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ELASTIC SCATTERING OF PROTONS BY DEUTERONS AT HIGH ENERGY

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The "phase shifts" which appear in an analysis of high energy $n-d$ or $p-d$ scattering must be complex numbers, since inelastic scattering also occurs. A modification of Born's approximation is presented which takes account of the effect of the inelastic scattering on the elastic scattering. From it, complex phase shifts are derived which satisfy requirements imposed on them by the condition that particles be conserved. These are used as a starting point for a phase shift analysis of the 95 Mev $p-d$ experimental data. Since the phase shifts are allowed to be complex, it is possible to analyze the elastic scattering data and the total cross section data simultaneously (however, no attempt is made to analyze the angular or energy distributions of inelastically scattered particles).

Satisfactory fits to the angular distribution for elastic scattering and the total scattering cross section are obtained by adjusting the S complex phase shifts, changing the complex P phase shifts slightly from their Born approximation values, and using phase shifts calculated in Born approximation for $\ell \geq 2$. While it is not necessary to use the

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complex phase shifts for $\ell \geq 2$ calculated from the modified Born approximation (it is sufficient to use real phase shifts calculated from the usual Born approximation), they do provide slightly better fits to the angular distribution for elastic scattering, and, more important, the contribution to the total cross section for $\ell \geq 2$ is sufficiently large to require them.

Central forces are used for the calculation. It is assumed that the effects of non-central forces are reproduced by the same "equivalent" central potential which reproduces their effects in high energy n - p and p - p scattering.

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I. Introduction.

The ideas underlying this work are the following.

(A) In calculating an angular distribution for a scattering process, it is best to use the exact expression for the angular distribution in terms of the quantities describing the asymptotic behavior of the partial waves even if these quantities have been calculated only approximately.

(B) Any approximate method for calculating these quantities should require them to satisfy conditions imposed on them by the requirement that particles be conserved in the process.

While the material presented in this section is more or less well known, it serves to illustrate these ideas and provides a basis for the generalizations of Section III.

The Schrodinger equation for the scattering of a particle of energy E off a potential V is

$$-\frac{\hbar^2}{2M} \nabla^2 \psi + V\psi = E\psi \quad (1)$$

This equation guarantees¹ that particles are conserved in

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1. See, for example, Mott and Massey, The Theory of Atomic Collisions, (Oxford University Press, London, 1949), second edition, Chapter I, § 7.
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The scattering process; that is, that

$$\text{div } \underline{j} = 0 \quad (2)$$

where

$$\underline{j} = \frac{\hbar}{2mi} (-\psi^* \nabla \psi - \psi \nabla \psi^*) \quad (3)$$

is the particle current.

For a partial wave of angular momentum ℓ , the asymptotic form of ψ is²

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2. See for example, Blatt and Weisskopf, Theoretical Nuclear Physics (John Wiley and Sons, New York, 1952), p. 320
-

$$\psi \sim (2\ell+1) i^\ell \left\{ \frac{1}{2} (1+\eta_\ell) j_\ell(kr) + i \frac{1}{2} (1-\eta_\ell) j_{-\ell}(kr) \right\} P_\ell \quad (4)$$

and the angular distribution for elastic scattering is

$$k^2 \sigma(\theta) = \left| \sum_{\ell=0}^{\infty} (2\ell+1) \frac{1}{2}(1 - \eta_{\ell}) P_{\ell}(\theta) \right|^2. \quad (5)$$

That is, the angular distribution is expressed in terms of the quantities η_{ℓ} describing the asymptotic behavior of the partial waves.

The contribution to the total elastic scattering from a partial wave of angular momentum ℓ is

$$k^2 \sigma_{\ell} = 4\pi(2\ell+1) \left| \frac{1}{2}(1 - \eta_{\ell}) \right|^2. \quad (6)$$

Conservation of particles requires that the net inward flux through a sphere containing the origin be zero, that is

$$-\int j_n dS = 0 \quad (7)$$

which gives

$$\eta_{\ell} n_{\ell}^* = 1. \quad (8)$$

This makes it possible to write $\eta_{\ell} = \exp(-2i\delta_{\ell})$ with δ_{ℓ} real, and it can be seen that

$$k^2 \sigma_{\ell} \leq 4\pi(2\ell+1). \quad (9)$$

To solve the scattering problem it is necessary to compute the η_{ℓ} from Schrodinger's Eq. (1). Since it is possible to do this only for two body collisions, approximate methods (such as Born approximations) must be developed for calculating the η_{ℓ} . These usually proceed by showing that if ψ_{ℓ} is any function which behaves asymptotically like Eq. (4), and in addition satisfies Schrodinger's Eq. (1), then

$$\frac{1}{2i}(1 - \eta_{\ell}) = -\frac{2m}{\hbar^2} \frac{k}{4\pi} \int j_{\ell}(kr) v u d\tau \quad (10)$$

where

$$\psi = i^{l(2l+1)} u P_l \quad (11)$$

(the proof of Eq. (30a) in Section III is carried out in this way.) Eq. (10) is an exact result, from which many Born approximations may be derived depending on what one takes for an approximate u . Two of these are described now.

(1) The first is based on substituting the "unperturbed" radial wave function

$$u = j_l(kr) \quad (12)$$

in Eq. (10), so that

$$\frac{1}{2i} (1 - \eta_l) = -\frac{2m}{\hbar^2} \frac{k}{4\pi} \int j_l^2 v d\tau = "d_l" \quad (13)$$

where " d_l " is real. This gives

$$\eta_l = 1 - 2i "d_l" \quad (14)$$

and

$$\eta_l \eta_l^* = 1 + 4 "d_l"^2. \quad (15)$$

Eq. (15) is not in accord with Eq. (8) and the underlying idea (B).

