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**EFFECTIVENESS OF REACTOR CONTROL
ABSORBERS SENSITIVE TO NEUTRONS
OF ALL ENERGIES**

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February 1, 1957

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A B S T R A C T

A criterion is established for the effectiveness of a reactor control absorber sensitive to neutrons of all energies from fission to thermal. This criterion can be used to determine an optimum alloy of absorber material, dependent on the absorption parameters of the elements and the slowing-down properties of the medium in which the absorber is used.

The analysis is based on multigroup diffusion theory. The effectiveness of an absorber is defined as the change in power output (as measured by thermal neutron content) produced by insertion of the absorber into an infinite uniform source medium having the same flux spectrum as the reactor in which the absorber is used. The same criterion can also be derived from perturbation theory.

A formula is developed for absorber effectiveness in terms of an equivalent diffusion length $\bar{L}$. It turns out that the effectiveness criterion for plate, cylinder, and sphere absorbers is basically different. For plate absorbers the effectiveness is independent of the thickness $R/\bar{L}$ of the plate but depends explicitly on the equivalent diffusion length. For this important case of cylindrical absorbers (control rods) the effectiveness depends on the size $R/\bar{L}$ of the absorber but is independent of the equivalent diffusion length. For spherical absorbers there is neither simplification and the effectiveness depends on both $R/\bar{L}$ and $\bar{L}$.

This effectiveness criterion describes the intermediate case between the limiting conditions of weak absorbers and black absorbers, which are known from simpler theory.

Introduction

The conventional analysis of control rod effectiveness is based on the assumption that the rod is black to thermal neutrons; this simplifying procedure contains two assumptions, (1) the control rod absorbs only thermal neutrons, (2) the rod is black, i.e., it absorbs all (thermal) neutrons entering it so that the flux is zero on its effective surface.

In this report this elementary theory will be quickly reviewed, for plane, cylindrical, and spherical absorbers, and then the theory will be extended to remove the two assumptions mentioned above.

The resultant effectiveness of an absorber control depends on three factors:

- a) The material composition of the rod
- b) The size and shape of the rod
- c) The medium in which the control rod is used

In the general situation mentioned above where the control rod is assumed to absorb neutrons over the complete energy spectrum from fission to thermal neutrons and is not necessarily black to neutrons at any energy level, it turns out that it is not possible to separate the effects due to these three factors, so that a criterion of control rod effectiveness can not be obtained which depends only on the composition of the rod and the properties of the medium in which it is used. For plate-type absorbers the size effect is not important and the effectiveness depends only on the number of absorber nuclei in the plate. For all types of absorbers the effectiveness is explicitly independent of the (slowing down properties) medium in the limiting case of weak absorbers.

Black Thermal Absorbers

In this section the conventional theory of black thermal absorbers will be briefly reviewed to examine its limitations and to prepare for a more general analysis. For completeness, plane, cylindrical, and spherical absorbers will be considered, since similar analytical developments are possible for these three cases. This analysis is based entirely on diffusion theory. More advanced theory, which accounts for the angular variation of neutron velocities, can be used to determine the equivalent parameters and effective boundaries.

The Plane Absorber

A plane absorber is shown (see Figure 1, page 31) immersed in a source medium of uniform strength "S". Before insertion of the absorber the neutron flux was everywhere equal to the uniform source flux; after insertion the neutron flux is zero at the surface of the black absorber,

and unaffected at large distances from the absorber. In particular the flux distribution is:

$$\phi = - S e^{-r/L} \quad (1.0)$$

where ϕ represents the flux measured from the level of the uniform source; the actual flux is $\phi' = \phi + S$. This flux distribution satisfies the conditions just mentioned; the diffusion length L enters through the differential equation or diffusion in the medium,

$$L^2 \nabla^2 \phi - \phi = 0 \quad (2.0)$$

The elementary absorptivity of the absorber is defined as the number of neutrons absorbed each second in a unit area of the absorber per unit source. This is represented by the shaded area in Figure 1, page 31. Thus,

$$\frac{-1}{S} \int \phi \, dv = \frac{2}{S} \int_0^\infty S e^{-r/L} \, dr = 2L \quad (3.0)$$

The interesting observation from this result is that the absorptivity of the plane absorber does not depend explicitly on the properties of the absorber material, but depends on the properties of the medium in which the absorber is used. This elementary theory makes it impossible to compare the effectiveness of absorbers made of different materials. Of course, the reason for this result is that the absorber has been assumed black, a perfect absorber, which is equivalent to using an infinite absorption cross section. The properties of this perfect absorber enter in formulating the boundary condition that the flux is zero at the surface of the absorber.

Before attempting more difficult analysis which will remove the limitations of the black thermal assumption, it is well to review the situation presented in Figure 1, page 31 for comparing control rod effectiveness. If even a small amount of absorber material is inserted into a critical reactor of finite size, the reactor will become subcritical and the flux will reduce to zero throughout the reactor, and thus this steady state situation cannot be used for comparing the effectiveness of various types of control rods. The situation given in Figure 1, page 31 has at least two interpretations. First, it can be considered as an ideal situation using a source medium setup for comparison of various types of absorbers. Second, it can be shown (see Appendix I, page 22) that a reactor perturbation theory leads to the same situation where the source S is the original flux at the point at which the absorber is inserted.

Furthermore, since the power output of a reactor depends on the number of fissions, which is proportional to the number of thermal neutrons by using the thermal neutron absorption cross section, the criterion in Equation (3.0) represents the power change produced by insertion of the absorber.

