ARGONNE NATIONAL LABORATORY
P. O. Box 299
Lemont, Illinois

A TWO-GROUP ITERATION METHOD FOR ANNULAR CYLINDERS

by

B. I. Spinrad and C. N. Kelber

Reactor Engineering Division

March, 1957

Operated by The University of Chicago
under
Contract W-31-109-eng-38

DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency Thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

DISCLAIMER

Portions of this document may be illegible in electronic image products. Images are produced from the best available original document.

A TWO-GROUP ITERATION METHOD FOR ANNULAR CYLINDERS

by

B. I. Spinrad and C. N. Kelber

ABSTRACT

A procedure is outlined for obtaining the fluxes and currents in the two-group approximations in an infinite annular cylinder, together with an estimate of k_{∞} by a process of iteration. Corresponding to each initial estimate of the slow flux there is an iterative procedure to obtain an estimate of the fast flux, a new estimate of the slow flux, and an estimate of k_{∞} . The new estimate of the slow flux, in turn, can be used to repeat the process and thus to secure better estimates of k_{∞} and the fluxes. There appears to be little difficulty in extending the procedure formally to more groups than two.

In the course of work on Argonaut, a low-power research reactor,¹ it became necessary to investigate the critical properties of a homogeneous annular region reflected externally and internally. An iterative technique, which was characterized by ease of application and rapidity of convergence, was evolved to solve these problems. While the idea of an iterative technique is by no means new,² the formalism used here is, to our knowledge, unique and well suited to this particular problem.

The critical properties of a homogeneous annulus may be found in a number of ways. The straightforward method of solution through matrix reduction of the equations arising from the boundary conditions can be adopted to this configuration. The result is a 4×4 critical determinant. Unfortunately, in many instances the calculations appear to be overly sensitive to minor errors in setting up the work, so that a great deal of labor is required to calculate the critical buckling accurately.

A second method which has been used with some success is an iterative technique. The equations used here differ slightly from others in use and hence are derived explicitly.

¹D. H. Lennox, C. N. Kelber, "Summary Report on the Hazards of the Argonaut Reactor," ANL-5647 (November, 1955).

²L. Noderer, "Some Methods of Solving Multi-Group Pile Equations," ORNL-291 (March 2, 1949) (Declassified January 15, 1955).

Consider the neutron conservation equation with source, Q , for an infinite, axially symmetric cylinder:

$$\left. \begin{aligned} \frac{1}{r} \frac{d}{dr}(rj) + \Sigma_a \phi &= Q \\ j &= -D \frac{d\phi}{dr} \end{aligned} \right\} \begin{aligned} a &\leq r \leq b \\ a &= r_0 \\ r_i &= r_0 + i \delta r \\ \delta r &= (b - a)/I \end{aligned} \quad (1)$$

where ϕ is the flux, j the current, Σ_a the removal cross section, D the diffusion coefficient, and I the number of equal intervals into which (a,b) is divided. If the problem is a separable cylindrical problem, the same formalism may be used provided that in Eq. (1):

$$\Sigma_a^* = \Sigma_a + DB_Z^2$$

is substituted for Σ_a , where B_Z^2 is the buckling along the axis.

Let

$$\delta r = (b - a)/I$$

$$r_i = a + i \delta r$$

$$\phi(r_i) = \phi_i$$

$$j(r_i) = j_i$$

Then, with the additional assumptions that

$$\left. \begin{aligned} \phi &= \phi_i + \frac{r - r_i}{\delta r} (\phi_{i+1} - \phi_i) \\ j &= j_i + \frac{r - r_i}{\delta r} (j_{i+1} - j_i) \end{aligned} \right\} r_i \leq r \leq r_{i+1}, \quad (2)$$

one finds

$$\phi_{i+1} = \phi_i - \frac{\delta r}{2D} (j_{i+1} + j_i) \quad (3)$$

$$j_{i+1} = j_i + \lambda_i (\alpha_i j_i - \beta_i \phi_i + q_i) \quad (4)$$

$$\lambda_i = \frac{\delta r}{1 - \theta(2 + C_i)} = \frac{\delta r}{C_3} \quad (5)$$