(2) This difficulty can be overcome by taking

$$u = \frac{1}{2}(1 + \eta_l) j_l(kr) \quad (16)$$

in Eq. (10). This is the same as Eq. (12) for weak interactions for which

$\eta_l \approx 1$. It is no more or no less justified than Eq. (12), but it gives

$$\frac{1 - \eta_\ell}{i(1 + \eta_\ell)} = \text{"}\delta_\ell\text{"}. \quad (17)$$

Then

$$\eta_\ell = \frac{1 - i \text{"}\delta_\ell\text{"}}{1 + i \text{"}\delta_\ell\text{"}} \quad (18)$$

and Eq. (8) is satisfied as required by the underlying idea (B), and so we would prefer this Born approximation to the first one.

Other Born approximations can be obtained by taking

$$u = f(\eta) j_\ell(kr)$$

with $f(\eta)$ any function which satisfies $f(1) = 1$. For example

$$f(\eta) = (\eta - 1)/\log \eta$$

gives

$$\eta = e^{-2i \text{"}\delta_\ell\text{"}} \quad (19)$$

a result which is also in agreement with Eq. (8) and underlying idea (B). (not all choices for $f(\eta)$ would be). Eq. (18) is no more justified than Eq. (19). They give approximately the same result for " δ_ℓ " small or $\eta_\ell \approx 1$. However, Born approximation (2) will be generalized in the following to cover those problems in which inelastic scattering occurs in addition to elastic scattering.

The exact expression Eq. (5) is used in computing the angular distributions according to underlying idea (A). The three dimensional Born approximation is based on Born approximation (1) above and Eq. (5). As is well known³,

³Reference 1, page 28

the sum over ℓ can be carried out in this case and leads to the three dimensional Born approximation

$$\sigma(\theta) = \left| \frac{1}{4\pi} \frac{m}{\hbar^2} \int (\exp i (\underline{k}_o - \underline{k}_f) \cdot \underline{r}) V(r) d\underline{r} \right|^2 \quad (20)$$

Thus according to Eq. (8), the three dimensional Born approximation is not in agreement with underlying idea (B); that is, it is not in agreement with general conservation theorems.

II. A Scattering Problem.

In this section we discuss a scattering problem with inelastic scattering. We are still laying the groundwork for Section III. For simplicity, the problem is chosen so that there is no Pauli principle to worry about and no continuum states.

If a particle 1 is bound in a potential $V(r)$, and there are two bound levels whose energies and wave functions are E_1, ϕ_1 and E_2, ϕ_2 ($E_1 > E_2$), then

$$-\frac{\hbar^2}{2m_1} \nabla_1^2 \phi_n + V \phi_n = -E_n \phi_n \quad n = 1, 2 \quad (21)$$

If a (distinguishable) particle 2 interacts with particle 1, the potential of the interaction being V_{12} , then Schrodinger's equation for the system of two particles is

$$-\frac{\hbar^2}{2m_1} \nabla_1^2 \psi + V \psi - \frac{\hbar^2}{2m_2} \nabla_2^2 \psi + V_{12} \psi = (E - E_1) \psi \quad (22)$$

for a scattering problem in which E is the energy of particle 2 incident on a target which is particle 1 in its ground state. Let E be such that

a transition from E_1 to E_2 is energetically possible (inelastic scattering) but no other inelastic process is energetically possible.

An approximate solution of Eq. (22) may be obtained by writing

$$\psi = f_1(2) \phi_1(1) + f_2(2) \phi_2(2) \quad (23)$$

Retention of the first term in Eq. (23) corresponds to a "no polarization" solution of the scattering problem like that often used in the n-d scattering problem 4, 5, 6.

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4. R. A. Buckingham and H. S. W. Massey, Proc. Roy. Soc. (London) A 179, 123 (1941).
 5. L. Motz and J. Schwinger, Phys. Rev. 58, 26 (1940).
 6. R. S. Christian and J. L. Gammel, Phys. Rev. 91, 100 (1953)
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Retention of the second term allows for the presence of inelastic scattering. Other terms not retained correspond to "polarization" of the target.

Substituting Eq. (23) into Eq. (22), taking account of Eq. (21), multiplying through by ϕ_n ($n=1,2$), and integrating over the coordinates of particle 1, remembering that ϕ_1 and ϕ_2 are orthogonal, we find

$$-\frac{\hbar^2}{2m_2} \nabla_2^2 f_1(2) + f_1(2) \int \phi_1(1) V(12) \phi_1(1) d\tau_1 + f_2(2) \int \phi_1(1) V(12) \phi_2(1) d\tau_1 \quad (24a)$$

$$-\frac{\hbar^2}{2m_2} \nabla_2^2 f_2(2) + f_1(2) \int \phi_2(1) V(12) \phi_1(1) d\tau_1 + f_2(2) \int \phi_2(1) V(12) \phi_2(1) d\tau_1 =$$

$$(E - E_1 + E_2) f_2(2) \quad (24b)$$

Following the usual principles of quantum mechanics, we define two currents as follows:

$$\underline{j}_1 = \frac{\hbar}{2im_2} (f_1^* \nabla_2 f_1 - f_1 \nabla_2 f_1^*) \quad (25)$$

$$\underline{j}_2 = \frac{\hbar}{2im_2} (f_2^* \nabla_2 f_2 - f_2 \nabla_2 f_2^*)$$

It is easily verified from Eqs. (24) and their complex conjugates that

$$\begin{aligned} \text{div } \underline{j}_1 &= \frac{\hbar}{2im_2} (f_1^* \nabla_2^2 f_1 - f_1 \nabla_2^2 f_1^*) \\ &= \frac{1}{i} (f_1^* f_2 - f_1 f_2^*) \int \phi_1(1) v(12) \phi_2(1) d\tau_1 \end{aligned} \quad (26a)$$

and

$$\text{div } \underline{j}_2 = \frac{1}{i} (f_2^* f_1 - f_2 f_1^*) \int \phi_2(1) v(12) \phi_1(1) d\tau_1 \quad (26b)$$

or that

$$\text{div } (\underline{j}_1 + \underline{j}_2) = 0 \quad (27)$$

This is to be interpreted as follows: Eq. (27) means that even though an exact solution of Eq. (24) gives only an approximate solution of Eq. (22), the approximation Eq. (23) has not forced us to violate the condition that particles be conserved. Eq. (26a) says that some particles are lost for the elastic scattering, but Eq. (26b) says these reappear as inelastically scattered particles.

