AN APPLICATION OF CONFORMAL SYMMETRY
TO QUANTUM ELECTRODYNAMICS[†]

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ABSTRACT

Conformal symmetry is used to determine the position space vertex function $\Gamma_\lambda(x, y; z)$ of massless quantum electrodynamics in terms of the current-current correlation function $\pi_{\mu\nu}(x, y)$ and coefficients E_1 and E_2 appearing in the Wilson expansion of the product $\bar{\psi}(y) \psi(x)$ of Fermi fields. This relationship holds order by order in perturbation theory for graphs containing a single electron line, evaluated in a special position dependent gauge. The relationship provides an explicit and perhaps simplified method for computing the vacuum polarization amplitude $F_1(\alpha)$ and is used to analyze the behavior of single electron loop amplitudes in quantum electrodynamics for values of the coupling constant α near a zero of $F_1(\alpha)$.

1 Introduction

We study the short distance behavior of the connected, improper vertex function $\Gamma_\lambda(x, y; z)$ computed in quantum electrodynamics from graphs containing a single electron line. Examples of such graphs, with the position coordinates x, y and z labeled, are shown in figure 1. Throughout this paper we will work with amplitudes in position space¹ and achieve the short distance limit by setting the electron mass equal to zero.

It is demonstrated in Section II that such zero-mass Feynman amplitudes, corresponding to graphs containing a single electron line are conformally covariant² provided they are computed in a special gauge³. In this gauge the free photon propagator is given by

$$D_{\mu\nu}^B(x, y) = -\frac{1}{2} \frac{(x-y)^2}{(x-B)^2 (y-B)^2} \frac{\partial}{\partial x_\mu} \frac{\partial}{\partial y_\nu} \frac{(x-B)^2 (y-B)^2}{(x-y)^2} \\ + \left(\frac{1}{4} \kappa - 1\right) \frac{\partial}{\partial x_\mu} \frac{\partial}{\partial y_\nu} \log(x-y)^2$$

The new gauge parameters B_μ , $1 \leq \mu \leq 4$, must be transformed as the coordinates of a position four vector in order to realize conformal covariance. In the limit $\kappa = 0$, $B_\mu \rightarrow \infty$ the above propagator reduces to the usual Feynman gauge propagator. The parameter κ is chosen as that power series in α

$$\kappa = 1 - \frac{3\alpha}{8\pi} + \dots$$

which makes the electron wave function renormalization constant Z_2 finite⁴.

In Section III the vertex function $\Gamma_\lambda(x, y; z)$ is first studied in the generalized Landau gauge with photon propagator

$$D_{\mu\nu}(x-y) = \frac{\delta_{\mu\nu}}{(x-y)^2} + \frac{1}{4} \kappa \frac{\partial}{\partial x_\mu} \frac{\partial}{\partial y_\nu} \log(x-y)^2. \quad (3)$$

Conformal symmetry is then used to determine the dependence of the improper vertex function $\Gamma_\lambda^B(x, y; z)$, evaluated in the gauge (1), on the gauge parameters B_μ . It is found that the limit $B \rightarrow z$ is closely related to the limit $x \rightarrow y$.

The connection between these two limits determines the vertex function

$\Gamma_\lambda^B(x, y; z)$ computed in the gauge $B = z$ in terms of i) the current -

current correlation function $\pi_{\mu\nu}(x, y)$ computed from graphs containing a single electron loop and ii) two constants E_1 and E_2 appearing in the Wilson expansion of the operator product $\bar{\psi}(y)\psi(x)$. We find

$$\begin{aligned} 4\pi^4 \Gamma_\lambda^z(x, y; z) = & i \frac{\gamma \cdot (x-z) \gamma_\lambda \gamma \cdot (z-y)}{[(x-z)^2 (y-z)^2]^2} E_1 F_1 \\ & + \frac{i \gamma \cdot (x-y)}{(x-y)^2 (x-z)^2 (y-z)^2} \left[\frac{(x-z)_\lambda}{(x-z)^2} - \frac{(y-z)_\lambda}{(y-z)^2} \right] E_2 F_1 \end{aligned} \quad (4)$$

where F_1 is the single electron loop vacuum polarization amplitude⁵ given by

$$\begin{aligned} \pi_{\mu\nu}(x, y) = & \langle 0 | T(j_\mu(x) j_\nu(y)) | 0 \rangle_1 \\ = & \frac{1}{\pi^4} \left\{ \frac{\delta_{\mu\nu}}{[(x-y)^2]^3} - 2 \frac{(x-y)_\mu (x-y)_\nu}{[(x-y)^2]^4} \right\} F_1(a). \end{aligned} \quad (5)$$

The quantity $F_1(a)$ and, in particular, its zeros are of central importance in determining the asymptotic behavior of complete amplitudes in quantum electrodynamics^{6, 7}. Equation (4) may provide a useful method to compute $F_1(a)$ in

perturbation theory. It also implies that a zero, α_∞ , of $F_1(\alpha)$ must be either a zero of the vertex function $\Gamma_\lambda^z(x, y; z)$ or else a singular point for the coefficients $E_1(\alpha)$ and $E_2(\alpha)$ described above⁸.