The Cylindrical Absorber

For this two-dimensional case of the previous situation (see Figure 2, page 31) the solution of the diffusion equation is

$$\phi = -S K_0 \left(\frac{r}{L} \right) / K_0 \left(\frac{R}{L} \right) \quad (4.0)$$

and the effectiveness of the absorber is

$$-\frac{1}{S} \phi \, dv = \frac{1}{S} \int_R^\infty S K_0 \left(\frac{r}{L} \right) / K_0 \left(\frac{R}{L} \right) 2\pi r \, dr \quad (5.0)$$

This can be evaluated by using the Bessel identities,

$$z K_{\gamma-1}(z) = -z K'_\gamma(z) - \gamma K_\gamma(z) \quad (6.0)$$

$$z K_{\gamma+1}(z) = -z K'_\gamma(z) + \gamma K_\gamma(z) \quad (6.1)$$

Finally, the result is

$$-\frac{1}{S} \int \phi \, dv = 2\pi L \cdot \left(\frac{R}{L} \right) \frac{K_1 \left(\frac{R}{L} \right)}{K_0 \left(\frac{R}{L} \right)} \cdot L \quad (7.0)$$

The Spherical Absorber

For this three-dimensional case of the previous situation (see Figure 3, page 31) the solution of the diffusion equation is

$$\phi = -S e^{-r/L} / r / L \bigg/ e^{-R/L} / R / L \quad (8.0)$$

and then

$$-\frac{1}{S} \int \phi \, dv = \frac{1}{S} \int_0^\infty \frac{e^{-r/L}}{e^{-R/L}} \frac{r/L}{R/L} 4\pi r^2 dr = 4\pi L^2 \cdot \left(\frac{R}{L}\right) \left(1 + \frac{R}{L}\right) \cdot L \quad (9.0)$$

Analytical Development of Solution of Diffusion Equation

In preparation for the more complex situation the formal analytical method for obtaining solutions to the diffusion equation for the 1, 2, and 3 dimensional absorbers just considered will be developed. In any case, the diffusion equation is

$$L^2 \nabla^2 \phi - \phi = A \delta(r) \quad (10.0)$$

where the absorber is represented by a plane, line, or point singularity. The strength of this negative source, or sink, is ΣA and is measured in neutrons per second (and per unit length in any direction in which the absorber extends indefinitely, as for line and plane absorbers). Temporarily this will be taken as a parameter but eventually this sink strength must be determined in terms of material properties of the absorber and the flux distribution in the medium.

The Fourier transformation is the general method for solving such equations; the n-dimensional direct and inverse Fourier transformations are

$$\begin{aligned} \bar{f}(\bar{\alpha}) &= \int e^{i\bar{\alpha} \cdot \bar{r}} d\bar{r} f(\bar{r}) \\ f(\bar{r}) &= \frac{1}{(2\pi)^n} \int e^{-i\bar{\alpha} \cdot \bar{r}} d\bar{\alpha} \bar{f}(\bar{\alpha}) \end{aligned} \quad (11.0)$$

Although these transforms are written in terms of vector notation, they do not necessarily involve any vector operations.

$$\bar{\alpha} \cdot \bar{r} = \alpha_1 r_1 + \alpha_2 r_2 + \dots + \alpha_n r_n$$

$$d\bar{r} = dr_1 dr_2 \dots dr_n$$

$$f(\bar{r}) = f(r_1, r_2, \dots, r_n)$$

For the diffusion equation with the absorber singularity the various direct transforms are

$$\overline{\phi(r)} = \int e^{i\vec{B} \cdot \vec{r}} d\vec{r} \quad \phi(r) = \tilde{\phi}(B)$$

$$\overline{\nabla^2 \phi(r)} = -B^2 \tilde{\phi}(B)$$

$$\overline{\delta(r)} = 1$$

The transformed equation,

$$-(1 + L^2 B^2) \tilde{\phi} = A$$

can then be solved algebraically for $\tilde{\phi}$

$$\tilde{\phi} = \frac{-A}{1 + L^2 B^2} \quad (12.0)$$

and then the flux ϕ can be found by means of the inverse transformation.

$$\phi = \frac{1}{(2\pi)^n} \int e^{-i \vec{B} \cdot \vec{r}} d\vec{B} \quad \frac{-A}{1 + L^2 B^2} \quad (13.0)$$

It can be shown that this integral introduces the various source (or sink) kernels for the 1, 2, and 3 dimensional absorbers. As mentioned in this equation the flux ϕ is measured from the value of the uniform source medium flux, and then the absorber can be considered as a negative source.

Multigroup Analysis

To account for the fact that all of the neutrons do not have the same energy, the multigroup energy analysis can be used (see Appendix II, page 26) for which the equations are

$$\begin{aligned} D_1 \nabla^2 \phi_1 - \Sigma_1 \phi_1 + k \Sigma_n \phi_n &= 0 \\ D_2 \nabla^2 \phi_2 - \Sigma_2 \phi_2 + \Sigma_1 \phi_1 &= 0 \\ &\vdots \\ D_n \nabla^2 \phi_n - \Sigma_n \phi_n + \Sigma_{n-1} \phi_{n-1} &= 0 \end{aligned} \quad (14.0)$$

In this system the various cross sections are slowing down cross sections, so that $\Sigma_1 \phi_1$ appear as both the number of neutrons which leave the i^{th} group and as a source term for neutrons reappearing in the $(i+1)^{\text{th}}$ group. In matrix form the transform of this system is

$$\begin{bmatrix} -(1+L_1^2 B^2) & 0 & k \frac{\Sigma_n}{\Sigma_1} \\ \frac{\Sigma_1}{\Sigma_2} & -(1+L_2^2 B^2) & 0 \\ 0 & \frac{\Sigma_2}{\Sigma_3} & -(1+L_3^2 B^2) \\ 0 & 0 & -(1+L_n^2 B^2) \end{bmatrix} \begin{bmatrix} \bar{\phi}_1 \\ \bar{\phi}_2 \\ \bar{\phi}_n \end{bmatrix} = 0 \quad (15.0)$$

where $L_i^2 = D_i / \Sigma_i$

or briefly the transformed system is

$$\bar{M} \bar{\phi} = 0 \quad (16.0)$$

for which the characteristic equation is

$$|\bar{M}| = 0 \quad (17.0)$$

This reduces to

$$k - (1+L_1^2 B^2)(1+L_2^2 B^2) - \dots - (1+L_n^2 B^2) = 0 \quad (18.0)$$

which can be solved for the reciprocal buckling values

$$(1+\bar{L}_1^2 B^2)(1+\bar{L}_2^2 B^2) - \dots - (1+\bar{L}_n^2 B^2) = 0 \quad (19.0)$$