We even go a little further in the approximation than Eqs. (24) and leave off the last term in Eq. (24b); however, these remarks still hold.

For partial waves of angular momentum ℓ , the asymptotic form of $f_{1\ell}$ is like Eq. (4), and the asymptotic form of $f_{2\ell}$ is

$$f_{2\ell} \sim (2\ell+1) i^\ell \frac{1}{2} \beta_\ell \left\{ j_\ell(k'r) + i j_{-\ell}(k'r) \right\} P_\ell \quad (28)$$

Conservation of particles requires that the net inward flux through a sphere containing the origin be zero, that is

$$-\int (\underline{j}_1 + \underline{j}_2)_n dS = 0$$

which gives

$$1 - |\eta_\ell|^2 = \frac{k}{k'}, |\beta_\ell|^2 \quad (29)$$

III. Born's Approximation.

In this section we want to develop a modification of Born's approximation (2), section I, for η_ℓ and β_ℓ such that these automatically satisfy Eq. (29).

First we show that if f_1 and f_2 are any two functions which satisfy Eq. (24) modified by the omission of the last term on the left of Eq. (24b), then

$$\frac{1 - \eta_\ell}{2i} = -\frac{2m_2}{\hbar^2} \frac{k}{4\pi} \iint j_\ell(k_2) u_1(2) \phi_1(1) v(12) \phi_1(1) d\tau_1 d\tau_2 \quad (30a)$$

$$- \frac{2m_2}{\hbar^2} \frac{k}{4\pi} \iint j_\ell(k_2) u_2(2) \phi_1(1) v(12) \phi_2(1) d\tau_1 d\tau_2$$

$$\frac{\beta_\ell}{2i} = -\frac{2m_2}{\hbar^2} \frac{k'}{4\pi} \iint j_\ell(k'_2) u_1(2) \phi_2(1) v(12) \phi_1(1) d\tau_1 d\tau_2 \quad (30b)$$

where

$$k^2 = \frac{2m_2}{\hbar^2} E \quad (31)$$

$$k'^2 = \frac{2m_2}{\hbar^2} (E - E_1 + E_2)$$

To prove Eq. (30a), for example, we observe that

$$-\frac{\hbar^2}{2m_2} \nabla_2^2 j_\ell(k_2) = E j_\ell(k_2) \quad (32)$$

Then we multiply Eq. (24a) by $j_\ell(k_2)$ and Eq. (32) by $f_1(2)$ and subtract.

This gives

$$\begin{aligned} & -j_\ell(k_2) \nabla_2^2 f_1(2) + f_1(2) \nabla_2^2 j_\ell(k_2) \\ & = \text{div} \left\{ -j_\ell(k_2) \nabla_2 f_1(2) + f_1(2) \nabla_2 j_\ell(k_2) \right\} \\ & = 4\pi \text{ (the integrand on the R.H.S. of Eq. (30a))}/k \end{aligned} \quad (33)$$

Integrating both sides of Eq. (33) and on the left side applying the divergence theorem to a sphere so large that on its surface f_1 has its asymptotic form we find

$$\begin{aligned} & 4\pi r_2^2 k \left[-j_\ell(k_2) \left\{ \frac{1}{2} (1 + \eta_\ell) j_\ell'(k_2) + \frac{1}{2} i(1 - \eta_\ell) j_{-\ell}'(k_2) \right\} \right. \\ & \quad \left. + \left\{ \frac{1}{2} (1 + \eta_\ell) j_\ell(k_2) + \frac{1}{2} i(1 - \eta_\ell) j_{-\ell}(k_2) \right\} j_\ell'(k_2) \right] \\ & = 4\pi \text{ R.H.S. of Eq. (30a)}/k \end{aligned} \quad (34)$$

But

$$j_\ell(k_2) j_{-\ell}'(k_2) - j_{-\ell}(k_2) j_\ell'(k_2) = 1/k^2 r_2^2$$

so that Eq. (34) is

$$\frac{1 - \eta_\ell}{2i} = \text{R.H.S. of Eq. (30a)}$$

which is Eq. (30a).

The Born approximation we wish to present is obtained by substituting

$$u_1 = \frac{1 + \eta_\ell}{2} j_\ell(k^2) \quad (35)$$

$$u_2 = \frac{1}{2} \beta_\ell j_\ell'(k'^2)$$

in the exact results Eq. (30a) and Eq. (30b).

In Eq. (35) the approximation for u_1 corresponds to Eq. (16) and causes no surprise. However, both this approximation and the approximation for u_2 can be justified a bit further as follows. The asymptotic form of u_2 is given by Eq. (28), and in fact we can write exactly

$$u_2 = \frac{1}{2} \beta_\ell f_1(2) j_\ell(k'^2) + \frac{1}{2} i \beta_\ell f_2(2) j_{-\ell}(k'^2)$$

where $f_2 = 0$ at the origin and $f_1 \rightarrow 1$ for $r_2 \rightarrow \infty$. The main contribution to the integrals Eq. (30a) and Eq. (30b) comes from near the origin, so that we would suppose that since $f_2 = 0$ there, we could drop the first term for a better approximation. In the absence of other information about f_1 , we use its asymptotic value $f_1 = 1$ and arrive at Eq. (35).