In section IV we expand such an argument to show that a zero, α_∞ , of $F_1(\alpha)$ will also be a zero of the single electron loop 2n-current correlation function provided a number of amplitudes, including E_1 and E_2 , are regular at α_∞ . This result, an extension of the Federbush, Johnson, Jost, Schroer theorem⁹ to single electron loop amplitudes, can be combined with an argument of S. Adler⁷ to show directly that a zero of $F_1(\alpha)$ is also a point of essential singularity. However, the same assumptions, if adopted for amplitudes containing the axial vector current $j_\lambda^5(z)$, imply that the single electron loop contribution to the matrix element

$$M_{\mu, \nu, \lambda}(x, y, z) = \langle 0 | T(j_\mu^5(x) j_\nu(y) j_\lambda(z)) | 0 \rangle_1 \quad (6)$$

must vanish at α_∞ , in conflict with the non-renormalization theorem of Adler and Bardeen¹⁰. Thus the assumptions necessary to derive the above result are by no means certain and the relation (4) appears to be consistent with a number of alternatives for the behavior of single electron loop amplitudes as α approaches a root of $F_1(\alpha)$ ¹¹.

II Conformal Symmetry

Let us begin by describing the conformal transformation properties of Feynman amplitudes in quantum electrodynamics. Consider an n^{th} order amplitude $F_{\lambda}(x, y; z)_{\alpha\beta}$ represented by a particular Feynman graph contributing to the time ordered product

$$\langle 0 | T \{ \psi_{\alpha}(x) \bar{\psi}_{\beta}(y) j_{\lambda}(z) \} | 0 \rangle \quad (7)$$

where $\psi_{\alpha}(x)$ is the electron field and $j_{\lambda}(z)$ the usual electromagnetic current. For simplicity we restrict our attention to three point functions. (Our results can be immediately extended to amplitudes containing more external electron lines and external electromagnetic vertices.) Let $f_{\lambda}(x, y; z; v_1, \dots, v_n)_{\alpha\beta}$ be the corresponding amplitude before integration over the positions $v_1, \dots, v_n$ of the n internal vertices so that

$$F_{\lambda}(x, y; z)_{\alpha\beta} = \prod_{i=1}^n \left\{ \int d^4 v_i \right\} f_{\lambda}(x, y; z; v_1, \dots, v_n)_{\alpha\beta} . \quad (8)$$

Figure 1b shows the graph representing a particular amplitude $f_{\lambda}(x, y; z; v_1, v_2)_{\alpha\beta}$.

Under special conformal transformation a position vector x_{μ} goes into the new position x'_{μ} :

$$x_{\mu} \rightarrow x'_{\mu} = \frac{x_{\mu} + b_{\mu} x^2}{1 + 2 b \cdot x + b^2 x^2} \quad (9)$$

The electromagnetic field $A_{\mu}(x)$ should transform as a (covariant) vector field of dimension -1 at x :

$$A_{\mu}(x) \rightarrow A'_{\mu}(x') = \frac{\partial x_{\nu}}{\partial x'_{\mu}} A_{\nu}(x) . \quad (10)$$

Similarly the current $j_{\mu}(x)$ should transform like a vector field at x of dimension -3:

$$j_{\mu}(x) \rightarrow j'_{\mu}(x') = \sigma_b(x)^2 \frac{\partial x_{\nu}}{\partial x'_{\mu}} j_{\nu}(x) \quad (11)$$

where

$$\sigma_b(x) = \det \left\{ \frac{\partial x_{\nu}}{\partial x'_{\mu}} \right\}^{1/4} = 1 + 2 b \cdot x + b^2 x^2 . \quad (12)$$

Finally the spinor field $\psi_{\alpha}(x)$ must transform in such a way as to insure the validity of equation (11):

$$\psi(x) \rightarrow \psi'(x') = \sigma_b(x) [1 + \gamma \cdot b \gamma \cdot x] \psi(x) . \quad (13)$$

It is not difficult to verify that the usual, free, massless spinor propagator

$$S_F(x-y) = \frac{\gamma \cdot (x-y)}{[(x-y)^2]^2} \quad (14)$$

is invariant under special conformal transformation,

$$S_F(x'-y') = \sigma_b(x) (1 + \gamma \cdot b \gamma \cdot x) S_F(x-y) (1 + \gamma \cdot y \gamma \cdot b) \sigma_b(y) . \quad (15)$$

However, the free propagator $D_{\mu\nu}(x-y)$ of equation (3) is not left invariant by the conformal transformation (9) and (10) but instead becomes the new function

$$\begin{aligned} D_{\mu\nu}^B(x', y') &= \frac{\partial x_{\rho}}{\partial x'_{\mu}} \frac{\partial y_{\sigma}}{\partial y'_{\nu}} D_{\rho\sigma}(x-y) \\ &= \frac{\delta_{\mu\nu}}{(x'-y')^2} + \frac{1}{4} \kappa \frac{\partial}{\partial x'_{\mu}} \frac{\partial}{\partial y'_{\nu}} \log(x'-y')^2 \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{2} \frac{\partial}{\partial x'^{\mu}} \log(1 - 2b \cdot x' + b^2 x'^2) \frac{\partial}{\partial y'^{\nu}} \log(1 - 2b \cdot y' + b^2 y'^2) \\
& + \frac{1}{2} \frac{\partial}{\partial x'^{\mu}} \log(1 - 2b \cdot x' + b^2 x'^2) \frac{\partial}{\partial y'^{\nu}} \log(x' - y')^2 \\
& + \frac{1}{2} \frac{\partial}{\partial x'^{\mu}} \log(x' - y')^2 \frac{\partial}{\partial y'^{\nu}} \log(1 - 2b \cdot y' + b^2 y'^2)
\end{aligned} \tag{16}$$

which differs from $D_{\mu\nu}(x' - y')$ by gauge terms. If we define

$$B_{\mu} = \frac{b_{\mu}}{b^2} \tag{17}$$

then equation (16) can be shown equivalent to the original definition (1) of $D_{\mu\nu}^B$.

