On Standing Wave Solutions to the Schrödinger Equation

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**Work supported in part by the U. S. Atomic Energy Commission

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Abstract

The standing wave solution to the Schrödinger equation defined in terms of the standing wave Green's function for the full Hamiltonian is discussed. This solution is compared with the more usual standing wave solution. The former is shown to be one-half the sum of the usual ingoing and outgoing wave solutions obeying Lippmann-Schwinger equations. Partial wave elements of the two solutions as well as of the two reaction (K) matrices are found to be related by a simple normalization factor, viz, $\cos^2 \delta_l$, where δ_l is the l th partial wave phase shift. Thus, either of the two standing wave solutions can be used to obtain the correct K matrix element, $\tan \delta_l$, since in each case it is the asymptotic ratio of the irregular to the regular solution.

1. Introduction

In a recent paper¹, a standing wave solution to the inhomogeneous equation describing identical particle scattering was developed [1], along with a reaction or K matrix which included the effects of the Pauli principle. Use was made of a Lippmann-Schwinger equation for the standing wave Green's function² $\hat{P}/(E-H)$ for the full Hamiltonian H. However, such use is incorrect, for the desired Lippmann-Schwinger equation does not exist. Thus, the transition from Eq. (I.14) to (I.15) is invalid. Nevertheless, the results based on (I.15) are correct, if the various K matrices are properly interpreted, as we show in detail in the succeeding paper. In the present work, we study the properties of the standing wave solution to the Schrodinger equation for potential well scattering as defined by use of $(E-H)^{-1}$. This new solution is one half of the sum of the usual ingoing and outgoing wave solutions. We compare this solution with the more conventional one involving the true reaction matrix, denoted K, whose partial wave matrix element is $\tan\delta_l$, where δ_l is the usual phase shift.

The new standing wave solution involves an altered K matrix, which we denote $\tilde{K}$. It is related, in the limit $\epsilon \rightarrow 0$, to the quantity $\tilde{r}$ defined by Tolocman and Nagarajan [2] in their study of the use of the K matrix in nuclear reaction theories. These authors have given a formal relation between K and $\tilde{K}$, although they do not discuss the standing wave solution in which $\tilde{K}$ occurs nor relate that solution to the more usual one. We do both in this paper, and derive another formal expression relating K and $\tilde{K}$. We also show that the partial wave elements of the two standing wave solutions are simply related. Thus if K has the matrix element $\tan\delta_l$, then $\tilde{K}$ has the matrix element $\sin\delta_l \cos\delta_l =$

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1. We refer to reference [1] as I, and equations from it as (I.1), (I.2), etc.
 2. As in I, we shall henceforth suppress the principal value symbol $\hat{P}$, and write, eg, $\hat{P}/(E-H)$ simply as $(E-H)^{-1}$.

$$\cos^2 \delta_l \cdot \tan \delta_l.$$

To our knowledge, these are new results. Because of this and also because the extension of these results to the case of the Pauli principle involves the justification of certain previous work, we discuss the potential scattering separately in this paper. It forms a basis for our later analysis, however, and is thus the introduction to the identical particle collision problem.

2. Formulation of the Problem.

We shall compare the usual standing wave solution to the Schrödinger equation with the sum of the ingoing and outgoing wave solutions. Our notation is such that the Hamiltonian H is given as the sum of H_0 , describing a free particle of mass M , and V , the interaction. Then

$$H_0 |\phi_{\underline{k}}\rangle = E |\phi_{\underline{k}}\rangle = \frac{\hbar^2 \underline{k}^2}{2M} |\phi_{\underline{k}}\rangle$$

where

$$\langle \underline{r} | \phi_{\underline{k}} \rangle = (2\pi)^{-3/2} e^{i \underline{k} \cdot \underline{r}}$$

describes the case of no interaction, and

$$(E - H) |\psi_{\underline{k}}\rangle = 0 \tag{1}$$

is the equation whose solutions of are of interest.