where

$$\alpha_i = S(2 + C_i) - C_2$$

$$L^2 = D/\Sigma_a^*$$

$$\beta_i = \frac{\Sigma_a^*}{2} (1 - C_i)$$

$$\theta = (\delta r)^2 / (12L^2)$$

$$q_i = \gamma_i Q_{i+1} + \epsilon_i Q_i$$

$$S = (\delta r) / (6L^2)$$

$$\gamma_i = \frac{2 + C_1}{6}$$

$$C_1 = r_i / r_{i+1}$$

$$\epsilon_i = \frac{1 + 2C_1}{6}$$

$$C_2 = 1/r_{i+1}$$

$$\Sigma_a^* = \Sigma_a + DB_Z^2$$

$$C_3 = 1 - \theta (2 + C_1)$$

Note that for a given problem C_1 , C_2 , γ_i and ϵ_i are functions of r_i only. The boundary conditions at a and b are of the form:

$$j_{f,s}(a) = (m_{f,s}) \left[\phi_{f,s}(a) \right] + d_{f,s} \quad (6a)$$

$$j_{f,s}(b) = (m'_{f,s}) \left[\phi_{f,s}(b) \right] + d'_{f,s} \quad (6b)$$

where the subscripts (f,s) refer to fast and slow neutrons, respectively. To see this, write, in $0 \leq r < a$,

$$\phi_f = AI_0(\mu_f r)$$

$$j_f = -D_f \mu_f AI_1(\mu_f r)$$

$$\phi_s = S_R \phi_f + BI_0(\mu_s r), \text{ etc.},$$

these being the solutions of the two-group equations in the region with

$$\mu_f^2 = \frac{1}{\tau} ; \mu_s^2 = \frac{1}{L^2} ; S_R = \frac{D_f}{D_s \left(\frac{\tau}{L^2} - 1 \right)}$$

the D 's being the diffusion coefficients in the internal region, L^2 the thermal diffusion area, and τ the neutron age from fission.

For a separable cylinder the μ 's can be replaced by

$$\sqrt{\frac{1}{\tau} + B_Z^2}, \sqrt{\frac{1}{L^2} + B_Z^2},$$

where B_Z^2 is the buckling in the axial direction in the internal region. Then

$$j_f(a) = -D_f \mu_f I_1(\mu_f a) \frac{\phi_f(a)}{I_0(\mu_f a)}$$

and

$$m_f = -D_f \mu_f \frac{I_1(\mu_f a)}{I_0(\mu_f a)}$$

$$d_f = 0.$$

Similarly,

$$j_s(a) = -D_s \mu_s \frac{I_1(\mu_s a)}{I_0(\mu_s a)} (\phi_s - S \phi_f(a)) + S_R \frac{D_s}{D_f} j_f(a)$$

and

$$m_s = -D_s \mu_s \frac{I_1(\mu_s a)}{I_0(\mu_s a)} (\mu_s a)$$

$$d_s = S_R \left(\frac{D_s}{D_f} m_f - m_s \right) \phi_f(a).$$

The two values of m' and d' may be found by similar analytical presentation of solutions in the outer reflector. Suppose that $\phi_{f,s}^H, j_{f,s}^H$ are the (homogeneous) solutions found when $Q \equiv 0$ and

$$j_{f,s}^H(a) = m \phi_{f,s}^H(a);$$

and $\phi_{f,s}^C, j_{f,s}^C$ are complete solutions (including source) obtained by starting with boundary conditions (Eq. 6a). Then parameters $\lambda_{f,s}$ can be determined such that

$$\phi_{f,s} + \lambda_{f,s} \phi_{f,s}^H$$

and

$$j_{f,s}^C + \lambda_{f,s} j_{f,s}^H$$

satisfy both sets of boundary conditions (Eqs. 6a and 6b).