It is interesting to note that the resulting propagator $D_{\mu\nu}^B(x, y)$ is invariant under the full conformal group provided the four additional parameters B_{μ} are transformed as the coordinates of a third position vector. The evident scale and Lorentz symmetry of $D_{\mu\nu}^B(x, y)$ stems from the scale and Lorentz invariance of equation (9) when rewritten in terms of B_{μ} . Likewise the translation invariance of $D_{\mu\nu}^B(x - y)$, not at all suggested by the form of equation (9), results from the translation, Lorentz and scale invariance of the original propagator $D_{\mu\nu}(x - y)$ and the property of the conformal transformation (9) that if its generator is commuted with a generator of translations one obtains a linear combination of the generators of scale and Lorentz transformations. Finally the conformal symmetry manifest in equation (16) can be traced to equation (9) and the definition (17) of B_{μ} . It can be seen¹² from equation (9) that a special conformal transformation specified by b_{μ} , followed by a second transformation given by c_{μ} is equivalent to a single conformal transformation with the

parameters $b_\mu + c_\mu$. Thus, if the function $D_{\mu\nu}^B(x, y)$ is transformed a second time using the parameters c_μ the function $D_{\mu\nu}^{B'}(x', y')$ will result, where

$$B'_\mu = \frac{b_\mu + c_\mu}{(b_\mu + c_\mu)^2} = \frac{B_\mu + c_\mu B^2}{1 + 2c \cdot B + c^2 B^2}, \quad (18)$$

consistent with equation (9). We conclude that the scale, translation and Lorentz invariance of $D_{\mu\nu}(x - y)$ imply directly the invariance of the transformed function $D_{\mu\nu}^B(x, y)$ under the full conformal group.

Now consider the function f^B that results from applying the conformal transformation (9), (11) and (13) to the unintegrated Feynman amplitude f :

$$f_\lambda^B(x', y'; z'; v'_1, \dots, v'_n)_{\alpha'\beta'} = \left[\sigma_b(z)^2 \frac{\partial z}{\partial z'} \frac{\lambda}{\lambda} \right] \left[\sigma(x) (1 + \gamma \cdot b - \gamma \cdot x) \right]_{\alpha'a} \\ \times f_\lambda(x, y; z; v_1, \dots, v_n)_{\alpha\beta} \left[\sigma_b(y) (1 + \gamma \cdot y - \gamma \cdot b) \right]_{\beta\beta'} \prod_{i=1}^n \left[\sigma_b(v_i)^4 \right], \quad (19)$$

where the factor $\sigma_b(v_i)^4$ has been introduced for each internal vertex v_i to compensate for the omitted integrations. The resulting function f^B is not the same as the original function f ; however, equations (11), (15) and (16) imply that this function can be computed in a manner identical to that used to obtain f provided each free photon propagator $D_{\mu\nu}(x - y)$ is replaced by the new propagator $D_{\mu\nu}^B(x, y)$ of equation (1). Again, the arguments of the preceding paragraph imply that any function f^B obtained from a translation, Lorentz and scale invariant function f by equation (19) must itself be invariant under the full conformal

group provided the additional vector B_μ is appropriately transformed. Alternatively, one can argue that since f^B can be computed using conformally invariant Feynman rules, it must also exhibit these symmetries.

Finally we discuss the conformal transformation properties of the integrated amplitude $F_\lambda(x, y; z)_{\alpha\beta}$ of equation (8). This amplitude will possess conformal transformation properties completely analogous to those just demonstrated for the unintegrated amplitude f provided the required integrations in equation (8) are well defined. We therefore consider only amplitudes which do not contain self energy corrections on internal photon lines and work in the gauge (3) to insure the finiteness of electron wave function renormalization and vertex corrections. However, even with these precautions the integrations of equation (8) will be only conditionally convergent and are not well defined unless a particular regulation scheme is adopted. For example we might compute the untransformed amplitude F_{Λ^2} using the "Pauli-Villars" propagator

$$D_{\mu\nu}(x-y, \Lambda^2) = D_{\mu\nu}(x-y) R((x-y)^2 \Lambda^2) \quad (20)$$

where the regulating factor $R((x-y)^2 \Lambda^2)$ is chosen to vanish as $(x-y)^2 \rightarrow 0$ and to approach 1 as $\Lambda^2 \rightarrow \infty$.

If this regulated propagator is transformed following equations (9) and (10) the resulting propagator is

$$D_{\mu\nu}^B(x, y, \Lambda^2) = D_{\mu\nu}^B(x, y) R\left(\frac{(x-y)^2 \Lambda^2 B^4}{(x-B)^2 (y-B)^2}\right) \quad (21)$$

Application of a second special conformal transformation yields a photon propagator differing from equation (21) only by the presence of an additional position independent

factor in the argument of R . If the amplitude $F_{\Lambda^2}^B$, computed using the regulated propagator (20), is transformed according to equations (9), (11) and (13) the resulting amplitude $F_{\Lambda^2}^B$, although not identical to $F_{\Lambda^2}^B$, can be obtained from the same combination of electron and photon propagators provided the transformed photon propagator (21) is used. In the limit $\Lambda^2 \rightarrow \infty$ the regulating factor $R((v_i - v_j)^2 \Lambda^2 B^4 / (v_i - B)^2 (v_j - B)^2)$ appearing in the amplitude $F_{\Lambda^2}^B$ influences the integration over v_i and v_j only when v_i and v_j each approach a third vertex position v_k . However, in such a region the additional factor $\Lambda^2 B^4 / (v_i - B)^2 (v_j - B)^2$ can be treated as a constant¹³ so that the resulting integral is identical to that obtained using the propagator

$$D_{\mu\nu}^B(x, y) = R((x - y)^2 \Lambda^2). \quad (22)$$

Thus if we apply a special conformal transformation to the amplitude

$$F = \lim_{\Lambda^2 \rightarrow \infty} F_{\Lambda^2}^B \quad (23)$$

the resulting amplitude F^B can be obtained as the limit $\Lambda^2 \rightarrow \infty$ of an amplitude computed in the same manner as $F_{\Lambda^2}^B$ except that the transformed photon propagator (22) must be used. The amplitude F^B so defined is invariant under the full conformal group provided the gauge four-vector B_μ is appropriately transformed.