Three types of solution to (1) may be distinguished, denoted here $|\psi_{\underline{k}}^+\rangle$, $|\psi_{\underline{k}}^-\rangle$ and $|\psi_{\underline{k}}^0\rangle$ corresponding as usual to outgoing, ingoing and standing waves, respectively. They obey Lippmann-Schwinger equations given by [3,4,5

$$|\psi_{\underline{k}}^+\rangle = |\phi_{\underline{k}}\rangle + (E^+ - H_0)^{-1} V |\psi_{\underline{k}}^0\rangle \tag{2}$$

and

$$|\psi_{\underline{k}}^0\rangle = |\phi_{\underline{k}}\rangle + (E - H_0)^{-1} V |\psi_{\underline{k}}^0\rangle, \tag{3}$$

where $E^\pm = E \pm i\epsilon$, $\epsilon \rightarrow 0^+$.

Transition and reaction operators $T^{\pm}$ and K are defined in the usual way via

$$T^{\pm}|\phi_k\rangle = V|\psi_k^{\pm}\rangle \quad (4)$$

and

$$K|\phi_k\rangle = V|\psi_k^0\rangle. \quad (5)$$

Substitution of these results into Eqs. (2) and (3) then gives

$$|\psi_k^{\pm}\rangle = |\phi_k\rangle + (E^{\pm} - H_0)^{-1}T^{\pm}|\phi_k\rangle \quad (6)$$

and

$$|\psi_k^0\rangle = |\phi_k\rangle + (E - H_0)^{-1}K|\phi_k\rangle. \quad (7)$$

Operation on both sides of (6) and (7) by V leads to the well-known results

$$T^{\pm} = V + V(E^{\pm} - H_0)^{-1}T^{\pm} \quad (8)$$

and

$$K = V + V(E - H_0)^{-1}K. \quad (9)$$

Equations (2) and (8) possess the formal solutions

$$|\psi_k^{\pm}\rangle = |\phi_k\rangle + (E^{\pm} - H)^{-1}V|\phi_k\rangle \quad (10)$$

and

$$T^{\pm} = V + V(E^{\pm} - H)^{-1}V. \quad (11)$$

These are verified to be solutions of (2) and (8) by making use of the Lippmann-Schwinger equation

$$(E^{\pm} - H)^{-1} = (E^{\pm} - H_0)^{-1} + (E^{\pm} - H_0)^{-1}V(E^{\pm} - H)^{-1} \quad (12)$$

for the interacting Green's functions. We shall use T in place of $T^{\pm}$, so that, from (11), $T^{\pm} = T$. A method for deriving (10) [or (18)] follows by writing

$$|\psi_k^{\pm}\rangle = |\phi_k\rangle + |\eta_k^{\pm}\rangle,$$

which when substituted into (1) leads to

$$(E-H)|\eta_k^+\rangle = V|\phi_k^+\rangle.$$

The solution to this equation is

$$|\eta_k^+\rangle = (E^+ - H)^{-1}V|\phi_k^+\rangle, \quad (13)$$

immediately leading to (10). Use of (12) in

$$|\psi_k^+\rangle = \pm i\epsilon(E^+ - H)^{-1}|\phi_k^+\rangle \quad (14)$$

leads to (2), and also to Eq. (10), as shown by Gell-Mann and Goldberger [6].

These are standard results, exhibited solely for notation and direct reference. We now wish to apply a similar procedure to Eqs. (3) and (7) and obtain explicit solutions analogous to Eqs. (10) and (11). Suppose we attempt this by writing

$$|\psi_k^0\rangle = |\phi_k^0\rangle + |\eta_k^0\rangle$$

and substitute into (1) to obtain

$$(E-H)|\eta_k^0\rangle = V|\phi_k^0\rangle.$$

By analogy with (13), we would put

$$|\eta_k^0\rangle = (E-H)^{-1}V|\phi_k^0\rangle,$$

principle value assumed in $(E-H)^{-1}$, and thus obtain

$$|\psi_k^0\rangle = |\phi_k^0\rangle + (E-H)^{-1}V|\phi_k^0\rangle.$$

In contrast to $|\eta_k^+\rangle$ and Eqs. (2), (10), (13) and (14), however, the above equation does not yield the same solution to (1) as that given by Eqs. (3) or (7).

