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NOMENCLATURE

- D = Flexural rigidity of shell = $Eh^3/12(1 - \nu^2)$
E = Modulus of elasticity
h = Thickness of shell
k = $\sqrt[4]{3(1 - \nu^2)} \sqrt{R/h}$
M = Bending moment per unit length (without subscript - meridional bending moment)
N = Stress resultant in meridional direction per unit length
P = Distributed load per unit length of circumference
Q = Transverse meridional shearing force per unit length
R = Radius of shell
T = Tangential stress resultant per unit length
V = - χ = Angle of rotation of tangent to meridian, positive for inward rotation with increasing meridional angle (ϕ)
w = Displacement perpendicular to the axis of symmetry, positive outward
Z = Total force
 ν = Poisson's ratio
 ϕ, θ = Spherical coordinates
 $2\phi_0$ = Total central angle subtended by opening in shell

Subscripts and Superscripts

- 0 = Junction of lower and upper shell portions
1 = Component of load P
d = Due to "discontinuity" forces
L = Lower portion of shell
U = Upper portion of shell
r = Edge load
 ϕ = Meridional direction
 θ = Tangential direction

Quantities taken directly from Ref. 1.

Constants

$A_1, A_2, B_1, B_2, C_1, C_2, D_1, D_2$

W_a, W_b, W_c, W_d

$\chi_a, \chi_b, \chi_c, \chi_d$

Variables (These are functions of $k\phi$)

n_a, n_b, n_c, n_d

m_a, m_b, m_c, m_d

t_a, t_b, t_c, t_d

THE STRESS ANALYSIS OF
A SPHERICAL SHELL LOADED ALONG A LATITUDE CIRCLE

This memorandum outlines the stress analysis of a spherical shell subjected to a line load acting along a latitude circle in a direction normal to the plane of that circle (see Fig. 1).

Figure 1

The appropriate equations for this system are obtained from equilibrium requirements. These are¹:

$$\left. \begin{aligned} \frac{d^2 Q_\phi}{d\phi^2} + \cot \phi \frac{dQ_\phi}{d\phi} - (\cot^2 \phi - \nu) Q_\phi &= E h \nu \\ \frac{d^2 \nu}{d\phi^2} + \cot \phi \frac{d\nu}{d\phi} - (\cot^2 \phi + \nu) \nu &= - \frac{d^2 Q_\phi}{D} \end{aligned} \right\} \quad (a)$$

and

which may be combined to give a fourth order equation in Q_ϕ only. This fourth order equation can then be reduced to the two equations*

$$\frac{d^2 Q_\phi}{d\phi^2} + \cot \phi \frac{dQ_\phi}{d\phi} - \cot^2 \phi + 2ik^2 Q_\phi = 0 \quad (b)$$

In the present treatment, we obtain the solutions** to these equations by applying Maria Esslinger's approximation². The approximation is made by replacing $\cot \phi$ in Eqs (b) by $1/\phi$, and the resulting general solution for Q is:

$$Q = AZ_1' (k\sqrt{2}\phi) + BZ_2' (k\sqrt{2}\phi) + CZ_3' (k\sqrt{2}\phi) + DZ_4' (k\sqrt{2}\phi)$$

The Z functions are Schleicher functions, and Z' denotes $dZ/d\phi$. A , B , C , and D are the integration constants.

Esslinger's relationships for the deformations and moment in terms of Q are:

$$w = \frac{k^2}{\sqrt{3(1-\nu^2)}} \cdot \frac{1}{E} \left(-\phi \frac{dQ}{d\phi} + \nu Q \right)$$

$$\chi = -\frac{1}{Eh} \left(\frac{d^2 Q}{d\phi^2} + \frac{1}{\phi} \frac{dQ}{d\phi} - \frac{1}{\phi^2} Q \right)$$

$$M = -\frac{hk}{4\sqrt{3(1-\nu^2)}} \cdot \frac{1}{(k\phi)^3} \left(\phi^3 \frac{d^3 Q}{d\phi^3} + \frac{5}{4} \phi^2 \frac{d^2 Q}{d\phi^2} - \frac{7}{4} \phi \frac{dQ}{d\phi} + \frac{7}{4} Q \right)$$

There are other solutions to the above equations;^{2,3,4} the most commonly used are due to J. W. Geckeler (see pages 20 and 21 of Ref. 2, and 469 and 470 of Ref. 3).