Control Rod Effectiveness

For this system which accounts for the varying energy levels of the flux, it is not apparent what the criterion for the effectiveness of the absorber should be. To generalize the elementary criterion to account for an absorber which is effective at above-thermal energy levels, the

effectiveness of the absorber will again be measured by the change in power output which it produces. Since the number of fissions is proportional to the number of thermal neutrons, the effectiveness of the absorber is determined by the total change in the thermal neutron content. Neutrons can be absorbed during slowing down as well as after being thermalized, the overall result being to change the thermal neutron content and thus the power level.

The ultimate purpose of this report is to examine the conditions as to whether it may be more effective to absorb epithermal neutrons before waiting for them to slow down. Such an epithermal absorber provides an anticipatory control, in that the neutrons are absorbed before they can thermalize and the competition between thermal absorption and fission begins. In principle, after having the results of the analysis, it should be possible to determine an optimum composition of the absorber to produce a maximum power change.

Normally the absorption cross section of a material is much larger for slow (thermal) neutrons since these slow neutrons spend more time in the vicinity of the absorber nucleus. However, if the properties of the reactor-medium are such that there are a large number of neutrons in one of the higher energy groups, then it appears that this will compensate for the decreasing cross section and a given volume of composite absorber material may be more effective if part of the material is sensitive to above thermal neutrons. The conditions under which this intuitive result holds will be determined by the general analysis.

Solution of the Multigroup Equations

The multigroup equations which include absorption at all energy levels are

$$\begin{aligned}
 D_1 \nabla^2 \phi_1 - \Sigma_1 \phi_1 + k \Sigma_n \phi_n &= A_1 \delta(r) \\
 D_2 \nabla^2 \phi_2 - \Sigma_2 \phi_2 + \Sigma_1 \phi_1 &= A_2 \delta(r) \\
 - - - & \\
 D_n \nabla^2 \phi_n - \Sigma_n \phi_n + \Sigma_{n-1} \phi_{n-1} &= A_n \delta(r)
 \end{aligned}
 \tag{20.0}$$

These equations can be Fourier transformed much the same as in the elementary case

$$\begin{aligned}
 -(1+L_1^2 B^2) \bar{\phi}_1 + k \frac{\Sigma_n}{\Sigma_1} \bar{\phi}_n &= \frac{A_1}{\Sigma_1} \\
 -(1+L_2^2 B^2) \bar{\phi}_2 + \frac{\Sigma_1}{\Sigma_2} \bar{\phi} &= \frac{A_2}{\Sigma_2} \\
 &\dots \\
 -(1+L_n^2 B^2) \bar{\phi}_n + \frac{\Sigma_{n-1}}{\Sigma_n} \bar{\phi}_{n-1} &= \frac{A_n}{\Sigma_n}
 \end{aligned}
 \tag{21.0}$$

By successive substitution these equations can be solved algebraically for the transformed thermal flux.

$$\begin{aligned}
 k \Sigma_n \bar{\phi}_n &= A_1 + (1+L_1^2 B^2) \bar{\phi}_1 \\
 &= A_1 + (1+L_1^2 B^2) A_2 + (1+L_1^2 B^2) (1+L_2^2 B^2) \Sigma_2 \bar{\phi}_2 \\
 &= A_1 + (1+L_1^2 B^2) A_2 + (1+L_1^2 B^2) (1+L_2^2 B^2) A_3 + \dots \\
 &\quad + (1+L_1^2 B^2) (1+L_2^2 B^2) \dots (1+L_n^2 B^2) \Sigma_n \bar{\phi}_n \\
 [k - (1+L_1^2 B^2) (1+L_2^2 B^2) \dots (1+L_n^2 B^2)] \Sigma_n \bar{\phi}_n &= \sum_{j=1}^n A_j \prod_{i=1}^{j-1} (1+L_i^2 B^2)
 \end{aligned}
 \tag{22.0}$$

After solving the characteristic equation the explicit expression for the transformed flux is

$$(k-1) \Sigma_n \bar{\phi}_n = \frac{\sum_{j=1}^n A_j \prod_{i=1}^{j-1} (1+L_i^2 B^2)}{\prod_{i=1}^n (1+L_i^2 B^2)}
 \tag{23.0}$$

which is, briefly

$$(k-1) \sum_n \bar{\theta}_n = \sum_{j=1}^n \frac{f_j(B^2)}{F(B^2)} A_j \quad (24.0)$$

where

$$f_j(B^2) = \prod_{i=1}^{j-1} (1 + L_i^2 B^2) \quad F(B^2) = \prod_{i=1}^n (1 + \bar{L}_i^2 B^2)$$

To extend the method of solution from the elementary case this can be expanded into the form

$$\frac{f_j(B^2)}{F(B^2)} = \frac{N_{j1}}{1 + \bar{L}_1^2 B^2} + \frac{N_{j2}}{1 + \bar{L}_2^2 B^2} + \dots + \frac{N_{jn}}{1 + \bar{L}_n^2 B^2} \quad (25.0)$$

To determine the coefficients N_{jk} from the expression

$$f_j(B^2) = F(B^2) \sum_{k=1}^n \frac{N_{jk}}{1 + \bar{L}_k^2 B^2}$$

the limit is taken as $B^2 \rightarrow B_k^2$ where $1 + \bar{L}_k^2 B_k^2 = 0$.