In summary, we have demonstrated that a) Feynman amplitudes computed¹⁴ in the gauge specified by the photon propagator $D_{\mu\nu}^B(x, y)$ will be conformally invariant if the gauge parameters B_μ are transformed as the coordinates of a position vector and b) the amplitude computed in this special gauge can be obtained

by first computing with the photon propagator $D_{\mu\nu}$ of equation (3) and then performing a special conformal transformation specified by the parameters

$b_\mu = B_\mu/B^2$. It is this last relationship that will be exploited in the remainder of the paper.

III Vertex Function

The behavior of quantum electrodynamics under conformal transformation studied in the preceding section will now be used to relate the electromagnetic vertex function, computed in a special gauge, with the current-current correlation function. Only amplitudes corresponding to graphs containing a single electron line will be considered. We begin with a general expression for the improper electromagnetic vertex function computed in the gauge (3).

$$\begin{aligned}
 4\pi^4 \Gamma_\lambda(x, y; z)_{\alpha\beta} &= 4\pi^4 \langle 0 | T \left\{ j_\lambda \psi_\alpha(x) \bar{\psi}_\beta(y) \right\} | 0 \rangle_1 \\
 &= \frac{i\gamma_\lambda(x-z) \gamma^\lambda \gamma(z-y)_{\alpha\beta}}{[(x-z)^2 (y-z)^2]^2} \left[G_1 + \frac{[(x-z)^2 (y-z)^2]^{1/4}}{\sqrt{(x-y)^2}} G_5 \right] \\
 &+ \frac{i\gamma_\lambda(x-y)_{\alpha\beta}}{(x-z)^2 (y-z)^2 (x-y)^2} \left[\frac{(x-z)_\lambda}{(x-z)^2} - \frac{(y-z)_\lambda}{(y-z)^2} \right] G_2 \\
 &+ \frac{i\gamma_\lambda(x+y-2z)_{\alpha\beta}}{[(x-z)^2 (y-z)^2]^{5/4}} \frac{1}{\sqrt{(x-y)^2}} \left[\frac{(x-z)_\lambda}{(x-z)^2} - \frac{(y-z)_\lambda}{(y-z)^2} \right] G_3 \\
 &+ \frac{i\gamma_\lambda(x-y)_{\alpha\beta}}{[(x-z)^2 (y-z)^2]^{5/4}} \frac{1}{\sqrt{(x-y)^2}} \left[\frac{(x-z)_\lambda}{(x-z)^2} + \frac{(y-z)_\lambda}{(y-z)^2} \right] G_4 \\
 &+ \frac{i\gamma_\lambda(x+y-2z)_{\alpha\beta}}{[(x-z)^2 (y-z)^2]^{5/4}} \frac{1}{\sqrt{(x-y)^2}} \left[\frac{(x-z)_\lambda}{(x-z)^2} + \frac{(y-z)_\lambda}{(y-z)^2} \right] G_5 \\
 &+ \frac{2i\gamma_{\alpha\beta}^\lambda}{[(x-z)^2 (y-z)^2]^{3/2}} \left[-\frac{1}{2} \frac{[(x-z)^2 (y-z)^2]^{1/4}}{\sqrt{(x-y)^2}} G_5 + G_6 \right] . \quad (24)
 \end{aligned}$$

The six form factors $G_i(u, v)$ are real, dimensionless functions of the two variables

$$u = \frac{(x-y)^2}{(x-z)^2} \quad \text{and} \quad v = \frac{(x-y)^2}{(y-z)^2} \quad (25)$$

Under the exchange of the arguments u and v the functions G_1, G_2, G_5 and G_6 are symmetric while G_3 and G_4 are anti-symmetric¹⁵.

If the improper, single electron line, vertex function (24) is conformally transformed following equations (9), (11) and (13) it becomes

$$\begin{aligned} 4\pi^4 \Gamma_\lambda^B(x, y; z)_{\alpha\beta} = & \frac{i\gamma_\alpha(x-z)\gamma^\nu\gamma_\beta(z-y)}{[(x-z)^2(y-z)^2]^2} \left\{ \delta^{\lambda\nu} \left[G_1 \right. \right. \\ & + \frac{[(x-B)^2(y-B)^2(x-z)^2(y-z)^2]^{1/4}}{\sqrt{(x-y)^2(z-B)^2}} G_5 \\ & + \frac{(x-z)^2(y-z)^2}{(x-y)^2} (X-Y)_\nu (X-Y)_\lambda G_2 \\ & + \frac{(x-z)^2(y-z)^2}{[(x-B)^2(y-B)^2(x-z)^2(y-z)^2]^{1/4}} \int \frac{(z-B)^2}{(x-y)^2} \left[-(X+Y)_\nu (X-Y)_\lambda G_3 \right. \\ & \left. \left. + (X-Y)_\nu (X+Y)_\lambda G_4 - \frac{1}{2} (X+Y)_\nu (X+Y)_\lambda G_5 \right] \right\} \\ & + \frac{i\gamma_\alpha(x-B)\gamma^\nu\gamma_\beta(B-y)}{[(x-z)^2(y-z)^2]^{3/2}} \frac{(z-B)^2}{\sqrt{(x-B)^2(y-B)^2}} \frac{\partial}{\partial B_\nu} \frac{\partial}{\partial z_\lambda} \log(z-B)^2 \left[-G_6 \right. \\ & \left. + \frac{[(x-z)^2(y-z)^2(x-B)^2(y-B)^2]^{1/4}}{2\sqrt{(x-y)^2(B-z)^2}} G_5 \right] \quad (26) \end{aligned}$$