* The subscript ϕ will not be used with Q in the remainder of the paper.

** Maria Esslinger sets the range of applicability for her equations as $0 \leq 2\phi_0 = 60^\circ$. However, a comparison made in Ref. 3 (p 443) indicates that the expressions give good results beyond this range.

The shell will be analyzed by first examining only the lower portion with the line load acting along its edge. See Fig. 2. Next, the upper and lower portions will be "joined" by applying the edge forces and moments necessary to provide compatible displacements and rotations for continuity at the junction. These forces and moments are shown in Fig. 3.

Figure 2

Consider a spherical segment loaded as in Fig. 2. The internal force and moment distributions as well as the edge displacement and rotation can be obtained from Bessinger's solution of the equilibrium equations. The equation applicable to this case is (33) on page² 18.

$$Q = CZ_3' (k\sqrt{2}\phi) + DZ_4' (k\sqrt{2}\phi) \quad 33$$

The boundary conditions used to evaluate the constants in Eq (33) are obtained from the loading shown in Fig. 3. These boundary conditions are:

$$M(\phi_0) = M_r \quad (1)$$

and

$$Q(\phi_0) = - N_r \tan \phi_0$$

or, for small ϕ_0 ,

$$Q(\phi_0) \approx - \phi_0 N_r \quad (2)$$

The resulting equations for the edge displacement and rotation of the lower portion of the shell become:

Figure 3

$$\frac{w_0}{h} = -W_c \frac{N_r}{Eh\phi_0} + W_d \frac{M_r}{Eh^2\phi_0}, \quad (3)$$

$$\chi_0 = \chi_c \frac{N_r}{Es\phi_0} - \chi_d \frac{M_r}{Eh^2\phi_0}, \quad (4)$$

The force and moment relationships are given as functions of ϕ by:

$$N = C^*n_c + D^*n_d, \quad (5)$$

$$T = C^*t_c + D^*t_d, \quad (6)$$

and

$$\frac{M}{h} = C^*m_c + D^*m_d, \quad (7)$$

where

$$C^* = C_1 \frac{N_r}{h} + C_2 \frac{M_r}{h},$$

and

$$D^* = D_1 \frac{N_r}{h} + D_2 \frac{M_r}{h}.$$

The constants W_c , W_d , χ_c , χ_d , C_1 , C_2 , D_1 and D_2 are functions of $k\sqrt{2}\phi_0$, and they are given by Maria Esslinger in Fig. 5 and Table 3. The variables n_c , n_d , t_c , m_c , and m_d are functions of $k\phi$ and are given in Table 4.

Since the forces and moments shown in Fig. 3 are those required to establish shell continuity, Eqs. (3) through (7) provide relationships needed for evaluating these forces. To clarify the nomenclature, the edge deformations produced by these discontinuity forces are designated as follows for the shell's lower portion:

$$w_0 = w_d^L$$

$$\chi_0 = \chi_d^L.$$

From Fig. 2,

$$N_1 = P \sin \phi_0 \quad (8)$$

$$Q_1 = P \cos \phi_0 \quad (9)$$

Thus,

$$N_1 = Q_1 \tan \phi_0$$

or, for ϕ_0 small,

$$N_1 \cong \phi_0 Q_1 = \phi_0 Q(\phi_0) \quad (10)$$

Also

$$M(\phi_0) = 0 \quad (11)$$

To correct for the change in boundary conditions between those previously used in evaluating the constants of Eq (33) and the ones applicable to the loading shown in Fig. 2, a relationship is needed between the meridional forces, N_r and N_1 . The elimination of $Q(\phi_0)$ from Eqs. (2) and (10) gives*

$$N_r = - \frac{1}{\phi_0^2} N_1 \quad (12)$$

This formula along with Eq. (11) provide the required relationships to be used in Eqs. (3) through (7).