Thus,

$$f_j(B_k^2) = N_{jk} \lim_{B^2 \rightarrow B_k^2} \frac{F(B^2)}{1 + \bar{L}_k^2 B^2} = N_{jk} \frac{F'(B_k)}{\bar{L}_k^2} = N_{jk} \prod_{i \neq k}^n (1 + \bar{L}_i^2 B_k^2) \quad (26.0)$$

After this preparation the transformed thermal flux becomes

$$(k-1) \sum_n \bar{\theta}_n = \sum_{j=1}^n A_j \sum_{k=1}^n e_{jk} \frac{1}{1 + \bar{L}_n^2 B^2} \left| e_{jk} = \frac{\prod_{i=1}^{j-1} (1 - L_i^2 / \bar{L}_k^2)}{\prod_{i \neq k}^n (1 - \bar{L}_i^2 / \bar{L}_k^2)} \right. \quad (27.0)$$

As in the elementary example the inverse Fourier transformation can be applied to each of the fractions in Equation (27.0) above

$$\frac{1}{1+L^2 B^2} \longrightarrow K\left(\frac{r}{L}\right)$$

For the 1, 2, and 3 dimensional cases these kernels are respectively,

$$\begin{aligned} K &= \frac{1}{2} e^{-r/L} \frac{1}{L} \\ K &= \frac{1}{2\pi L} K_0\left(\frac{r}{L}\right) \frac{1}{L} \\ K &= \frac{1}{4\pi L^2} e^{-r/L} \frac{1}{r/L} \end{aligned} \quad (28.0)$$

The expression for the total thermal flux is obtained by superposition

$$(k-1)\Sigma_n \phi_n = \sum_{j=1}^n \sum_{k=1}^n e_{jk} A_j K\left(\frac{r}{L_k}\right) \quad (29.0)$$

For this immediate application only the thermal flux is required; if needed other fluxes can be found by successive substitution into the multigroup equations.

The total effectiveness of the absorber can be found by volume integrating the thermal flux $\bar{\phi}_n = \int \phi_n dv$; this simplifies due to the property of unit kernels,

$$\int K\left(\frac{r}{L}\right) dv = 1 \quad (30.0)$$

and so the total change in thermal flux is:

$$(k-1)\Sigma_n \bar{\phi}_n = \sum_{j=1}^n \sum_{k=1}^n e_{jk} A_j = \sum_{j=1}^n e_j A_j \quad (31.0)$$

For one and two groups the weighting coefficients $\mathcal{C}_j$ are readily found to be

$$\begin{array}{lll} n = 1 & \mathcal{C}_1 = 1 & \\ n = 2 & \mathcal{C}_1 = 1 & \mathcal{C}_2 = 1 \end{array}$$

After these elementary tests the conjecture,

$$\mathcal{C}_j = \sum_{k=1}^n \mathcal{C}_{jk} = \sum_{k=1}^n \frac{\prod_{i=1}^{j-1} (1 - L_1^2 / \bar{L}_k^2)}{\prod_{i \neq k} (1 - \bar{L}_1^2 / \bar{L}_k^2)} = 1 \quad (32.0)$$

may be made. This result is valid; a formal proof is given in Appendix III. This result can also be shown to be an obvious result of neutron balance and can be demonstrated without explicitly solving the system of n partial differential equations of the multigroup analysis.

$$(k-1)\Sigma_n \bar{\phi} = \sum_{j=1}^n A_j \quad (33.0)$$

General Theory

Returning to the general multigroup equations,

$$D_1 \nabla^2 \phi_1 - \Sigma_1 \phi_1 + k \Sigma_n \phi_n = 0$$

$$D_2 \nabla^2 \phi_2 - \Sigma_2 \phi_2 + \Sigma_1 \phi_1 = 0$$

- - -

$$D_n \nabla^2 \phi_n - \Sigma_n \phi_n + \Sigma_{n-1} \phi_{n-1} = 0$$

where ϕ is measured from the uniform source medium flux, $\phi = \phi' - S$, and ϕ' is the actual flux; these equations will be applied to the region excluding the absorber and proper boundary conditions introduced at the surface of the absorber. Without solving the equations, considerable information can be obtained, particularly about the power-change produced by the absorber. By volume integrating these multigroup diffusion equations, the

large scale form of the multigroup equations can be obtained. After denoting $\bar{\phi}_1 = \int \phi_1 dv$ the equations become:

$$\begin{aligned}
 D_1 \int \nabla^2 \phi_1 dv - \Sigma_1 \bar{\phi}_1 + k \Sigma_{n \rightarrow n} \bar{\phi}_n &= 0 \\
 D_2 \int \nabla^2 \phi_2 dv - \Sigma_2 \bar{\phi}_2 + \Sigma_1 \bar{\phi}_1 &= 0 \\
 - - - \\
 D_n \int \nabla^2 \phi_n dv - \Sigma_n \bar{\phi}_n + \Sigma_{n-1} \bar{\phi}_{n-1} &= 0
 \end{aligned}
 \tag{35.0}$$

These equations can be expressed in terms of the neutron current through the surface of the absorber, since

$$D_1 \int \nabla^2 \phi_1 dv = \int \nabla \cdot D_1 \nabla \phi_1 dv = - \int \nabla \cdot J_1 dv = - \int J_1 \cdot ds
 \tag{36.0}$$

The final large scale form of the multigroup equations is

$$\begin{aligned}
 \int \bar{J}_1 \cdot ds + \Sigma \bar{\phi} - k \Sigma_{n \rightarrow n} \bar{\phi}_n &= 0 \\
 \int \bar{J}_2 \cdot ds + \Sigma \bar{\phi} - \Sigma_1 \bar{\phi}_1 &= 0 \\
 - - - \\
 \int \bar{J}_n \cdot ds + \Sigma_n \bar{\phi}_n - \Sigma_{n-1} \bar{\phi}_{n-1} &= 0
 \end{aligned}
 \tag{37.0}$$