We therefore define a new solution of (1), denoted $|\chi_k^0\rangle$, by

$$|\chi_k^0\rangle = |\phi_k^0\rangle + (E-H)^{-1}V|\phi_k^0\rangle; \quad (15)$$

our intent is to compare Eqs. (7) and (15).

The reason that $|\chi_k^0\rangle$ and $|\psi_k^0\rangle$ differ is that $(E-H)^{-1}$ does not have a Lippmann-Schwinger iteration, as noted, for example, by Newton [4]. That is,

$$(E-H)^{-1} = (E-H_0)^{-1} + (E-H_0)^{-1}V(E-H)^{-1} \quad (16)$$

is an invalid result. Were (16) to be correct, then its use in (15) would

lead to

$$|x_k^0\rangle = |\phi_k\rangle + (E-H_0)^{-1} \tilde{K} |\phi_k\rangle \quad (17)$$

(also an invalid result) with

$$\tilde{K} = V + V(E-H)^{-1}V. \quad (18)$$

We shall see however that $|x_k^0\rangle$ still yields the proper K matrix elements, provided they are defined as the ratio of the coefficients of the irregular to the regular solution of the unperturbed Hamiltonian.

The correction to (17) resulting from the fact that (16) is invalid can be determined in various ways. One method is to note that $(E-H)^{-1} = \{ (E^+-H)^{-1} + (E^--H)^{-1} \} / 2$, so that iteration plus the relation $(E^+ - H)^{-1} = (E-H)^{-1} + i\pi\delta(E-H)$ yields

$$(E-H)^{-1} = (E-H_0)^{-1} [1 + V(E-H)^{-1}] - \pi^2 \delta(E-H_0) V \delta(E-H). \quad (19)$$

Substituting (19) into (15) then gives

$$|x_k^0\rangle = [1 - \pi^2 \delta(E-H_0) V \delta(E-H) V] |\phi_k\rangle + (E-H_0)^{-1} \tilde{K} |\phi_k\rangle \quad (20)$$

as the replacement for (17), with $\tilde{K}$ defined by (18). It might be thought that the delta function term in (20) is zero since $V|\phi_k\rangle = (E-H)|\phi_k\rangle$ and $\delta(E-H)(E-H)$ seems like $x\delta(x)$. However, $\delta(E-H)$ can only be evaluated by inserting a complete set of states of H to the right of $\delta(E-H)$. Doing so, we find terms such as $\langle \psi_k^+ | H | \phi_k \rangle$, where H acts to the right, and H cannot be made to act to the left without giving an additional contribution because of the continuum nature of the states. That is, as is well known in various contexts, H is not Hermitian between continuum states [6].

The delta function term can be evaluated eventually in terms of either K matrix elements or the optical theorem. Rather than do this at once, we go back to (15) and notice that $|x_k^0\rangle$ is also given by

$$|x_k^0\rangle = [|\psi_k^+\rangle + |\psi_k^-\rangle] / 2 \quad (21)$$

which follows from $2(E-H)^{-1} = \{ (E^+-H)^{-1} + (E^--H)^{-1} \}$ and Eq. (10).

Thus, a comparison of $|x_k^0\rangle$ and $|\psi_k^0\rangle$, the two "obvious" forms of standing wave solution to Eq. (1), is actually a comparison of $|\psi_k^0\rangle$ with the outgoing and ingoing wave solution $|\psi_k^\pm\rangle$. That such a comparison of $|x_k^0\rangle$ and $|\psi_k^0\rangle$ should yield a simple relation is therefore not surprising, but as noted above, we have not found any previous statements of such a relationship in the literature. We first use Eq. (21) and the formal solutions for $|\psi_k^\pm\rangle$ in terms of $T^\pm$ [ie, Eqs. (6)] along with the formal Heitler damping equation, to evaluate $|x_k^0\rangle$. We then use Eq. (20) and a partial wave expansion to re-evaluate $|x_k^0\rangle$. These methods will be useful for the analysis of the inhomogeneous equation standing wave solutions.