where

$$X_\mu = \frac{1}{2} \frac{\partial}{\partial z_\mu} \log \left(\frac{(z - B)^2}{(z - x)^2} \right)$$

$$Y_\mu = \frac{1}{2} \frac{\partial}{\partial z_\mu} \log \left(\frac{(z - B)^2}{(z - y)^2} \right).$$

The form factors $G_i(u, v)$ in equation (26) are to be evaluated at the harmonic ratios

$$u = \frac{(x - y)^2 (z - B)^2}{(x - z)^2 (y - B)^2} \quad \text{and} \quad v = \frac{(x - y)^2 (z - B)^2}{(y - z)^2 (x - B)^2} \quad (27)$$

Although equation (26) can be obtained by direct conformal transformation of equation (24), it can be most easily derived by observing that a) the amplitude $\Gamma_{\alpha\beta}^B(x, y; z)_\lambda$ defined by equation (26) is the most general possible conformal invariant function of its four arguments and b) the expression (26) reduces to (24) in the limit $B_\mu \rightarrow \infty$ (or $b_\mu \rightarrow 0$). The results of Section II imply that the amplitude $\Gamma_{\alpha\beta}^B(x, y; z)_\lambda$ of equation (26) can be obtained by directly computing the vertex function using the photon propagator (1).

We now observe that if the gauge four vector B_μ in equation (26) is set equal to z_μ , the harmonic ratios u and v of equation (27) each become zero, considerably simplifying the form of the vertex function (26). This value of u and v also occurs if the positions x and y are equated in either equation (26) or equation (24). Therefore, let us first consider the limit x equals y for the vertex function (24). This can be conveniently analyzed by using a Wilson expansion¹⁶ for the short distance behavior of the product of fields

$$\psi_\alpha(x) \bar{\psi}_\beta(y):$$

$$\psi_\alpha(x) \bar{\psi}_\beta(y) \simeq \frac{+i}{4} (\gamma^\mu)_{\alpha\beta} j_\mu(x) \left\{ E_1 \delta_{\mu\nu} + E_2 \frac{(x-y)_\mu (x-y)_\nu}{(x-y)^2} \right\}$$

+ ...

(28)

where operators which do not contribute to the matrix element (24) and terms vanishing more rapidly as x approaches y have been dropped. The requirement that the amplitude (24) correspond to graphs containing a single electron line implies that the operators appearing in the expansion (28) must contain at least two electron fields. Lorentz invariance dictates that the operator must also be a Lorentz vector. The current $j_\mu(x)$ is the operator of highest dimension satisfying these requirements. The γ_5 -invariance of massless quantum electrodynamics implies that the four by four spinor matrix $\Gamma_\lambda(x, y; z)_{\alpha\beta}$ must anti-commute with γ_5 . This condition, when combined with parity conservation determines the dependence of $\psi_\alpha(x) \bar{\psi}_\beta(y)$ on α and β to be that shown in equation (28). Finally, the scale invariance of single electron line amplitudes in the gauge (3) implies that E_1 and E_2 are constants, independent of $(x-y)^2$. E_1 and E_2 can be computed in perturbation theory and through second order in α ($= \frac{e^2}{4\pi}$)

$$E_1 = 1 - \frac{\alpha}{8\pi}, \quad E_2 = \frac{\alpha}{2\pi}.$$

(29)

As has been emphasized by K. Johnson¹⁷, matrix elements of the current $j_\lambda(z)$ computed by regulating all internal photon lines, as in equation (20), and then letting the regulator mass Λ^2 approach infinity will in general differ from matrix elements obtained by separating the arguments of the spinor fields

$i\bar{\psi}(z + \frac{\epsilon}{2}) \gamma_{\mu} \psi(z - \frac{\epsilon}{2})$ appearing in the definition of the current and then letting $\epsilon \rightarrow 0$, after all regulator masses have been sent to infinity. Thus we expect

$$j_{\lambda}(z) \neq \lim_{\epsilon \rightarrow 0} \frac{1}{2\pi^2} \int d\Omega_{\epsilon} i\bar{\psi}(z + \frac{\epsilon}{2}) \gamma^{\lambda} \psi(z - \frac{\epsilon}{2}), \quad (30)$$

which from equation (28) can be written

$$j_{\lambda}(z) \neq (E_1 + \frac{1}{4}E_2) j_{\lambda}(z). \quad (31)$$

Matrix elements of the right and left hand sides of equation (30) should not differ to order α since, for a single loop diagram, the two regulating procedures are equivalent. In fact, from equation (28), we see that $E_1 + \frac{1}{4}E_2 = 1$ through order α . However, two loop diagrams, such as that shown in figure 1d, should contribute differently to the right and left hand sides of equation (30).