Similarly for the absorber the large scale equations are

$$\begin{aligned}
 \int \bar{J}_1 \cdot ds - \Sigma_1^* \bar{\phi}'_1 &= 0 \\
 \int \bar{J}_2 \cdot ds - \Sigma_2^* \bar{\phi}'_2 &= 0 \\
 - - - \\
 \int \bar{J}_n \cdot ds - \Sigma_n^* \bar{\phi}'_n &= 0
 \end{aligned}
 \tag{38.0}$$

The large scale form of the equations for the absorber are simply statements that the net number of neutrons which flow into the absorber are absorbed. In this form the exact shape of the absorber material need not be specified although the most important cases are those where the absorber is a plane, cylinder, or sphere. The only leakage of neutrons from the medium is through the surface of the absorber and so the surface integrals in the two sets of equations are the same. Furthermore, for the cases considered, the volume of the absorber is small (in the preceeding analysis the absorber has been considered as approaching a singularity), and the flux can be taken as constant throughout the absorber. If this assumption has to be improved this can be done by modifying the absorption cross section to compensate for the depression of the flux in an absorber of finite size. By using this condition,

$$\sum_1^* \bar{\phi}'_1 = \sum_1^* \int \phi'_1 dv = \sum_1^* v \phi'_1 = N \sigma_1 v \phi'_1 = n \sigma_1 \phi'_1 \quad (39.0)$$

where $n=Nv$ is the total number of absorber nuclei in the absorber no matter what the shape. Then the large scale form of the equations for the absorber is,

$$\begin{aligned} \int \bar{J}_1 \cdot ds &= n \sigma_1 \phi'_1 \\ \int \bar{J}_2 \cdot ds &= n \sigma_2 \phi'_2 \\ \int \bar{J}_n \cdot ds &= n \sigma_n \phi'_n \end{aligned} \quad (40.0)$$

Power Change Produced by Absorber

The total change in power as measured by the total change in thermal neutron content can be obtained immediately by adding the Equations (36.0) together.

$$(k-1) \sum_n \bar{\phi} = \int \bar{J}_1 \cdot d\bar{s} + \int \bar{J}_2 \cdot d\bar{s} + \dots + \int \bar{J}_n \cdot d\bar{s} = \int \bar{J} \cdot d\bar{s} \quad (41.0)$$

The result which this equation provides is clear: the total change in the number of thermal neutrons is equal to the total number of neutrons absorbed. By combining the large scale equations for the medium and the absorber, the effectiveness "P" of the absorber, which is the power change produced by the absorber in the proper system of units, can be found from

$$P = -(k-1) \sum_n \bar{\phi} = \int \bar{J} \cdot d\bar{s} = \sum_1^n n \sigma_1 \phi'_1 \quad (42.0)$$

Solution for the Power Change

The exact solution of the system (Equation 37.0) and (Equation 40.0) for plane, cylinder, and sphere absorbers is essentially contained in the section titled "Control Rod Effectiveness", page 6, where the solution has been developed by using Fourier transform methods. For even a three or four group calculation the results are quite complicated, and the parameters involved are difficult to measure experimentally. It is known that the exact solution is a superposition of kernels $K(r/L_k)$ which decay at different rates; to obtain useable results it will be assumed that ϕ_i decays at a single effective rate $1/\bar{L}_i$. By substituting into the multi-group equations, it can then be found that each ϕ_i decays at the same rate, say, $1/\bar{L}$; this effective decay rate can be determined after solving Equation (20.0) or estimated from the buckling of the reactor in which the control absorber is used.

$$\phi_i = a_i K \frac{r}{L} \quad (43.0)$$

Thus, immediately,

$$\int \bar{J}_i \cdot d\bar{s} = \frac{D_i}{L} K' \left(\frac{R}{L} \right) s a_i \quad (44.0)$$

With this substitution the i^{th} absorber equation becomes

$$a_i \frac{D_i}{L} s K' = n \rho_i (S_i + a_i K) \quad (45.0)$$

By using this procedure it can be found that for all cases:

$$\bar{J} \cdot d\bar{s} = \sum_{i=1}^n \frac{S_i}{\frac{1}{n \rho_i} + \frac{1}{D_i \frac{G}{L}}} \quad (46.0)$$

where G is a factor depending on the geometry.

Evaluation of Power Change

In any case the large scale equations for the medium and absorber can be combined to determine the power change produced by the absorber.

$$P = -(k-1) \sum_n \phi_n = \int \vec{J} \cdot d\vec{s} = \sum_{i=1}^n n \sigma_i \phi_i' \quad (47.0)$$

To determine the coefficients a_i for the three cases of plane, cylindrical, and spherical absorbers each case must be considered separately. It will be found that in all three cases the effectiveness of the absorber is represented in the form:

$$P = \sum_{i=1}^n \frac{S_i}{\frac{1}{n\sigma_i} + \frac{1}{D_i} \frac{G}{\bar{L}}} \quad (48.0)$$

where G is a factor depending on the geometry, which is for the 1, 2, and 3 dimensional cases:

- 1) $G = 2$
 - 2) $G = 2 \pi \bar{L} \left(\frac{R}{\bar{L}} \right) K \left(\frac{R}{\bar{L}} \right) / K_0 \left(\frac{R}{\bar{L}} \right)$
 - 3) $G = 4 \pi \bar{L}^2 \left(\frac{R}{\bar{L}} \right) \left(1 + \frac{R}{\bar{L}} \right)$
- (49.0)

The Plane Absorber

For the case of a plane absorber the flux is:

$$\phi_i = a_i e^{-r/\bar{L}} \quad (50.0)$$

and the neutron current into the absorber is:

$$\int \vec{J}_i \cdot d\vec{s} = 2 \frac{D_i}{\bar{L}} e^{-R/\bar{L}} a_i \quad (51.0)$$

By applying Equation (47.0) the coefficient a_i can be determined and the effectiveness of the absorber calculated.

$$P = \sum_{i=1}^n \frac{S_i}{\frac{1}{n\sigma_i} + \frac{1}{D_i \frac{G}{L}}} \quad G = 2 \quad (52.0)$$

It should be noted that the thickness of the absorber as measured by R has dropped out so that the effectiveness of a plane absorber is independent of the thickness of the absorber and depends only on the total number of absorber nuclei. It will be found in the next two sections that this result does not hold for cylindrical and spherical absorbers; their effectiveness is dependent on the size as well as the number of absorber nuclei.