Substituting Eq. (35) into Eq. (30a) and Eq. (30b) gives

$$\begin{aligned} \frac{1 - \eta_\ell}{1(1 + \eta_\ell)} &= - \frac{2m}{\hbar^2} \frac{k}{4\pi} \iint j_\ell^2(k^2) \phi_1(1) V(12) \phi_1(1) d\tau_1 d\tau_2 \\ &- \frac{2m}{\hbar^2} \frac{k}{4\pi} \frac{\beta_\ell}{(1 + \eta_\ell)} \iint j_\ell(k^2) j_\ell(k'^2) \phi_1(1) V(12) \phi_2(1) d\tau_1 d\tau_2 \\ &= " \delta_\ell " + \frac{1}{2i} \frac{k}{k'} \frac{\beta_\ell}{1 + \eta_\ell} " \beta_\ell " * \end{aligned} \quad (36a)$$

$$\frac{\beta_\ell}{1(1+\eta_\ell)} = \frac{2m_2}{\hbar^2} \frac{k'}{4\pi} \iint j_\ell(k'2) j_\ell(k2) \phi_2(1) v(12) \phi_1(1) d\tau_1 d\tau_2$$

$$\equiv \frac{1}{2i} " \beta_\ell "$$
(36b)

Putting Eq. (36b) into Eq. (36a) gives

$$\frac{1 - \eta_\ell}{1(1+\eta_\ell)} = " \delta_\ell " - i \frac{k}{4k'} | " \beta_\ell " |^2$$
(37)

so that

$$\eta_\ell = \frac{(1 - \frac{k}{4k'} | " \beta_\ell " |^2) - i " \delta_\ell "}{(1 + \frac{k}{4k'} | " \beta_\ell " |^2) + i " \delta_\ell "}$$
(38)

Then putting this back into Eq. (36b) gives

$$\beta_\ell = \frac{" \beta_\ell "}{(1 + \frac{k}{4k'} | " \beta_\ell " |^2) + i " \delta_\ell "}$$
(39)

From these it can be verified that Eq. (29) is satisfied.

It is interesting to note that " δ_ℓ " is the phase shift calculated in Born approximation for the elastic scattering with neglect of the inelastic scattering. " β_ℓ "/2 is the amplitude for inelastic scattering calculated in Born approximation.

Thus, for high energy p-d or n-d scattering we calculate the phase shift " δ_ℓ " for elastic scattering neglecting inelastic scattering in the same way Christian and Gammel did⁶. Then we calculate the amplitude for absorption by inelastic scattering from a partial wave of angular momentum ℓ , and use this for " β_ℓ "/2 in Eq. (38)

IV. Calculation of the " δ_l " for High Energy n-d and p-d Scattering.

Hereafter we follow the notation of reference 6. As in reference 6, we use a central Yukawa potential adjusted to fit the low energy n-p and p-p data and the high energy n-p data. Thus we take

$$\begin{aligned} 3V_{np}^+ &= 1 & 3V_{np}^- &= 0 & 1V_{np}^+ &= .69 & 1V_{np}^- &= 0 \\ 1V_{nn}^+ &= .69 & 1V_{nn}^- &= 0. \end{aligned} \quad (40)$$

$$U(r) = 68 \text{ Mev} \left\{ \exp - (r/1.18 \times 10^{-13} \text{ cm}) \right\} / (r/1.18 \times 10^{-13} \text{ cm})$$

In reference 6 are formulae for J_1 and J_2 for this potential, but not J_3 . As no method of calculating this integral has ever been described in the literature (it is customary to assume it is spherically symmetrical and so ignore it or to state that it has been evaluated numerically) we want to give one. According to reference 6, Eq. (5)

$$J_3 = \frac{4M}{3\hbar^2} \iint \exp \left\{ -i\mathbf{k}' \cdot \mathbf{q} \right\} U\left(\mathbf{q} + \frac{1}{2}\mathbf{r}\right) \exp \left\{ -i\mathbf{k} \cdot \left(\frac{1}{2}\mathbf{q} + \frac{3}{4}\mathbf{r}\right) \right\} \phi\left(\left|\mathbf{q} - \frac{1}{2}\mathbf{r}\right|\right) d\mathbf{r} d\mathbf{q} \quad (41)$$

Let

$$\begin{aligned} \mathbf{r} &= \mathbf{z} \\ \mathbf{q} - \frac{1}{2}\mathbf{r} &= \mathbf{w} \end{aligned} \quad (42)$$

Then

$$J_3 = \frac{4M}{3\hbar^2} \iint \exp \left\{ -i\mathbf{k}' \cdot \left(\mathbf{w} + \frac{1}{2}\mathbf{z}\right) \right\} \phi(\mathbf{z}) U(|\mathbf{w} + \mathbf{z}|) \exp \left\{ -i\mathbf{k} \cdot \left(\frac{1}{2}\mathbf{w} + \mathbf{z}\right) \right\} \phi(\mathbf{w}) d\mathbf{w} d\mathbf{z} \quad (43)$$

The potential may be written

$$U(r) = U_0 \frac{e^{-\mu r}}{r} = \frac{U_0}{2\pi^2 \mu} \int \frac{\exp(i\mathbf{K} \cdot \mathbf{r})}{\mu^2 + K^2} d\mathbf{K} \quad (44)$$

This gives

$$J_3 = \frac{4M}{3\hbar^2} \frac{U_0}{2\pi\nu} \int \frac{d\mathbf{K}}{\nu^2 + K^2} \int \exp i(\mathbf{K} - \mathbf{k}' - \frac{1}{2} \mathbf{k}) \cdot \mathbf{w} \phi(\mathbf{w}) d\mathbf{w} \quad (45)$$

$$\cdot \int \exp i(\mathbf{K} - \frac{1}{2} \mathbf{k}' - \mathbf{k}) \cdot \mathbf{z} \phi(\mathbf{z}) d\mathbf{z}$$

We use for $\phi(\mathbf{w})$ the approximation⁷

⁷G. F. Chew, Phys. Rev. 74, 809 (1948).