The edge displacement and rotation as given by Eqs. (3), (4), (11), and (12) for that part of the shell shown in Fig. 2 are:

$$\begin{aligned} \frac{w_0}{h} &= -w_c \frac{1}{Eh\phi_0} \left(-\frac{1}{\phi_0^2} N_1 \right) \\ &= w_c \frac{P}{Eh\phi_0} \frac{\sin \phi_0}{\phi_0^2} = \frac{w_c L}{h} \end{aligned} \quad (13)$$

* Eq. (12) is merely an artifice for using the existing solution for Q , and it cannot be used for evaluating either N_r or N_1 .

$$\begin{aligned}\chi_0 &= \chi_c \frac{1}{Eh\phi_0} \left(-\frac{1}{\phi_0^2} N_1 \right) \\ &= -\chi_c \frac{P}{Eh\phi_0} \frac{\sin \phi_0}{\phi_0^2} = \chi^L\end{aligned}\quad (14)$$

These are the displacement and rotation that determine the magnitudes of the discontinuity forces.

The equations for the internal forces and moment are the same as Eqs. (5), (6) and (7). However, the constants C^* and D^* for this case are:

$$C^* = C_1 \frac{N_1}{\phi_0^2} \qquad D^* = D_1 \frac{N_1}{\phi_0^2}$$

The edge deformations of the upper portion of the shell, loaded as shown in Fig. 3, are given by:

$$\frac{w_0}{h} = w_a \frac{N_r}{Eh\phi_0} + w_b \frac{M_r}{Eh^2\phi_0} = \frac{w_d^U}{h} \quad (15)$$

and

$$\chi_0 = \chi_a \frac{N_r}{Eh\phi_0} + \chi_b \frac{M_r}{Eh^2\phi_0} = \chi_d^U \quad (16)$$

The internal forces and moment are

$$N = A^*n_a + B^*n_b, \quad (17)$$

$$T = A^*t_a + B^*t_b, \quad (18)$$

and

$$\frac{M}{h} = A^*m_a + B^*m_b. \quad (19)$$

In these equations

$$A^* = A_1 N_r + A_2 \frac{M_r}{h}$$

and

$$B^* = B_1 N_r + B_2 \frac{M_r}{h}$$

The constants are given in Fig. 4 and Table 1 of Ref. 2; the variables are given in Table 2.

The expressions for continuity between the two shell segments are obtained by equating the algebraic sums of the discontinuity displacements and rotations to those caused by the segment loading shown in Fig. 2.

$$w_d^U - w_d^L = w^L \quad (20)$$

and

$$\chi_d^U - \chi_d^L = \chi^L \quad (21)$$

Since the discontinuity forces are the same for both portions of the shell, Eqs. (3), (4), (13), (14), (15) and (16) determine the force system. Hence,

$$(W_a + W_c)N_r + (W_b - W_d)\frac{M_r}{h} = W_c \frac{\sin \phi_0}{\phi_0^2} P \quad (20a)$$

$$(\chi_a - \chi_c)N_r + (\chi_b + \chi_d)\frac{M_r}{h} = -\chi_c \frac{\sin \phi_0}{\phi_0^2} P \quad (21a)$$

The tangential, discontinuity force common to both edges is given by

$$T_r = (A_1 N_r + A_2 \frac{M_r}{h})t_a(k\phi_0) + (B_1 N_r + B_2 \frac{M_r}{h})t_b(k\phi_0) \quad (22)$$

The stresses at the junction are:

$$\sigma_\phi = \frac{N}{h} \pm \frac{6M_r}{h^2} \quad (23)$$

and

$$\sigma_\theta = \frac{T_r}{h} \pm \frac{6M_\theta}{h^2} \quad (24)$$

The tangential bending moment, M_θ , is given by

$$M_\theta = \frac{Eh^3}{12R(1-\nu^2)} \left(\nu \frac{d\chi}{d\phi} + \frac{1}{\phi} \chi \right)$$

$$= \nu M_r + \frac{Eh^3}{12R} \frac{1}{\phi} \chi.$$

Writing it in terms of the quantities already defined,

$$\frac{M_\theta}{h} = \nu \frac{M}{h} + \frac{1}{12} \sqrt{3(1-\nu^2)} \frac{1}{(k\phi)^2} (\chi_a N_r + \chi_b \frac{M_r}{h}) \quad (25)$$

for the upper portion of the shell. For the lower portion,

$$\frac{M_\theta}{h} = \nu \frac{M}{h} + \frac{1}{12} \sqrt{3(1-\nu^2)} \frac{1}{(k\phi)^2} (\chi_c N + \chi_d \frac{M_r}{h}). \quad (26)$$