The Cylindrical Absorber

For this case of a cylindrical absorber the flux is:

$$\phi_1 = a_1 K_0 \left(\frac{R}{L} \right) \quad (53.0)$$

and the total neutron current into the absorber is:

$$\int \bar{J}_1 \cdot d\bar{s} = 2\pi \bar{L} \frac{D_1}{L} \left(\frac{R}{L} \right) K_1 \left(\frac{R}{L} \right) a_1 \quad (54.0)$$

By applying Equation (47.0) the coefficient a_1 can be determined and the effectiveness of the absorber calculated.

$$P = \sum_{i=1}^n \frac{S_i}{\frac{1}{n\sigma_i} + \frac{1}{D_i \frac{G}{L}}} \quad G = 2\pi \bar{L} \frac{\left(\frac{R}{L} \right) K_1 \left(\frac{R}{L} \right)}{K_0 \left(\frac{R}{L} \right)} \quad (55.0)$$

It should be observed that the effectiveness of the absorber not only depends on the total number of absorber nuclei but on the size of the cylindrical absorber. The factor G is a monotone increasing function of the radius so that the effectiveness of the rod increases as the geometrical size becomes larger. For small R/L this geometrical factor is a function which varies rapidly; thus it is generally important to know the effective radius of the rod accurately; for weak absorbers the $n\sigma_i$ effect predominates and the effective radius of the rod is less important in determining overall performance.

The Spherical Absorber

For this case of a spherical absorber the flux is:

$$\phi_1 = a_1 \frac{e^{-r/\bar{L}}}{r/\bar{L}} \quad (56.0)$$

and the total neutron current into the absorber is:

$$\int \bar{J}_1 \cdot d\bar{s} = 4\pi \bar{L}^2 \frac{D_1}{\bar{L}} \left(\frac{R}{\bar{L}} \right) \left(1 + \frac{R}{\bar{L}} \right) \frac{e^{-R/\bar{L}}}{R/\bar{L}} a_1 \quad (57.0)$$

By applying Equation (47.0) the coefficient a_1 can be determined and the effectiveness of the absorber calculated.

$$P = \sum_{i=1}^n \frac{S_i}{\frac{1}{n\sigma_1} + \frac{1}{D_1 \frac{G}{\bar{L}}}} \quad G = 4\pi \bar{L} \left(\frac{R}{\bar{L}} \right) \left(1 + \frac{R}{\bar{L}} \right) \quad (58.0)$$

Again in this case the geometrical factor depends on the size of the absorber, but for weak absorbers the $n\sigma_1$ effect predominates in determining the overall performance of the absorber control.

S U M M A R Y

Plane, Cylinder, and Sphere Absorbers

For all three cases the effectiveness of the absorber, which is not necessarily black, and which absorbs neutrons over the complete energy range is given by the relation:

$$P = \sum_{i=1}^n \frac{S_i}{\frac{1}{n\sigma_1} + \frac{1}{D_1 \frac{G}{\bar{L}}}} \quad (59.0)$$

The geometrical factors are

$$\begin{aligned}
 G &= 2G' & G &= 2\pi \bar{L} G' & G &= 4\pi \bar{L}^2 G' \\
 G' &= 1 & G' &= \frac{\left(\frac{R}{L}\right) K_1 \left(\frac{R}{L}\right)}{K_0 \frac{R}{L}} & G' &= \left(\frac{R}{L}\right) \left(1 + \frac{R}{L}\right)
 \end{aligned} \tag{60.0}$$

where G' is a dimensionless factor depending on the size (R/L) of the absorber; this dimensionless factor is presented in Figure 4, page 31; it can be seen that for $R/\bar{L} \approx 0.6$, $G' \approx 1$.

For strong and weak absorbers the effectiveness of the absorber can be simplified.

$$\text{Strong:} \quad P = \frac{G}{L} \sum_{i=1}^n D_i \tag{61.0}$$

$$\text{Weak:} \quad P = n \sum_{i=1}^n \sigma_i S_i \tag{61.1}$$

These final formulas based on n -group analysis can now be generalized as $n \rightarrow \infty$ to account for the continuous energy variation of the parameters.

$$P = \int \frac{S}{\frac{1}{n\sigma} + \frac{D}{L} \frac{G}{L}} dE \tag{62.0}$$

$$\text{Strong:} \quad P = \frac{G}{L} \int_0^\infty D dE \tag{63.0}$$

$$\text{Weak:} \quad P = n \int_0^\infty \sigma S dE$$

In conclusion, for the practically important case of a cylindrical absorber, the three effects due to 1) material properties of the absorber, 2) size of the control rod, and 3) properties of the medium in which the control rod is used, cannot be separated; except in limiting

cases. Furthermore, the effectiveness of a control rod depends strongly on the effective radius of the rod as well as the total number of absorber nuclei in the rod. The formula in Equation (62.0) accounts for all three of these effects for an absorber sensitive but not necessarily black to neutrons of all energy levels. In the limiting cases of strong and weak absorption simplified formulas can be used.

Application

For the case of weak absorption, the formula in Equation (63.1) for absorber effectiveness is intuitively obvious. It states that the most effective absorber is the one for which the number of absorptions ΣS is greatest. This criterion is explicitly independent of the slowing-down properties of the medium, and could have been given immediately even without setting an exact definition of absorber effectiveness. However, any absorption of neutrons disturbs the original distribution of neutrons $S(E)$, and if this absorption is not weak, the criterion for absorber effectiveness is not clear without the analysis.