$$\phi(\mathbf{w}) = N \frac{e^{-\alpha \mathbf{w}} - \hat{\mathbf{e}} e^{-\beta \mathbf{w}}}{\mathbf{w}} \quad (46a)$$

where

$$\alpha = .2316 \times 10^{+13} \text{ cm} \quad \beta = 7.0\alpha \quad (46b)$$

$$N = \left[\frac{\alpha\beta(\alpha+\beta)}{2\pi(\alpha-\beta)^2} \right]^{1/2}$$

Thus J_3 consists of four terms, one of which becomes on carrying out the integrals over $\mathbf{w}$ and $\mathbf{z}$,

$$J_3 = \frac{4M}{3\hbar^2} \frac{U_0}{2\pi\nu} N^2 (4\pi)^2 \int \frac{d\mathbf{K}}{\left\{ \nu^2 + K^2 \right\} \left\{ \alpha^2 + K^2 + \left| \mathbf{k}' + \frac{1}{2} \mathbf{k} \right|^2 - 2\mathbf{K} \cdot \left(\mathbf{k}' + \frac{1}{2} \mathbf{k} \right) \right\}} \quad (47)$$

$$\cdot \left\{ \beta^2 + K^2 + \left| \frac{1}{2} \mathbf{k}' + \mathbf{k} \right|^2 - 2\mathbf{K} \cdot \left(\frac{1}{2} \mathbf{k}' + \mathbf{k} \right) \right\}$$

Using the result

$$\int_0^1 \frac{dx}{[d+(c-d)x]^2} = \frac{1}{cd} \quad (48)$$

Eq. (47) becomes

$$J_3 = \frac{4M}{3h^2} \frac{U_0}{2\pi^2} N^2 (4\pi)^2 \int \frac{d\underline{k}}{y^2 + \underline{k}^2} \int_0^1 \frac{dx}{\left[(\alpha^2 + \underline{k}^2 + \left| \underline{k} + \frac{1}{2} \underline{k}' \right|^2 + (\beta^2 - \alpha^2)x - 2\underline{k} \cdot \left\{ (\underline{k}' + \frac{1}{2} \underline{k}) + \frac{1}{2}(\underline{k} - \underline{k}')x \right\} \right]^2} \quad (49)$$

where we have also used

$$\left| \frac{1}{2} \underline{k}' + \underline{k} \right| = \left| \underline{k}' + \frac{1}{2} \underline{k} \right| \quad (50)$$

Let

$$\alpha^2 + \left| \underline{k} + \frac{1}{2} \underline{k}' \right|^2 + (\beta^2 - \alpha^2)x + A^2 \quad (51)$$

$$\left(\underline{k}' + \frac{1}{2} \underline{k} \right) + \frac{1}{2}(\underline{k} - \underline{k}')x = \underline{B}$$

Then

$$J_3 = \frac{4M}{3h^2} \frac{U_0}{2\pi^2} N^2 (4\pi)^2 \int \frac{d\underline{k}}{y^2 + \underline{k}^2} \int_0^1 \frac{dx}{(A^2 + \underline{k}^2 - 2\underline{k} \cdot \underline{B})^2} \quad (52)$$

$$= \frac{4M}{3h^2} \frac{U_0}{2\pi^2} N^2 (4\pi)^2 \int_0^\infty \frac{K^2 dK}{y^2 + K^2} \int_0^1 \frac{dx}{(A^2 + K^2)^2 - 4K^2 B^2}$$

It can be verified that

$$(A^2 + K^2)^2 - 4K^2 B^2 > 0 \quad (53)$$

The integral over K can be done by extending the limits to $-\infty$ to $+\infty$ (the integrand depends only on K^2) and using the theory of residues. The roots of the denominator in Eq. (52) are

$$K = \pm i \sqrt{A^2 - B^2} \quad K = \pm B + i \sqrt{A^2 - B^2} \quad (54)$$

and it can be verified that

$$A^2 - B^2 > 0 \quad (55)$$

The result of the integration is

$$J_3 = \frac{4M}{3\hbar^2} \frac{U_0}{2\pi^2} N^2 (4\pi)^2 4\pi \frac{\pi}{4} \int_0^1 \frac{dx}{\sqrt{A^2 - B^2} (A^2 + \nu^2 + 2\nu \sqrt{A^2 - B^2})} \quad (56a)$$

or in the notation of reference 1

$$\frac{k}{8\pi} J_3 = \frac{1}{2} k g \int_0^1 \frac{dx}{\sqrt{A^2 - B^2} (A^2 + \nu^2 + 2\nu \sqrt{A^2 - B^2})} \quad (56b)$$

Since

$$A^2 = \alpha^2 + \frac{5}{4} k^2 + k^2 \mu + (\beta^2 - \alpha^2)x \quad (57)$$

$$A^2 - B^2 = \alpha^2 + (\beta^2 - \alpha^2)x + \frac{1}{2}(k^2 - k^2 \mu)(x - x^2)$$

Eq. (56) is a simple numerical integral for J_3 .

J_1 , J_2 , and J_3 are tabulated in Table I. As in reference 6, a partial wave analysis is made of these

$$\delta_i^l = \int_{-1}^1 \frac{k}{8\pi} J_i(\cos \theta) P_l(\cos \theta) d(\cos \theta) \quad (58)$$

and

$$" \delta_l " = \alpha \delta_1^l + \beta \delta_2^l + \gamma \delta_3^l \quad (59)$$

where α , β , and γ are given for $S = 3/2$ by Eq. (9), reference 6, and for $S = 1/2$, by Eq. (10), reference 6. Table II gives values of the δ_1^l and " δ_l ".

V. Amplitude for Absorption by Inelastic Scattering from a Partial Wave of Angular Momentum l .

A calculation of the inelastic p-d and n-d scattering at low energies has been made which is very crude and therefore very simple⁸.