The homogeneous and complete solutions are obtained by use of the work sheets included at the end of this paper. Briefly the steps are as follows:

1. Fill out a set-up sheet for each group.
2. The constants m , d , m' , d' , for each group are calculated and inserted in the appropriate spaces on Sheet 4.
3. It is usually convenient to take $\phi_0^H = 1$ for each group: from m , j_0^H is determined for each group and the remainder of Sheet 2 may be filled out. Thus, ϕ_f^H and ϕ_s^H are obtained.

Sheet 4 contains instructions for filling out Sheet 3. For a thin annulus, the assumption that $\phi_s(r) \equiv 1$ in the annulus is often adequate to establish an initial set of values of $Q_i(\text{fast})$.

The equations are iterated again using the same sources but with

$$\phi_f^C + \lambda_f \phi_f^H$$

as a new starting value, and the procedure continued until convergence is secured.

In this process the source, Q_i , does not change with each iteration but is left unchanged. The result is a convergent series of estimates of the fast flux based on the initially assumed values of the source, $Q_i(\text{fast})$. This procedure is used to guard against error produced by roundoff in the course of a single iteration and is basically a check. The last estimate is then used to determine a source for the slow flux, $Q_i(\text{slow})$. Following the same procedure as before, a convergent series of estimates of the slow flux is obtained. The last estimate of the slow flux does not, in general, have the same shape as the original estimate used in obtaining $Q_i(\text{fast})$. If the shapes are not radically different, however, an estimate of k_∞ may be obtained from the ratios of the integral of $\phi_s \phi_s^\dagger$ over the annulus. We take ϕ_f to be a good estimate of $\phi_s^\dagger$. The integral is performed by taking Eq. (2) literally, integrating over each sub-interval and summing the results. Space is provided on Sheet 5 for this work.

The process just outlined completes a master iteration which is made up of individual iterations such as iteration 1, 1', etc. If, now, the last estimate of ϕ_s obtained is used to give a new estimate of $Q_i(\text{fast})$, then the entire procedure can be followed once again to yield a new estimate of k_∞ . In this way a sequence of estimates of k_∞ can be obtained. In general, this sequence converges rapidly and the first estimate is often good enough.

ANNULAR CYLINDRICAL CORE
SET-UP SHEET

Problem:

Group:

Geometry:

$r_0 =$ _____

$r_i =$ _____ cm

Intervals (I) = _____

$\delta_r = (r_i - r_0)/I =$ _____ cm

Group Parameters:

$\Sigma_a =$ _____ cm^{-1}

$\Sigma_a^* = \Sigma_a + DB_Z^2 =$ _____ cm^{-1}

$L^2 = D/\Sigma_a^* =$ _____ cm^2

Set-up Parameters:

$\eta = \delta_r/2D =$ _____

$\theta = (\delta_r)^2/(12L^2) =$ _____

$S = (\delta_r)/(6L^2) =$ _____

i	(r_i) $r_0 + i(\delta_r)$	(C_1) r_i/r_{i+1}	(C_2) $1/r_{i+1}$	(C_3) $1 - \theta(2 + C_1)$	(λ_i) δ_r/C_3	(μ_i) $S(2 + C_1) - C_2$	(β_i) $\Sigma_a^*(1 + C_1)/2$	(γ_i) $(2 + C_1)/6$	(ϵ_i) $(1 + 2C_1)/6$
0									
1									
↓									
I									

ANNULAR CYLINDRICAL CORE
ITERATION CONSTANTS

Problem: _____

Homogeneous Initial Conditions:

Fast: $j_0^H =$

Thermal: $j_0^H =$

i	λ_i	α_i	β_i	ϕ_i^H	j_i^H
Fast Group; $\eta =$					
0					
1					
↓					
I					
Thermal Group; $\eta =$					
0					
1					
↓					
I					

Formulae:

$$j_{i+1} = j_i + \lambda_i (\alpha_i j_i - \beta_i \phi_i)$$

$$\phi_{i+1} = \phi_i - \eta(j + j_{i+1})$$

ANNULAR CYLINDRICAL CORE

AUXILIARY INSTRUCTIONS AND CONSTANTS

This sheet contains instructions for filling out Sheet No. 3 and provides space, if desired, for performing auxiliary computations.