If the Wilson expansion (28) is substituted into the definition (24) of the vertex function and the limit $x \rightarrow y$ taken, the resulting equation relates the current-current correlation function (5) and the values $G_i(0, 0)$ of the form factors. We find that in the limit $x \rightarrow y$ with

$$\eta_{\mu} = \frac{(x-y)}{\sqrt{(x-y)^2}} \quad (32)$$

held fixed, equations (5), (24) and (28) reduce to

$$\begin{aligned} & \left[\gamma^{\lambda} - 2 \frac{(x-z)_{\lambda} (x-z) \cdot \gamma}{(x-z)^2} \right] E_1 F_1 + \eta \cdot \gamma \left[\eta_{\lambda} - 2 \frac{(x-z)_{\lambda} (x-z) \cdot \eta}{(x-z)^2} \right] E_2 F_1 \\ &= \left[\gamma^{\lambda} - 2 \frac{(x-z)_{\lambda} (x-z) \cdot \gamma}{(x-z)^2} \right] G_1 + \eta \cdot \gamma \left[\eta_{\lambda} - 2 \frac{(x-z)_{\lambda} (x-z) \cdot \eta}{(x-z)^2} \right] G_2 \end{aligned}$$

$$\begin{aligned}
& + 2 \frac{\gamma \cdot (x-z)}{\sqrt{(x-z)^2}} \left[\eta_\lambda - 2 \frac{(x-z)_\lambda (x-z) \cdot \eta}{(x-z)^2} \right] G_3 + 2 \gamma \cdot \eta \frac{(x-z)_\lambda}{\sqrt{(x-z)^2}} G_4 \\
& + \frac{1}{\sqrt{(x-z)^2}} \epsilon_{\lambda\mu\nu\sigma} \eta_\mu (x-z)_\nu \gamma_5 \gamma^\sigma G_5 + 2 \gamma^\lambda G_6
\end{aligned} \tag{33}$$

where all the form factors G_i are to be evaluated at the point $u = 0, v = 0$.

Equating the coefficients of the independent tensors in equation (33) yields

$$\begin{aligned}
G_1(0, 0) &= E_1 F_1, \quad G_2(0, 0) = E_2 F_1 \\
G_3(0, 0) &= G_4(0, 0) = G_5(0, 0) = G_6(0, 0) = 0
\end{aligned} \tag{34}$$

These relations can now be used to explicitly evaluate the vertex function

$\Gamma_\lambda^B(x, y; z)_{\alpha\beta}$ in the gauge $B = z$. If we take the limit $B_\mu \rightarrow z_\mu$ in the expression (26) holding the quantity

$$\xi_\mu = \frac{(z-B)_\mu}{\sqrt{(z-B)^2}} \tag{35}$$

fixed, we find

$$\begin{aligned}
4\pi^4 \lim_{B \rightarrow z} \Gamma_\lambda^B(x, y; z) &= \frac{i \gamma \cdot (x-z) \gamma^\lambda \gamma \cdot (z-y)}{[(x-z)^2 (y-z)^2]^{1/2}} G_1 + 2 G_6 \\
&- \left[\frac{(x-z)^2 (y-z)^2}{(x-y)^2} \left(\frac{\xi \cdot (z-x)}{(z-x)^2} + \frac{\xi \cdot (z-y)}{(z-y)^2} \right) G_5 \right] \\
&+ \frac{i \gamma \cdot (x-y)}{(x-z)^2 (y-z)^2 (x-y)^2} \left[\left(\frac{(x-z)_\lambda}{(x-z)^2} - \frac{(y-z)_\lambda}{(y-z)^2} \right) G_2 + 2 \frac{(x-z)^2 (y-z)^2}{(x-y)^2} \xi_\lambda G_4 \right]
\end{aligned}$$

$$\begin{aligned}
& - \frac{i \gamma \cdot (x-z) \gamma^\xi \gamma \cdot (z-y)}{[(x-z)^2 (y-z)^2]^{3/2} \sqrt{(x-y)^2}} \left[+ 2 \left(\frac{(x-z)_\lambda}{(x-z)^2} - \frac{(y-z)_\lambda}{(y-z)^2} \right) G_3 \right. \\
& + \left. \left(\frac{(x-z)_\lambda}{(x-z)^2} + \frac{(y-z)_\lambda}{(y-z)^2} \right) G_5 + 4 \sqrt{\frac{(x-y)^2}{(x-z)^2 (y-z)^2}} \xi_\lambda G_6 \right] \\
& + \frac{i}{[(x-z)^2 (y-z)^2]^{3/2} \sqrt{(x-y)^2}} \left[\gamma^\xi \gamma^\lambda \gamma \cdot (y-z) + \gamma \cdot (x-z) \gamma^\lambda \gamma^\xi \right. \\
& \left. + \gamma \cdot (2z-x-y) \xi_\lambda \right] G_5
\end{aligned} \tag{36}$$

where the form factors G_i are to be evaluated at the point $u = 0, v = 0$. If equation (34) is used to determine these quantities $G_i(0, 0)$, equation (36) becomes

$$\begin{aligned}
4\pi^4 \Gamma_\lambda^z(x, y; z) &= \frac{i \gamma \cdot (x-z) \gamma^\lambda \gamma \cdot (z-y)}{[(x-z)^2 (y-z)^2]^2} E_1 F_1 \\
&+ \frac{i \gamma \cdot (x-y)}{(x-y)^2 (y-z)^2 (z-y)^2} \left[\frac{(x-z)_\lambda}{(x-z)^2} - \frac{(y-z)_\lambda}{(y-z)^2} \right] E_2 F_1.
\end{aligned} \tag{4}$$

The simple form of equation (4) makes direct calculation of the improper vertex function in the gauge $B_\mu \rightarrow z_\mu$ quite straight forward. From the diagrams in figure 1a) and b), we find

$$G_1(0, 0) = E_1 F_1 = 1 + \frac{5a}{8\pi} \tag{37a}$$

$$G_2(0, 0) = E_2 F_1 = -\frac{a}{2\pi} + \tag{37b}$$

accurate to first order in α . Equations (29) and (37a) can be used to find F_1 to first order in α

$$F_1 = 1 + \frac{3\alpha}{4\pi}, \quad (38)$$

which is just the Jost-Luttinger¹⁸ result.