⁸R. M. Frank and J. L. Gammel, Phys. Rev. , (195).

cross section is directly proportional to that for elastic scattering.

In this problem, continuum states are present, so that we must integrate over them to get the total absorption from a given l by inelastic scattering; that is, we must consider

$$\int_0^{k''_{\max}} \frac{dk''}{k'} \left| \frac{1}{2} " \beta_l " \right|^2$$

In reference 8, the three dimensional Born approximation was used. To break this up into contributions from different partial waves, we must

calculate

$$\frac{1}{2} \beta_{\ell} \sim \int_{-1}^1 d(\cos \underline{k}, \underline{k}') P_{\ell}(\cos \underline{k}, \underline{k}') J$$

where J is the scattering amplitude for inelastic scattering. But according to reference 8, Eq. (31), the scattering amplitude for inelastic scattering is proportional to that for elastic scattering, and the factor of proportionality is independent of the angle $\underline{k}, \underline{k}'$.

According to reference 8, Eq. (31)

$$\frac{1}{2} \beta_{\ell} \sim \sqrt{\frac{2\pi}{\alpha}} \frac{\{\exp(2i \delta_0(k'')) - 1\}}{2ik''} \int d(\cos \underline{k}, \underline{k}') P_{\ell}(\cos \underline{k}, \underline{k}') J_{\text{elastic}}$$

or

$$\frac{1}{2} \beta_{\ell} = \sqrt{\frac{2\pi}{\alpha}} \frac{\{\exp(2i \delta_0(k'')) - 1\}}{2ik''} \delta_{\ell}$$

whatever factors left out in $\sim$ being the same for β_{ℓ} and δ_{ℓ} .

Comparing this with Eq. (32) and subsequent equations, reference 8, it is clear that

$$\frac{1}{\delta_{\ell}^2} \int_0^{k'' \max} dk'' \frac{k}{k'} \left| \frac{1}{2} \beta_{\ell} \right|^2$$

is precisely the ratio of inelastic to elastic cross sections calculated in reference 8.

The crucial point is that the ratio of the scattering amplitudes for inelastic and elastic scattering is independent of the direction of $\underline{k}'$. Were this not the case, it would be extremely difficult to pass from the three dimensional Born approximation to the partial wave Born approximation. It is possible to carry through the three dimensional Born approximation with the highly accurate wave functions described in footnote 4,

reference 8, but it is impossible (that is, extremely difficult) to break the result down into contributions from different partial waves; that is, to calculate the " β_ℓ ".

Our result, therefore, is necessarily crude, being based on a zero range approximation. However, a calculation⁸ of the inelastic n-d scattering at low energies (< 14 Mev) is in good agreement with what is known experimentally, even though the zero range approximation is used. In reference 6, it was found that calculations with a zero range potential gave nearly identical results with more elaborate calculations at low energies on the elastic scattering.

It must be remembered that in analyzing the data, we use the η 's calculated here only as a starting point (certainly we don't believe our results for $\ell = 0$). Also, several starting points and methods of analysis are used. For example, an analysis is carried out with real phase shifts ($|\eta_\ell| = 1$ for all ℓ) to see whether it is necessary to use complex phase shifts. Also an analysis is carried out with complex phase shifts for $\ell = 0$, and real phase shifts for $\ell \geq 1$. In this type analysis, it is assumed that all of the inelastic scattering comes from $\ell = 0$. Finally, the best fit is obtained with the η_ℓ calculated here for $\ell \geq 2$; that is, the analysis of the data justifies the calculation made here (it might be expected that as ℓ increases the validity of the Born and zero range approximations improves, because the angular momentum barrier prevents penetration of the potential).

Thus we use the result given in footnote 7, reference 8

$$\frac{1}{\delta_\ell^2} \int_0^{k_{\max}} d\underline{k} \frac{k}{4k'} |\beta_\ell|^2$$

$$= \frac{1}{2\pi} \int_0^{E'_{\max}} dE' \frac{[E'(E_{\text{lab}} - \frac{3}{2}E' - \frac{3}{2}E_b)]^{\frac{1}{2}}}{[E_b E_{\text{lab}}]^{\frac{1}{2}}} \left\{ \frac{1}{D_{\text{triplet}}} + \frac{(V_{\text{np}}^+)^2}{3} \frac{1}{D_{\text{singlet}}} \right\} \quad (60)$$

The $\frac{1}{2}(1 - \eta_\ell)$ calculated with assumption, the " δ_ℓ " given in Table II, and Eq. (38) are given in Table II.

VI. Results Calculated in Born Approximation.

The results presented in this section are based on Born approximations. No modification has been made even for $\ell = 0$.

A graph of the angular distribution calculated from the three dimensional Born approximation corresponding to Eq. (20) is shown in Figure 1. That it lies well above the experimental points^{9,10} is explained by

9. M. O. Stern, Phys. Rev. 85, 752(1952). More detailed information on the 95 Mev p-d scattering -- actually he observed 190 Mev d-p scattering -- can be found in his Ph.D. Thesis, 1951, UCRL-1440. Angular distributions at other energies have been measured. A useful summary of these is found in D. D. Clark (Ph.D. Thesis, 1953), UCRL-2255.
10. The total cross section at 95 Mev is 93 ± 7 mt. A. L. Bloom, Phys. Rev. 85, 752 (1952).

Eq. (15) which says that this method of calculating the angular distribution does not take account of the necessity of conserving particles, and in fact says that more elastically scattered particles appear than is possible. Of course, the effect of the inelastic scattering is not taken into account in this calculation.

If we use Eq. (18) to calculate the η_ℓ and use these in Eq. (5)

(that is, we use real phase shifts δ_ℓ , $\tan \delta_\ell = "j_\ell"$), we get an angular distribution also shown in Figure 2. This is in far better agreement with the experimental points.