For the other limiting case of strong absorption (Equation 63.0) the absorber effectiveness is independent of the absorption cross section, $\Sigma(E)$ and the flux distribution $S(E)$. For one-group theory $\bar{L} = L$ and the elementary results can be checked.

The general formula for absorber effectiveness is Equation (62.0) which depends on the energy-dependent parameters:

- Σ Equivalent cross section of composite absorber
- D Diffusion coefficient for medium in which absorber is used
- S Original flux at point which absorber is inserted

and on the energy-independent parameters:

- n Number of absorber nuclei
- L Equivalent diffusion length
- G Geometric factor which includes size and shape of absorber

To use formula (Equation 62.0) for absorber effectiveness, the composition of the absorber can be chosen and the cross section $\Sigma(E)$ found from the cross sections of the alloying elements by weighting these cross sections according to the composition and superposing. For the case of weak absorption, this Σ, E diagram can be multiplied by the S, E diagram and integrated to find the effectiveness. For the general case of intermediate absorption, a similar but more complicated combination can be made of the Σ, E ; D, E ; S, E diagrams to find the effectiveness. The

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material composition can be varied to find a composition which maximizes the effectiveness.

It turns out that the relations for the effectiveness of plane, cylinder and sphere absorbers have basically different features.

For the one-dimensional case of a plane absorber, the effectiveness depends on the number of absorber nuclei but is independent of the thickness of the plane. This can readily be seen from Figure 4, page 31, which shows that the geometrical effectiveness factor is independent of $R/\bar{L}$ for the one-dimensional case. However, the absorber effectiveness depends on the equivalent diffusion length, $\bar{L}$, which is estimated from multigroup analysis.

For the practically important two-dimensional case of a cylinder or rod absorber, the effectiveness, P , depends on the number of absorber nuclei and the size of the rod. However, there is simplification for this case in that $G/\bar{L}$ depends only on the $R/\bar{L}$ ratio. For example, if an $R/\bar{L}$ ratio is chosen, the effectiveness can be found without knowing L . For relatively weak absorption, the effectiveness is not too sensitive to the $R/\bar{L}$ ratio and an estimate can be made for this factor and the best absorber determined by varying the composition parameter, n .

For the three-dimensional case, neither of the previous two simplifications is available and both $R/\bar{L}$ and $\bar{L}$ must be known to find the effectiveness of the absorber.

APPENDIX I

PERTURBATION ANALYSIS

In this section a simplified perturbation analysis will be developed to determine the change in reactor buckling produced by the insertion of an absorber control. Before insertion of the absorber the flux is determined from:

$$\nabla^2 \phi + B^2 \phi = 0 \quad (1)$$

After insertion of the absorber the flux is determined from:

$$\nabla^2 \phi' + B'^2 \phi' = 0 \quad (2)$$

so that the respective bucklings are.

$$B^2 = - \frac{\nabla^2 \phi}{\phi} \quad B'^2 = - \frac{\nabla^2 \phi'}{\phi'}$$

By cross mixing these expressions for the bucklings can be imbedded in a volume integral

$$B^2 = - \frac{\int \phi' \nabla^2 \phi \, dv}{\int \phi \phi' \, dv} \quad B'^2 = - \frac{\int \phi \nabla^2 \phi' \, dv}{\int \phi \phi' \, dv} \quad (3)$$

The change in buckling is:

$$B^2 - B'^2 = \frac{\int [\phi \nabla^2 \phi' - \phi' \nabla^2 \phi] \, dv}{\int \phi \phi' \, dv} \quad (4)$$

The difference volume integral can be transformed into a surface integral:

$$\int [\phi \nabla^2 \phi' - \phi' \nabla^2 \phi] \, dv = \int [\phi \nabla \phi' - \phi' \nabla \phi] \cdot d\vec{s}$$

Furthermore, the surface integral can be reduced to one over just the surface of the absorber, if on the surface of the reactor

$$\phi, \phi' = 0$$

which holds for a bare reactor, or if

$$\phi - \phi' = 0$$

which holds approximately if the absorber is not inserted too near the surface of the reactor. The change in buckling becomes

$$\delta B^2 = \frac{\int [\phi \nabla \phi' - \phi' \nabla \phi] \cdot d\vec{s}}{\int \phi \phi' dv} \quad (5)$$

Furthermore, since the size of the absorber is small in comparison with the dimensions of the reactor,

$$\int \phi' \nabla \phi \cdot d\vec{s} = \phi' \int \nabla \phi \cdot d\vec{s} = 0$$

the expression for the change in buckling becomes:

$$\delta B = \frac{\phi \int \nabla \phi' \cdot d\vec{s}}{\int \phi \phi' dv} \quad (6)$$

The restrictions made in deriving Equation (6) have been mild, and this result is quite accurate. Since, the effect of the absorber is mainly local the volume integral is approximately independent of the perturbed flux and the change in buckling reduces to:

$$\delta B \propto \phi \int \nabla \phi' \cdot d\vec{s} \quad (7)$$

This integral is just the one which has been used in comparing the effectiveness of various types of control absorbers in the text; ϕ takes the part of the source term in that analysis.

