cards numbered ANC20300, C2030001, ..., C2031169.
- 2) Absolute binary deck: 118 cards numbered ANC20300, C2030001, ..., C2030117.

Timing is dependent on n, M and the options exercised. As an example. when n = 4, M = 10 and all sense switches are down, the time is 15 seconds.

DETERMINATION OF M, DELTA, EPSILON (In general, this section need not be consulted).

- a) The constants M and DELTA (δ) are determined from Table I. (For an n > 30, consult the Reference). First, the separation coefficient η is estimated as well as possible.

$$\eta = \min (\eta_{\rho}, \eta_{\alpha})$$

$$\eta_{\rho} = \frac{\min (\rho_k - \rho_h)}{\rho_k} \quad \text{where } \rho_k > \rho_h,$$

and ρ stands for the absolute value of a zero.

$$\eta_{\alpha} = \frac{\min |x_{\ell} - x_m|}{|x_{\ell}|} \quad \text{where } |x_{\ell}| > |x_m|,$$

and x_l, x_m are the real parts of zeros with equal absolute values,

$$|x_l + iy_l| = |x_m + iy_m| = \rho.$$

Example:

The zeros, α_i , of a fifth degree polynomial are:

$$\begin{aligned} \alpha_1 &= 6 & \rho_1 &= 6 \\ \alpha_{2,3} &= 2 \pm 1i & \rho_2 = \rho_3 = \rho_4 = \rho_5 &= 5 \\ \alpha_{4,5} &= 1 \pm 2i \end{aligned}$$

Hence:

$$\eta_\rho = \frac{6 - 5}{6} = 0.167$$

$$\eta_\alpha = \frac{2 - 1}{2} = 0.5$$

$$\eta = \min(\eta_\rho, \eta_\alpha) = \underline{\underline{0.16}}$$

We read from Table I for $n = 6$:

$$M = 7, \delta > .0214 \text{ for } \epsilon < 10^{-6}.$$

- b) In general, the relative accuracy, $\text{EPSILON}(\epsilon)$, of the absolute values is greater than 10^{-6} , as is indicated in Table I. If less than n zeros (roots) appear in the final output, increase EPSILON from 10^{-8} to 10^{-6} , etc. Since the final multiplicity of a zero (root) is determined by synthetic division, using this epsilon to test the remainder, this change does not influence the accuracy of the zero.
- c) The code is capable of solving a large range of exceptional polynomial equations with high or low accuracy, according to desire. A study of the theoretical background of the code (See Reference) is recommended, if one has many exceptional polynomials to solve. For occasional use, however, the adjustment of the constants may be done by trial, following instructions a) and/or b).

Approved *Richard King*
Date *May 13, 1959*

TABLE I *

For $(\epsilon) \leq 10^{-6}$

n	η	M	δ greater than
2	.5	5	.0001
	.1	7	.0025
	.01	10	.0117
	.001	13	.2531
	.0001	16	.3741
4	.5	5	.0002
	.1	7	.0095
	.01	10	.0472
	.001	14	.1361
	.0001	17	.3198
6	.5	5	.0004
	.1	7	.0214
	.01	10	.1080
	.001	14	.3223
	.0001	18	.0258
8	.5	5	.0006
	.1	7	.0382
	.01	10	.1964
	.001	15	.0089
	.0001	18	.0460
10	.5	5	.0009
	.1	7	.0561
	.01	11	.0018
	.001	15	.0138
	.0001	18	.0724
12	.5	5	.0012
	.1	8	.0011
	.01	11	.0025
	.001	15	.0199
	.0001	19	.0002
14	.5	5	.0016
	.1	8	.0002
	.01	11	.0336
	.001	16	.0001
	.0001	19	.0003
16	.5	5	.0021
	.1	8	.0003
	.01	12	.0001
	.001	16	.0001
	.0001	19	.0004
20	.5	6	.0001
	.1	8	.0004
	.01	12	.0001
	.001	16	.0001
	.0001	19	.0005
30	.5	6	.0070
	.1	9	.0007
	.01	12	.0154
	.001	17	.0001
	.0001	20	.0001

- * n degree of polynomial
 η separation coefficient
 ϵ tolerated relative error in modulus
M number of root-squaring iterations
 δ lower pivotal bound

Illustrative Problem:

Find zeros of $P(x) = x^6 - 34x^5 + 487x^4 - 3836x^3 + 17983x^2 - 48706x + 60025$

The zeros are:

$$x_1 = x_2 = x_3 = x_4 = 7, \quad x_{5,6} = 3 \pm 4i.$$

The output, including all options, is as follows:

INTERMEDIATE PRINTOUT

REAL PART		IMAGINARY PART	
REMAINDER TERM		MODULUS	
.30000000000000000482	00001	.3999999999999999596	00001
-.33483212404962491928	-00012	.4999999999999999966	00001
MULTIPLICITY OF MODULUS	02		
.69999999999999999699	00001	.18621903463553626714	-00007
.29646153153900512335	-00013	.6999999999999999946	00001
MULTIPLICITY OF MODULUS	04		
.6999999999999999946	00001	.0000000000000000000	00000
.11180834903756764475	-00013	.0000000000000000000	00000
MULTIPLICITY OF MODULUS	04		

MULTIPLICITY	REAL PART	IMAGINARY PART	REMAINDER TERM
04	.7000000000 00001	.0000000000 00000	.1118083490 -00013
02	.3000000000 00001	.4000000000 00001	-.3348321240 -00012

COEFFICIENTS A SUB I

I	ORIGINAL	GENERATED	ADJUSTED BY A SUB 0
00	.1000000000 00001	.1000000000 00001	.1000000000 00001
01	-.3400000000 00002	-.3400000000 00002	-.3400000000 00002
02	.4870000000 00003	.4870000000 00003	.4870000000 00003
03	-.3836000000 00004	-.3836000000 00004	-.3836000000 00004
04	.1798300000 00005	.1798300000 00005	.1798300000 00005
05	-.4870600000 00005	-.4870600000 00005	-.4870600000 00005
06	.6002500000 00005	.6002500000 00005	.6002500000 00005