3. Formal Method.

From Eqs. (6) it follows that

$$\begin{aligned} |x_k^0\rangle &= |\phi_k\rangle + \{(E^+ - H_0)^{-1}T + (E^- - H_0)^{-1}T^\dagger\} |\phi_k\rangle / 2 \\ &= |\phi_k\rangle + \{(E - H_0)^{-1}(T + T^\dagger) - i\pi\delta(E - H_0)(T - T^\dagger)\} |\phi_k\rangle / 2, \end{aligned}$$

or

$$\begin{aligned} |x_k^0\rangle &= [1 - \frac{i\pi}{2}\delta(E - H_0)(T - T^\dagger)] |\phi_k\rangle + (E - H_0)^{-1} [\frac{1}{2}(T + T^\dagger)] |\phi_k\rangle \\ &\equiv P |\phi_k\rangle + (E - H_0)^{-1} \tilde{K} |\phi_k\rangle, \end{aligned} \quad (22)$$

which defines P (and $\tilde{K}$). Comparing the first line of Eq. (22) with Eq. (20), we see that the δ -function products and the factor $V + V(E - H)^{-1}V$ are also given in terms of T and $T^\dagger$.

We now show that

$$\tilde{K} = PK,$$

where K is the true K matrix, as appears in Eq. (7) for $|\psi_k^0\rangle$. To do this we introduce the Heitler damping equation in operator form (e.g. [5]):

$$T = K - i\pi K\delta(E-H_0)T = K - i\pi T\delta(E-H_0)K, \quad (23)$$

whose solution is

$$T = \{1 + i\pi K\delta(E-H_0)\}^{-1}K.$$

From (23) we easily find that

$$\frac{1}{2}(T+T^\dagger) = \{1+\pi^2[K\delta(E-H_0)]^2\}^{-1}K$$

and

$$\frac{1}{2}(T-T^\dagger) = -\pi i\{1+\pi^2[K\delta(E-H_0)]^2\}^{-1}K\delta(E-H_0)K.$$

Substitution into Eq. (22) yields

$$|x_k^0\rangle = \{1+\pi^2[K\delta(E-H_0)]\}^{-1}|\phi_k\rangle + (E-H_0)^{-1}\{1+\pi^2[K\delta(E-H_0)]^2\}^{-1}K|\phi_k\rangle, \quad (25)$$

from which we see by comparison with (22) that

$$P = \{1+\pi^2[K\delta(E-H_0)]^2\}^{-1}$$

and

$$\tilde{K} = PK,$$

as anticipated.

Although it may not be obvious from these results, $\{P, H_0\} = 0$, so that

$$P^{-1}|x_k^0\rangle = |\phi_k\rangle + (E-H_0)^{-1}K|\phi_k\rangle,$$

ie,

$$P^{-1}|x_k^0\rangle = |\psi_k^0\rangle. \quad (26)$$

We prove this in the next section, and also show that P is a sum of projection operators onto the angular momentum states $|\ell m\rangle$, so that (26) is actually a relation between partial wave elements of $|x_k^0\rangle$ and $|\psi_k^0\rangle$.

We remarked above that Tobocean and Nagarajan [2] have derived a formal relation between K and $\tilde{K}$. In terms of our notation, this relation can be reduced to that implied in Eq. (20), with

$$P = 1 - \pi^2 \delta(E-H_0) V \delta(E-H) V.$$

Since we have derived formal relations between K and $\tilde{K}$, it is interesting to note that $|x_k^0\rangle$ can be used to obtain an inhomogeneous integral equation for $\tilde{K}$ directly. Multiplication of both sides of Eq. (15) by V leads to

$$\tilde{K}|\phi_k\rangle = V|x_k^0\rangle$$

Using this result in Eq. (20), we find

$$\tilde{K} = V + V(E-H_0)^{-1}\tilde{K} - \pi^2 V\delta(E-H_0)V\delta(E-H)V.$$

An analogous relation is obtained from Eq. (25).