At points some distance away from the apex of the sphere, the tangential moment is approximately

$$M_\theta = \nu M. \quad (27)$$

In Eqs. (23), (24), (25) and (26), the total moments and forces are to be used. For the lower part of the sphere, N represents the sum of the discontinuity force N_r and the component N_1 due to P ; thus,

$$N = N_r + N_1 \quad (28)$$

The octahedral shear stress for a biaxial stress system is

$$\tau_{oct} = \frac{\sqrt{2}}{3} \sqrt{\sigma_\phi^2 + \sigma_\theta^2 - \sigma_\phi \sigma_\theta} \quad (29)$$

In order to obtain the maximum accuracy for a given case, the constants and variables can be calculated from the formulas given by Maria Esslinger. The relationships for the constants and variables are given in terms of the Schleicher functions for $K\sqrt{2} \phi \leq 6$ and in terms of the asymptotic expansions of these functions for $K\sqrt{2} \phi > 6$. Table I provides an index for locating the expressions given in Ref. 2.

A discussion of the Schleicher functions along with a tabulation of their values is given by Timoshenko on pages 417 through 420 of Ref. 1. These functions are also discussed in Ref. 5, and a very complete tabulation of the functions is given in Table III beginning on page 246 of this reference.

TABLE I

Quantity	Equation(s) No.		Page No.			
	$K\sqrt{2} \phi \leq 6$	$K\sqrt{2} \phi > 6$	$K\sqrt{2} \phi \leq 6$ Book	$K\sqrt{2} \phi \leq 6$ CF 57-12-37	$K\sqrt{2} \phi > 6$ Book	$K\sqrt{2} \phi > 6$ CF 56-12-37
Constants:						
A_1	22	30	12	14	15	18
$\left. \begin{matrix} A_2 \\ B_1 \\ B_2 \end{matrix} \right\}$	22	30	12	14	15	19
$\left. \begin{matrix} C_1 \\ C_2 \end{matrix} \right\}$	38	46	19	22	22	26
$\left. \begin{matrix} D_1 \\ D_2 \end{matrix} \right\}$	38	46	19	23	22	26
$\left. \begin{matrix} W_a \\ W_b \end{matrix} \right\}$	25	31	13	15	16	20
$\left. \begin{matrix} W_c \\ W_d \end{matrix} \right\}$	41	47	20	23	23	27
$\left. \begin{matrix} \chi_a \\ \chi_b \end{matrix} \right\}$	25	31	13	15	16	20
$\left. \begin{matrix} \chi_c \\ \chi_d \end{matrix} \right\}$	41	47	20	23	23	27

TABLE I (Cont)

Quantity	Equation(s) No.		$K\sqrt{2} \phi < 6$		$K\sqrt{2} \phi > 6$	
	$K\sqrt{2} \phi \leq 6$	$K\sqrt{2} \phi > 6$	Book	CP 57-12-37	Book	CP 56-12-37

Variables:

$\left. \begin{matrix} n_a \\ n_b \end{matrix} \right\}$	27	32	13	15	17	20
$\left. \begin{matrix} n_c \\ n_d \end{matrix} \right\}$	43	48	20	24	25	29
$\left. \begin{matrix} m_a \\ m_b \end{matrix} \right\}$	27	32	13	15	17	21
$\left. \begin{matrix} m_c \\ m_d \end{matrix} \right\}$	43	48	20	24	25	29
t_a	27	32	13	15	17	20
t_b	27	32	13	15	17	21
$\left. \begin{matrix} t_c \\ t_d \end{matrix} \right\}$	43	48	20	24	25	29

Example:

An application in which a spherical shell is loaded as described in this paper will now be considered. A structure inside the shell has a running load along its outer edge. This structure bears against and transmits load to the spherical shell along a latitude circle. See Fig. 4. Using the above equations, the stresses due to the bearing load will be calculated.