It seems to us that the calculation of Bethe and Gluckstern¹¹ and calculations using the impulse approximation^{12,13} are more closely related to the three dimensional Born approximation than the calculation

11. R. L. Gluckstern and H. A. Bethe, Phys. Rev. 81, 761 (1951).

12. G. F. Chew, Phys. Rev. 84, 710 (1951).

13. G. F. Chew, Phys. Rev. 84, 1057 (1951).

described in the previous paragraph. In fact, the angular distribution calculated by Gluckstern and Bethe is similar to our three dimensional Born approximation curve (it is possibly a little better in the vicinity of $\theta = 90^\circ$ but is still too high by a factor of 2).

The calculations of Horie, Tamura, and Yoshida¹⁴, which takes account

14. Horie, Tamura, and Yoshida, Prog. Theor. Phys. 8, 341 (1952).

of tensor forces, also employs the three dimensional Born approximation. Their calculation gives an angular distribution very similar to the one described in the previous paragraph, which is a considerable improvement over the results of Bethe and Gluckstern. This makes it appear that the tensor part of the two body forces is important in p-d scattering at 95 Mev (in the sense that it cannot be replaced by the same "effective" central potential which reproduces its effects in the high energy n-p and p-p scattering.) This conclusion is not firm, being based on the three dimensional Born approximation. Our analysis is based on the assumption

that we can use "effective" central potentials. Bransden¹⁵ has also made

15. B. H. Bransden, Proc. Royal Soc. 209A, 380 (1951).

a calculation with tensor forces. In the last paragraph of their paper, Horie, Tamura, and Yoshida¹⁵ compare his work with theirs.

In Figure 2, a graph of the angular distribution calculated using the modified Born approximation developed in previous sections and Eq. (5) is shown.

None of these really agree with the experimental data, probably because Born's approximation fails for states of low angular momentum, especially $\ell = 0$.

VII. Analysis of the 95 Mev p-d Experimental Data.

At this point we abandon our previous work entirely, except insofar as we use it as a starting point for analyzing the data. We do not abandon it in the sense that we believe Born's approximation should become more valid as the angular momentum increases.

However, we want to avoid the use of complex phase shifts if possible in order to test whether or not they are required to fit the data. Thus several types of analyses were carried out as follows. These are classified according to whether real or complex phase shifts are used for $\ell = 0$, $\ell = 1$, and $\ell \geq 2$.

Analysis	$\ell = 0$	$\ell = 1$	$\ell \geq 2$
I	real*	real	real
II	complex*	real	real
III	complex*	complex*	real
IV	complex*	complex	complex
V	complex*	complex*	complex

An asterisk means that the phase shifts for that ℓ were adjusted in trying to fit the data. The method of analysis was the same as that of Dodder and Gammel¹⁶.

16. D. C. Dodder and J. L. Gammel, Phys. Rev. 88, 520 (1952).

Analysis II assumes that the entire contribution to the inelastic scattering comes from $\ell = 0$; Analysis III, from $\ell = 0, 1$; Analysis IV and V that the inelastic scattering is distributed among all ℓ (for $\ell \geq 1$ or 2) according to Eq. (60).

Results of the analyses for the $\frac{1}{2}(1 - \eta_\ell)$ are given in Table III, and Figures 2 and 3.

It is not possible to find real phase shifts which fit the data (Analysis I). It can be seen how the situation is improved as the S and P phase shifts are allowed to become complex (Analysis II, III), even though higher phase shifts are allowed to remain real.

Analysis IV shows that a good fit to the data can be obtained by adjusting the S phase shifts only provided for $\ell \geq 1$ the modified Born approximation is used.

Analysis V shows that the η 's for $\ell = 1$ need be changed only slightly from the values calculated with the modified Born approximation to obtain still better fits to the data (though the total cross section data is not fit quite so well).

Table IV shows the analysis of the total cross section data. Again, Analysis I is ruled out, and Analyses IV and V spoken for.

Analyses III and V bring out an interesting thing. The values of the η 's for $\ell = 0$ and 1 turn out to be about the same whether we take those

for $\ell \geq 2$ real or complex. In fact, Analyses II-V show the η 's for $\ell = 0$ to be about the same in all cases. This seems to us a fortunate thing. If we had some auxiliary theory for the energy dependence of these quantities (such as an effective range theory or a resonance theory) we could try to find a connection between these results and the low energy results⁶. This would require an examination of the other data available of course (32 - 340 Mev). This is an ambitious program which we hope to carry out in the future.

The question naturally arises whether or not other solutions can be found for the η 's which fit the data. The answer is yes. We may think about it this way: Suppose we are analyzing for the η 's for $\ell = 0$. All the other η 's are set on their Born approximation values. Now suppose we have fit four points:

$$\frac{2}{3}(x + B_1(\theta))^2 + \frac{2}{3}(y + B_2(\theta))^2 + \frac{1}{3}(z + B_3(\theta))^2 + \frac{1}{3}(w + B_4(\theta))^2 = \sigma(\theta)$$

and four similar equations for the four other points. (x is the real part of $\frac{1}{2}(1 - \eta)$ for $\ell = 0$ and $S = 3/2$, w is the imaginary part of $\frac{1}{2}(1 - \eta)$ for $\ell = 0$ and $S = 1/2$. The B's are the real or imaginary part of

$$\sum_{\ell=1}^{\infty} (2\ell + 1) \left(\frac{1}{2} - \frac{1}{2} \eta_{\ell} \right) P_{\ell}(\cos \theta)$$

The B's are of course, known functions of θ since we are supposing the η 's known for $\ell \geq 1$. This set of equations has two sets of solutions for x, y, z, w. The second set for analyses II, IV are shown in Table V. The total cross section data does not help very much in distinguishing the two sets, as shown in Table VI. Again, a knowledge of the probable energy dependence is required to resolve this problem.