We may also write a formal solution $|x_k^{(1)}\rangle$ to Eq. (1) of the form

$$|x_k^{(1)}\rangle = |\phi_k\rangle + (E-H_0)^{-1}\tilde{K}|\phi_k\rangle,$$

which defines $|x_k^{(1)}\rangle$ since $\tilde{K}$ is assumed known. An inhomogeneous integral equation for $|x_k^{(1)}\rangle$ involving $(E-H_0)^{-1}$ can be derived using Eq. (22) and the relation $\tilde{K}|\phi_k\rangle = V|x_k^{(1)}\rangle$. A similar result holds for $|x_k^{(0)}\rangle$, though we do not display these as they are of no special interest here.

4. Partial Wave Method

In this section we evaluate $|x_k^0\rangle$ using Eq. (20). The quantity of interest is P , as given by

$$P = 1 - \pi^2 \delta(E-H_0)V\delta(E-H)V \equiv 1-Q.$$

The determination of P is made by using

$$\begin{aligned} \delta(E-H) &= \sum_n |\psi_n\rangle \delta(E-E_n) \langle \psi_n| + \int d^3k' |\psi_{k'}^+\rangle \delta(E-E_{k'}) \langle \psi_{k'}^+| \\ &\rightarrow \int d^3k' |\psi_{k'}^+\rangle \delta(E-E_{k'}) \langle \psi_{k'}^+|, \end{aligned} \quad (27)$$

where $|\psi_{k'}^+\rangle$ are continuum eigenstates of H and $|\psi_n\rangle$ are the bound states, with energies $E_{k'}$ and E_n respectively. We specifically assume that the continuous spectra of H and H_0 coincide, so that $E_n < E$ for all E_n , which leads to the second

line of (27). Introducing the notation $\rho_k = dE_k/dk' = Mk'/\hbar$ and denoting the angles of k' by Ω_k , (27) becomes

$$\delta(E-H) = \rho_k \int |\psi_k^+ \rangle d\Omega_k \langle \psi_k^+ |, \quad k' = k, \quad (28)$$

and similarly,

$$\delta(E-H_0) = \rho_k \int |\phi_k^+ \rangle d\Omega_k \langle \phi_k^+ |, \quad k' = k. \quad (29)$$

Let us now recall that the matrix elements of T are given by

$$T_{k''k} = \langle \phi_{k''}^+ | V | \psi_k^+ \rangle.$$

Then substituting Eqs. (28) and (29) into the expression for Q , we have

$$Q|\phi_k^+ \rangle = -(\pi\rho_k)^2 \int d\Omega_k d\Omega_{k''} |\phi_{k''}^+ \rangle T_{k''k} T_{kk''}^+; \quad k=k'=k''. \quad (30)$$

The TT^+ factor in Q can be calculated using the optical theorem. From Eq. (11),

$$T-T^+ = -2\pi i V \delta(E-H) V,$$

leading to

$$T_{k''k} - T_{k''k}^+ = -2\pi i \rho_k \int d\Omega_{k''} T_{k''k''} T_{kk''}^+, \quad k = k' = k'',$$

so that Eq. (30) can be written as

$$Q|\phi_k^+ \rangle = -\frac{\pi i}{2} \rho_k \int d\Omega_{k''} |\phi_{k''}^+ \rangle [T_{k''k} - T_{k''k}^+]. \quad (31)$$

The right hand side of (31) can be evaluated using the damping equation as is done in section 3, but it is simpler in this case of pure elastic scattering (although the results are generalizable) to use the phase shift expansion of $T_{k''k}$:

$$T_{k''k} = \frac{M}{2\pi\hbar^2} T_{k''k} = (2\pi)^{-3} \frac{4\pi}{k} \sum_{lm} e^{i\delta_l} \sin\delta_l Y_l^m(\Omega_{k''}) Y_l^m(\Omega_k). \quad (32)$$

The angular integration in (31) can be carried out after substitution of (32) for the term in square brackets if we employ the expansion

$$|\phi_{k''}^+ \rangle = \sum_{lm} |klm\rangle Y_l^m(\Omega_{k''}), \quad k''=k,$$

where

$$\langle r | klm \rangle = \sqrt{2/\pi} j_l(kr) i^l Y_l^m(\Omega_r).$$

This yields

$$Q|\phi_k\rangle = \sum_{lm} \sin^2 \delta_l |klm\rangle Y_l^m(\Omega_k). \quad (33)$$