Figure 4

The data required to determine the stresses are:

$$\begin{aligned} R &= 30.375'' \\ h &= 1.0'' \\ \phi_0 &= 35^{\circ}09' = 0.611 \text{ radians} \\ \cos \phi_0 &= 0.818 \\ \sin \phi_0 &= 0.576 \\ k &= \sqrt{3(1-\nu^2)} \sqrt{R/h} = 7.07 \\ k\phi_0 &= 4.32 \\ k\sqrt{2} \phi_0 &= 6.11 \\ P &= 346 \text{ lb/in} \\ \nu &= 0.30 \end{aligned}$$

The values of the constants to be used are taken from Fig. 4 on page 17, Fig. 5 on page 24, Table 1 on page 97, and Table 2 on page 98 of Ref. 2.

$$\chi_a = 40 \quad W_a = 100 \quad A_1 = -0.399 \quad t_a(k\phi_0) = -12.14$$

$$\chi_b = 30 \quad W_b = 40 \quad A_2 = -0.303 \quad t_b(k\phi_0) = -9.78$$

$$\chi_c = 35 \quad W_c = 100 \quad B_1 = -0.352$$

$$\chi_d = 27 \quad W_d = 35 \quad B_2 = 0.013$$

From Eqs. (20a) and (21a)

$$M_r = -1.02 P$$

$$N_r = 0.80 P$$

Using these values in Eq. (22)

$$T_r = 3.10 P$$

Equation (8) gives:

$$N_1 = 0.58 P$$

The stresses in the upper portion of the shell at the junction become

$$\sigma_\phi = \frac{N_r}{h} \pm \frac{6M_r}{h^2} = -5.32 P \text{ or } 6.92 P$$

$$\sigma_\theta = \frac{T_r}{h} \pm \frac{6\psi M_r}{h^2} = -1.25 P \text{ or } 4.94 P$$

For the lower portion:

$$\sigma_\phi = \frac{1}{h} (N_r + N_1) \pm \frac{6M_r}{h^2} = -4.75 P \text{ or } 7.49 P$$

$$\sigma_\theta = -1.26 P \text{ or } 4.94 P.$$

The maximum single as well as combined stress exists in the lower portion. Thus, the maximum combined edge stress is

$$\tau_{oct} = 3.11 P = 1076 \text{ lb/in}^2 .$$

Since it is difficult to accurately determine values for W_a , W_b , W_c , W_d , χ_a , χ_b , χ_c , and χ_d from Fig. 4 and Fig. 5 of Ref. 2, these quantities were calculated from the given equations (see Table I). The values thus obtained for the constants are:

$\chi_a = 40.13$	$W_a = 93.77$
$\chi_b = 30.96$	$W_b = 40.49$
$\chi_c = 36.32$	$W_c = 105.73$
$\chi_d = 27.51$	$W_d = 35.97 .$

These constants together with A_1 , A_2 , B_1 , B_2 , $t_a(k\phi_0)$ and $t_b(k\phi_0)$ as given above yield the following forces and moments:

$$M_r = -1.013 P ,$$

$$N_r = 0.840 P ,$$

and

$$T_r = 3.27 P .$$

A comparison of the results shows that satisfactory values for the forces and moment were obtained by using Figs. 4 and 5 of Ref. 2. However, this fact may not apply for other values of $k\phi_0$.

References:

1. S. Timoshenko, Theory of Plates and Shells, McGraw-Hill Book Company, Inc., New York, 1940, pp. 467-472.
2. Maria Esslinger, Statische Berechnung von Kesselboden, Julius Springer, Berlin, Germany, 1952, pp. 1-25 and pp. 97-100.

English Translation:

Static Calculation of Boiler Bottoms,
ORNL CF 56-12-37, August 22, 1957.

3. G. D. Galletly, "Influence Coefficients for Hemispherical Shells With Small Openings at the Vertex", Journal of Applied Mechanics Vol. 22 (1955), pp. 20-24 and pp. 443-444.
4. F. J. Stanek, Spherical Segment With Circular Hole at Vertex Loaded Axisymmetrically Along the Edges, ORNL-2207 (December 19, 1956) pp. 5-8.
5. M. Hetenyi, Beams on Elastic Foundation, Ann Arbor; The University of Michigan Press, 1946, pp. 103-105.