Problem: _____

1. γ_i and ϵ_i were computed on Sheet No. 1. They need not be computed for each iteration.

2. $Q_i^f = \sum_a^{\text{th}} \phi_i^{\text{th}}$, where ϕ_i^{th} is the most recently computed set; similarly,

$$Q_i^{\text{th}} = \sum_a^f \phi_i^f .$$

$$3. \quad q_i = \gamma_i Q_{i+1} + \epsilon_i Q_i .$$

4. ϕ_0^C and j_0^C must satisfy the relations:

$$j_0^C = m \phi_0^C + b.$$

The constants m and b are characteristic of the problem and group, and are listed in the accompanying box.

Group	m	b
Fast		
Thermal		

5. Given ϕ_0^C and j_0^C , the iteration formulae are:

$$j_{i+1}^C = j_i^C + \lambda_i (\alpha_i j_i^C - \beta_i \phi_i^C + q_i)$$

$$\phi_{i+1}^C = \phi_i^C - \eta(j_i^C + j_{i+1}^C)$$

The constants λ_i , α_i , and β_i can be obtained by folding Sheet No. 2 and clipping to Sheet No. 1.

6. After ϕ_I^C and j_I^C have been computed, a constant "a" must be computed. The formula is:

$$a = \frac{j_I^C - m' \phi_I^C - b'}{m' \phi_I^H - j_I^H}$$

To aid this computation, the following box contains the fixed constants

Group	ϕ_I^H	j_I^H	m'	b'
Fast				
Thermal				

Unless $|a\phi_0^H| < 0.01 \phi_0^C$, the iteration is to be repeated, using as new starting values:

$$\phi_0^C + a\phi_0^H,$$

$$j_0^C + aj_0^H,$$

with the "a" being that first computed. Space for this iteration is provided on Sheet No. 3. The following box is used for computation of "a."

COMPUTATION OF "a"

	Iteration 1			Iteration 1'			Iteration 2			Iteration 2'		
Group	ϕ_I^C	j_I^C	a	ϕ_I^C	j_I^C	a	ϕ_I^C	j_I^C	a	ϕ_I^C	j_I^C	a
Fast												
Thermal												

7. Following the second determination of "a" (or the first, if $|a\phi_0^H| < 0.01 \phi_0^C$), ϕ_i on Sheet No. 3 is computed from

$$\phi_i = \phi_i^C + a\phi_i^H$$

where ϕ_i^C and a are the most recently computed values.

ANNULAR CYLINDRICAL CORE COMPUTATION OF k

Problem:

An estimate of k is obtained by completing the following tabulation. The superscripts "f" and "th" refer to fast and thermal groups, respectively.

Iteration No. 1						Iteration No. 2					
i	$\phi_i^f r_i$	Q_i^f	ϕ_i^{th}	$\phi_i^f r_i Q_i^f$	$\phi_i^f r_i \phi_i^{th}$	$\phi_i^f r_i$	ϕ_i^f	ϕ_i^{th}	$\phi_i^{th} r_i Q_i^f$	$\phi_i^f r_i \phi_i^{th}$	
0											
1											
2											
3											
4											
I											
Sum A						Sum A					
First Term x ($\frac{1}{2} - \delta r / 6r_0$)											
Last Term x ($\frac{1}{2} - \delta r / 6r_I$):											
Sum B						Sum B					
Sum A minus Sum B						Sum A minus Sum B					
				u	v					u	v

$$k = \frac{u}{\sum_{i=0}^{I-1} v} = \underline{\hspace{2cm}}$$

$$\underline{\hspace{2cm}}$$