Since the analysis in the text shows that the integral in Equation (7) depends linearly on the original flux, this result confirms the well known fact that the change in buckling produced by absorber control depends on the square of the original flux at the point at which the absorber is inserted.

$$\delta B \propto \phi^2$$

(8)

APPENDIX II

CRITIQUE OF THE MULTIGROUP METHOD

In the text the conventional use of the multigroup method has been made. In particular there are two simplifying aspects to this method which are generally used without comment but which can be clarified and justified. These two inherent conditions can be observed from the conventional multigroup equations:

$$D_1 \nabla^2 \phi_1 - \Sigma_1 \phi_1 + k \Sigma_n \phi_n = 0 \quad (1)$$

$$D_2 \nabla^2 \phi_2 - \Sigma_2 \phi_2 + \Sigma_1 \phi_1 = 0$$

- - -

$$D_n \nabla^2 \phi_n - \Sigma_n \phi_n + \Sigma_{n-1} \phi_{n-1} = 0$$

1) Only two neutron groups are used in each equation; this follows from the assumption that there is a gradual degradation of the neutrons from one energy group to the next lower, or equivalently the energy loss per collision is a small fraction of the neutron energy so that no neutrons skip over a group or groups in slowing down. If just a few groups are used this is not a severe restriction, and if the effect is important more terms can obviously be introduced into the multigroup equations to describe this effect at the expense of increased complexity.

2) From the i^{th} and $(i+1)^{\text{th}}$ group equations of the system,

$$D_i \nabla^2 \phi_i - \Sigma_i \phi_i + \Sigma_{i-1} \phi_{i-1} = 0 \quad (2)$$

$$D_{i+1} \nabla^2 \phi_{i+1} - \Sigma_{i+1} \phi_{i+1} + \Sigma_i \phi_i = 0$$

it is apparent that the absorption term in the i^{th} equation is the same as the source term in the $(i+1)^{\text{th}}$ equation; this means that all the neutrons which disappear from the i^{th} group reappear in the $(i+1)^{\text{th}}$ group; this implies that no neutrons are absorbed in slowing down, which is an obvious limitation, since the multigroup method is introduced to determine the resonance absorption during slowing down. Fortunately, this effect can be accounted for by using the form of Equation (1) with properly defined parameters.

2
1
2

Absorption during slowing down can be determined by using the system:

$$\begin{aligned}
 D_1 \nabla^2 \phi_1 - \Sigma_1 \phi_1 + p_n k' \Sigma_n \phi_n &= 0 \\
 D_2 \nabla^2 \phi_2 - \Sigma_2 \phi_2 + p_1 \Sigma_1 \phi_1 &= 0 \\
 - - - \\
 D_n \nabla^2 \phi_n - \Sigma_n \phi_n + p_{n-1} \Sigma_{n-1} \phi_{n-1} &= 0
 \end{aligned} \tag{3}$$

where p_i represents the resonance escape probability from the i^{th} to the $(i+1)^{\text{th}}$ group; i.e., it represents the percentage of the neutrons which degrade from the i^{th} to the $(i+1)^{\text{th}}$ group. By expanding the determinant of the transformed system the critical equation can be found.

$$p_1 p_2 - - - p_n k' - (1 + L_1^2 B^2)(1 + L_2^2 B^2) - - - (1 + L_n^2 B^2) = 0 \tag{4}$$

Since the definition of k includes the resonance escape probability $k = p_1 p_2 - - p_n k'$ it is apparent that the usual multigroup analysis based on Equation (1) provides the correct eigenvalues. However, the distribution of neutrons as determined by Equations (1) and (3) will be different.

The Equations (3) which include absorption during slowing down can be used to determine absorber effectiveness in much the same way as the analysis based on Equation (1). Thus, by weighting the equations with the factors

$$p'_1 = p_1 p_{1+1} - - - p_{n-1}$$

the system in Equation (1) becomes:

$$\begin{aligned}
 p'_1 D_1 \nabla^2 \phi_1 - p'_1 \Sigma_1 \phi_1 + k \Sigma_n \phi_n &= 0 \\
 p'_2 D_2 \nabla^2 \phi_2 - p'_2 \Sigma_2 \phi_2 + p'_1 \Sigma_1 \phi_1 &= 0 \\
 D_n \nabla^2 \phi_n - \Sigma_n \phi_n + p'_{n-1} \Sigma_{n-1} \phi_{n-1} &= 0
 \end{aligned} \tag{5}$$

and continuing the analysis similar to that presented for the case without explicit mention of absorption during slowing down in the medium, the final expression for control absorber effectiveness is:

$$(k-1)\Sigma_n \bar{\phi}_n = \sum_{i=1}^n p_i' \int \bar{J}_i \cdot d\bar{s} \quad (6)$$

APPENDIX III

THE $\mathcal{E}_j$ IDENTITY

In this section a brief formal proof will be given for the identity used in the text.

$$\mathcal{E}_j = \sum_{k=1}^n \frac{\prod_{i=1}^{j-1} \left(1 - \frac{L_i^2}{L_k^2}\right)}{\prod_{i \neq j} \left(1 - \frac{L_i^2}{L_k^2}\right)} = 1 \quad (1)$$

To prove this result abstractly appears quite formidable, but by using the relations from which it was derived the proof can be carried through directly. Analytically this is equivalent to introducing a parameter and manipulating in terms of that parameter to obtain the desired results. Originally,

$$(k-1) \sum_n \phi_n = \frac{\sum_{j=1}^n A_j \prod_{i=1}^{k-1} (1 + \frac{L_i^2}{L_k^2} B^2)}{\prod_{i=1}^n (1 + \frac{L_i^2}{L_k^2} B^2)} = \sum_{j=1}^n A_j \sum_{k=1}^n \mathcal{E}_{jk} \frac{1}{1 + \frac{L_k^2}{L_j^2} B^2} \quad (2)$$

The partial fraction expansion is an identity which holds for all values of B^2 ; in particular for $B^2 = 0$,

$$\sum_{j=1}^n A_j = \sum_{j=1}^n A_j \sum_{k=1}^n \mathcal{E}_{jk} = \sum_{j=1}^n \mathcal{E}_j A_j \quad (3)$$

$$\mathcal{E}_j = 1$$

Figure 1. The Plane Absorber

Figure 2. The Cylinder Absorber

Figure 3. The Sphere Absorber

Figure 4. Dimensionless Geometrical Effectiveness factor for (1) Plane, (2) Cylinder, (3) Sphere Absorbers.