A further point of interest concerns the figure 29 ± 3 mb given by

Stern⁹ for the total elastic cross section. One finds that the total elastic cross section comes from quite small angles, and even at 15° the effect of the Coulomb scattering is not negligible (more accurately, its interference with the nuclear scattering). It reduces the cross section at this angle. Therefore, we may conclude that the figure 29 ± 3 mb, which is actually

$$\int_{10^\circ}^{180^\circ}$$

should be lower than the n-d total elastic scattering at 95 Mev.

VII. Conclusion.

The defects of this analysis are only too clear. Chief among them is the neglect of tensor forces and the crude calculation of the inelastic scattering amplitudes.

However, the experiments are nearly as crude. We hope the analysis will encourage more accurate experiments.

We consider the chief merits of this work to be

- (1) It shows how complex phase shifts can be calculated when inelastic scattering is present.
- (2) The analyses show that the η 's for $\ell = 0$ and $\ell = 1$ are reasonably well determined, and could be used to check a theory of their energy dependence.

IX. Acknowledgements.

I would like to thank Dr. R. S. Christian for helpful discussions about the calculation.

- TABLE I. Scattering Amplitudes for Three Dimensional Born Approximation.
- TABLE II. Phase Shifts Calculated in Born Approximation.
- TABLE III. Results of Phase Shift Analysis.
- TABLE IV. Total Cross Sections.
- TABLE V. Second Roots for Analysis.
- TABLE VI. Total Cross Sections for Second Roots.
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- FIGURE 1. Angular Distributions Calculated in Born Approximation.
- FIGURE 2. Angular Distributions Resulting from Analyses I, II, and III.
- FIGURE 3. Angular Distributions Resulting from Analyses IV and V.

TABLE I.

θ	$\frac{k}{4\pi} J_1$	$\frac{k}{4\pi} J_2$	$\frac{k}{4\pi} J_3$
0°	4.9783	.026443	2.1295
10°	4.4102	.026870	2.0838
20°	3.2782	.028186	1.9035
30°	2.1823	.030542	1.7381
40°	1.4089	.034183	1.5095
50°	.91603	.039523	1.3277
60°	.61079	.047217	1.2032
70°	.42085	.058315	1.1079
80°	.26484	.074403	1.0361
90°	.22353	.098231	.99361
100°	.17012	.13331	.98189
110°	.13479	.18647	.98628
120°	.11040	.26717	.99801
130°	.093374	.38899	1.0200
140°	.081544	.56834	1.0522
150°	.073372	.81552	1.1050
160°	.068084	1.1125	1.1636
170°	.065106	1.3811	1.2075
180°	.064145	1.4938	1.2222

These results are calculated for 90 Mev.

TABLE II

ℓ	δ_1	δ_2	δ_3	" $4\delta_\ell$ "	" $2\delta_\ell$ "	$\frac{1}{2}(1 - 4m_\ell)$	$\frac{1}{2}(1 - 2m_\ell)$
0	.58206	.22706	1.1720	.64997	1.6944	.42124 + i .2625	.78591 + i .09200
1	.36220	-.13305	.10555	.36693	.35503	.20402 + i .25665	.19408 + i .25339
2	.20667	.067999	.081297	.085985	.29353	.01141 + i .074685	.14346 + i .23101
3	.11377	-.032798	.019010	.099188	.10327	.019065 + i .09540	.021005 + i .10001
4	.061384	.015500	.011250	.020817	.073211	.000433 + i .020808	.01072 + i .07203
5	.033553	-.007426	.0021002	.025253	.028419	.00637 + i .025237	.000807 + i .028397
6	.019824	.0038071	.0027311	.0074705	.022231	.000056 + i .0074705	.000494 + i .022227

TABLE III

Analysis	S = 3/2		S = 1/2		RMS
	$\frac{1}{2} - \frac{1}{2} \eta_0$	$\frac{1}{2} - \frac{1}{2} \eta_1$	$\frac{1}{2} - \frac{1}{2} \eta_0$	$\frac{1}{2} - \frac{1}{2} \eta_1$	
I	$\delta_0 = 31.75^\circ$		$\delta_0 = 45.29^\circ$		44.5%
II	.21413 + .36593 i		.26013 - .01369 i		23%
III	.21704 + .34577 i	.22884 + .15724 i	.25001 - .00213 i	.25667 + .17754 i	13.2%
IV	.21939 + .37671 i		.22373 - .10556 i		18.2%
V	.24261 + .35485 i	.24982 + .14471 i	.25032 - .07934 i	.22320 + .12517 i	12.7%
				experimental	11.7%

TABLE IV

Analysis	RMS.	$\sigma_{\text{elastic}} \text{ (mb)}$	$\sigma_{\text{inelastic}} \text{ (mb)}$	$\sigma_{\text{total}} \text{ (mb)}$
experimental	11.7%	29 ± 3		92 ± 7
real Born		68.1	0	68.1
complex Born		60.6	45.5	106.1
I	44.5%	62.1	0	62.1
II	23.0%	48.4	10.6	59.0
III	13.2%	41.3	28.7	70.0
IV	18.2%	45.9	45.3	91.2
V	12.7%	40.3	42.0	82.2

TABLE V

	$S = 3/2$	$S = 1/2$
Solution	$\frac{1}{2} - \frac{1}{2} \eta_0$	$\frac{1}{2} - \frac{1}{2} \eta_0$
II'	$.27037 + i .36981$	$-.19560 + i .16190$
IV'	$.31594 + i .32398$	$-.14448 + i .21263$

TABLE VI

Solution	$\sigma_{eL} \text{ (mb)}$	$\sigma_{in} \text{ (mb)}$	$\sigma_{total} \text{ (mb)}$
II'	49.6	19.3	68.9
IV'	46.6	49.87	96.5

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- REAL PHASE SHIFTS (44.5%)
- - - COMPLEX S, REST REAL (23%)
- · - COMPLEX S,P, REST REAL (13.2%)

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1-34

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