Since $P = 1-Q$, we thus obtain

$$P|\phi_k\rangle = \sum_{lm} \cos^2 \delta_l |klm\rangle Y_l^m(\Omega_k)$$

or

$$P = \sum_{lm} |lm\rangle \cos^2 \delta_l \langle lm|, \quad (34)$$

$$\text{since } |lm\rangle \langle lm| \phi_k\rangle = |klm\rangle Y_l^m(\Omega_k).$$

Using this in Eq. (20) we finally see that

$$\begin{aligned} |\chi_k^0\rangle &= \sum_{lm} \cos^2 \delta_l |klm\rangle Y_l^m(\Omega_k) + (E-H_0)^{-1} K |\phi_k\rangle \\ &= P|\phi_k\rangle + (E-H_0)^{-1} PK |\phi_k\rangle. \end{aligned} \quad (35)$$

Since $[H_0, |lm\rangle \cos^2 \delta_l \langle lm|] = 0$, we get

$$P^{-1} |\chi_k^0\rangle = |\phi_k\rangle + (E-H_0)^{-1} K |\phi_k\rangle \equiv |\psi_k^0\rangle, \quad (36)$$

$$\text{where } P^{-1} = \sum_{lm} |lm\rangle \sec^2 \delta_l \langle lm|.$$

We thus conclude that

$$\sec^2 \delta_l \langle lm | \chi_k^0 \rangle = \langle lm | \psi_k^0 \rangle, \quad (37)$$

showing that partial wave elements of $|\psi_k^0\rangle$ and $|\chi_k^0\rangle$ differ only by an overall normalization factor. From this result we see that either $|\psi_k^0\rangle$ or $|\chi_k^0\rangle$ will yield the correct K-matrix elements asymptotically. That is, the ratio of the irregular to the regular portions of $|\chi_k^0\rangle$ also yields $\tan \delta_l$ directly. This relationship between $|\psi_k^0\rangle$ and $|\chi_k^0\rangle$, Eqs. (36) or (37), can be thought of as a means of proceeding from $|\chi_k^0\rangle$ to the correct K matrix solution $|\psi_k^0\rangle$, and we shall exploit this in establishing the correct solution for the inhomogeneous equations in the succeeding paper.

5. Concluding Remarks

The results of this work display the consistency of the formalism used, viz, quantum mechanics. The operators T and K can be introduced solely with reference to the S matrix, K being assured of existence because of the unitarity of S. On the other hand, starting from the outgoing wave $[|\psi_k^+>]$ and standing wave $[|\psi_k^0>]$ solutions, we can define T and K, whose relation via the damping equation (23) is the same as obtained from unitarity of S, thus establishing the consistency. We use this argument ab initio in the next paper to verify the proper form of standing wave solution to the inhomogeneous equations.

Acknowledgements

One of us (FSL) thanks Professors F. Auluck and A. N. Mitra for their hospitality at University of Delhi in January and February, 1973, when most of this paper was written. He also thanks the National Science Foundation and the Council for Scientific and Industrial Research for providing the opportunity to visit Delhi University under the auspices of the India-United States Exchange of Scientists Program.

The other of us (DJK) thanks the University of Houston for a Faculty Development Leave and the Department of Chemistry at the University of Illinois for their hospitality and partial support of this work during the period Fall 1972. Helpful conversations with Professor Don Secrest are gratefully acknowledged. Partial support of this work by the Stiftung Volkswagenwerk is also gratefully acknowledged.

References

1. F. S. Levin, Ann. Phys. (NY) 72 (1972), 497.
2. W. Tobocman and M. A. Nagarajan, Phys. Rev. 163 (1967), 1011.
3. M. L. Goldberger and K. M. Watson, "Collision Theory", Wiley, New York, 1964.
4. R. G. Newton, "Scattering Theory of Waves and Particles", McGraw-Hill, New York, 1966.
5. L. S. Rodberg and R. M. Thaler, "Introduction to the Quantum Theory of Scattering", Academic Press, New York, 1967.
6. M. Gell-Mann and M. L. Goldberger, Phys. Rev. 91 (1953), 398.