Thus when evaluated in the position dependent gauge $B_\mu = z_\mu$, the vertex function $\Gamma_\lambda^B(x, y; z)$ of quantum electrodynamics is determined in terms of the current-current correlation function $\pi_{\mu\nu}(x - y)$ and the coefficients E_1 and E_2 appearing in the Wilson expansion (28). This relationship, (4), holds order by order in perturbation theory for amplitudes corresponding to graphs containing a single electron line and has been verified explicitly in lowest non-trivial order.

IV Applications

Let us now consider two applications of the relation (4) between the electromagnetic vertex $\Gamma_{\lambda}^B(x, y; z)$ and the current-current correlation function $\pi_{\mu\nu}(x, y)$: first the use of equation (4) for explicit perturbation theory calculation of $F_1(\alpha)$ and second its non-perturbative implications for the interesting case that $\pi_{\mu\nu}(x, y)$ vanishes for a particular value of the coupling constant.

1. Calculation of $F_1(\alpha)$. The relation (4) determines the first n terms in a perturbation expansion of the vacuum polarization amplitude $f_1(\alpha) = \frac{\alpha}{3\pi} F_1(\alpha)$ in terms of the quantities $E_1(\alpha)$ and $G_1(\alpha)$ computed to order $n-1$ in α . Thus, as we saw in the previous section, the fourth order Jost-Luttinger contribution to $f_1(\alpha)$ can be obtained by computing $E_1(\alpha)$ and $G_1(\alpha)$ to second order. This explicit procedure for computing the perturbation expansion for $f_1(\alpha)$ in terms of lower order contributions to the electromagnetic vertex function is similar in many respects to a method developed by K. Johnson and co-workers¹⁹ some time ago.

2. Implications of a zero of $F_1(\alpha)$. As is well known, the existence of a zero of $F_1(\alpha)$ for real positive α affects significantly the asymptotic behavior of quantum electrodynamics when summed to all orders in perturbation theory^{6, 7}. We now study the consequences of such a zero which are implied by the relation (4)²⁰. As is clear from equation (4), if $F_1(\alpha)$ has a root α_{∞} ,

$$F(\alpha_{\infty}) = 0, \quad (39)$$

then either $E_1(\alpha_{\infty})$ or $E_2(\alpha_{\infty})$ is divergent or the vertex function $\Gamma_{\lambda}^Z(x, y; z)$ vanishes for $\alpha = \alpha_{\infty}$. The first possibility requires anomalous behavior of the

Wilson expansion of the product $\psi(x) \bar{\psi}(y)$ similar to that deduced by Adler, Callan, Gross and Jackiw²¹ for the product of two vector currents. We will not consider this alternative further.

The second possibility implies the vanishing at $\alpha = \alpha_\infty$ of all amplitudes in which the improper, single electron line vertex function $\Gamma_\lambda^z(x, y; z)$ appears. For example, consider a complete amplitude $F_\lambda^B(z)$ with an external electromagnetic vertex at z , evaluated in the gauge (1). (The dependence on the variables corresponding to all other external lines has been suppressed.) The perturbation theory sum defining $F_\lambda^B(z)$ can be so arranged that the amplitude is written as an integral of the product of $\Gamma_\lambda^B(x, y; z)_{\alpha\beta}$ and a second amplitude $H^B(x, y)_{\alpha\beta}$, represented as a sum of graphs which lack the external vertex at z but contain two more (amputated) external electron lines:

$$F_\lambda^B(z) = \int d^4x \int d^4y \Gamma_\lambda^B(x, y; z)_{\alpha\beta} H^B(y, x)_{\beta\alpha} . \quad (40)$$

Equation (40) is represented graphically in figure 2. The amplitude $H^B(y, x)_{\beta\alpha}$ contains no term corresponding to a graph which can be divided into two disjoint subgraphs by cutting two electron lines in which one subgraph contains the points x and y while all the other external lines enter the other subgraph. It is not difficult to prove that this two-particle irreducibility of $H(y, x)_{\beta\alpha}$ insures sufficient convergence of the integral over x and y , for x and y near z , that the limit $B \rightarrow z$ and the x and y integration are interchangeable.

Therefore

$$\begin{aligned} \lim_{B \rightarrow z} F_{\lambda}^B(z) &= \int d^4 x \int d^4 y \Gamma_{\lambda}^z(x, y; z)_{\alpha\beta} H^z(y, \alpha)_{\beta\alpha} \\ &= G_1(0, 0) H_{1, \lambda}(z) + G_2(0, 0) H_{2, \lambda}(z) \end{aligned} \quad (41)$$

where

$$\begin{aligned} H_{1, \lambda}(z) &= \frac{i}{4\pi^4} \int d^4 x \int d^4 y \frac{\gamma_{\alpha}(x-z) \gamma^{\lambda} \gamma_{\beta}(z-y)_{\alpha\beta} H^z(y, x)_{\beta\alpha}}{[(x-z)^2 (y-z)^2]^2} \\ H_{2, \lambda}(z) &= \frac{i}{4\pi^4} \int d^4 x \int d^4 y \frac{\gamma_{\alpha}(x-y)_{\alpha\beta} H^z(y, x)_{\beta\alpha}}{(x-z)^2 (z-y)^2 (y-x)^2} \\ &\quad \times \left[\frac{(x-z)_{\lambda}}{(x-z)^2} - \frac{(y-z)_{\lambda}}{(y-z)^2} \right]. \end{aligned} \quad (42)$$

The vanishing of the coefficients $G_1(0, 0)$ and $G_2(0, 0)$ appearing in $\Gamma_{\lambda}^z(x, y; z)$ for $\alpha = \alpha_{\infty}$ implies the vanish of $F_{\lambda}^z(z)$ for $\alpha = \alpha_{\infty}$ provided the integrals $H_{1, \lambda}(z)$ and $H_{2, \lambda}(z)$ multiplying $G_1(0, 0)$ and $G_2(0, 0)$ in equation (41) remain finite at $\alpha = \alpha_{\infty}$. Thus the assumption that the vanishing of $F_1(\alpha)$ at $\alpha = \alpha_{\infty}$ is not accompanied by the divergence of the quantities $E_1, E_2, H_{1, \lambda}, H_{2, \lambda}$ implies that the original single electron line amplitude $F_{\lambda}^B(z)$ must vanish in the gauge $B = z$ for $\alpha = \alpha_{\infty}$.

The somewhat unusual gauge requirement $B = z$ can of course be eliminated if we choose for $F_{\lambda}^B(z)$ the single electron loop amplitude

$$\pi_{\lambda_1, \lambda_2, \dots, \lambda_{2n}}(z_1, z_2, \dots, z_{2n}) = \langle 0 | T \{ j_{\lambda_1}(z_1) j_{\lambda_2}(z_2) \dots j_{\lambda_{2n}}(z_{2n}) \} | 0 \rangle_1 \quad (43)$$

which contains $2n$ current vertices and no external electron lines. Such an amplitude is gauge invariant and therefore independent of B . Thus the preceding argument implies that if the current-current correlation function $\pi_{\mu\nu}(x, y)$ vanishes for $\alpha = \alpha_\infty$ then the $2n$ current correlation function (43) will also vanish at $\alpha = \alpha_\infty$ provided E_1, E_2 and the appropriate coefficients $H_{1,\lambda}$ and $H_{2,\lambda}$ are finite at $\alpha = \alpha_\infty$. This connection²¹ between the vanishing of $\pi_{\mu\nu}(x, y)$ and the vanishing of $\pi_{\lambda_1, \dots, \lambda_{2n}}(z_1, \dots, z_{2n})$ is of particular interest because, if combined with the formula

$$\begin{aligned} \frac{\partial}{\partial \alpha} \pi_{\lambda_1, \dots, \lambda_{2n}}(z_1, \dots, z_{2n}) \\ = \frac{1}{8\pi^2} \int d^4 z_{2n+1} \int d^4 z_{2n+2} D_{\lambda_{2n+1}, \lambda_{2n+2}}(z_{2n+1} - z_{2n+2}) \\ \times \pi_{\lambda_1, \dots, \lambda_{2n+2}}(z_1, \dots, z_{2n+2}) \end{aligned} \quad (44)$$

of S. Adler⁷, it directly implies that if $F_1(\alpha_\infty) = 0$ then $\frac{\partial^n}{\partial \alpha^n} F_1(\alpha_\infty) = 0$ so that any root of $F_1(\alpha)$ must be a point of essential singularity for that function.

Finally let us apply the arguments of the preceding two paragraphs to matrix elements containing the axial vector current

$$j_\lambda^5(z) = i \bar{\psi}(z) \gamma_\lambda \gamma_5 \psi(z) ; \quad (45)$$

for example the single electron loop amplitude

$$M_{\mu, \nu, \lambda}(x, y; z) = \langle 0 | T \left\{ j_{\mu}^5(x) j_{\nu}(y) j_{\lambda}(z) \right\} | 0 \rangle_1. \quad (6)$$

Since this is a gauge invariant amplitude with at least one external electromagnetic vertex, the preceding assumptions imply that $M_{\mu, \nu, \lambda}(x, y; z)$ should vanish for $\alpha = \alpha_{\infty}$. However, Adler and Bardeen¹⁰ have shown that in our one electron loop approximation $M_{\mu, \nu, \lambda}(x, y; z)$ vanishes²² for each order of perturbation theory above the lowest. Thus they show that $M_{\mu, \nu, \lambda}(x, y; z)$ is independent of α , given by the two zeroth order graphs of figure 3, and therefore will not vanish at $\alpha = \alpha_{\infty}$. Unfortunately, there appear to be many possible resolutions to this conflict, among them:

- i) $F_1(\alpha)$ has no real positive root α_{∞}
- ii) A coefficient E_1 or E_2 appearing in the Wilson expansion (28) or an integral $H_{1, \lambda}$ or $H_{2, \lambda}$ in equation (42) becomes infinite at $\alpha = \alpha_{\infty}$.
- iii) The convergence of the perturbation series for $M_{\mu, \nu, \lambda}(x, y; z)$ is sufficiently poor that the order of summation implied by the gauge dependent factorization of the amplitude specified in equation (40) may define an amplitude different from that determined by the gauge invariant procedure necessary to obtain the Adler-Bardeen result²³.

It is quite possible, as with ii) and iii) above, that this conflict is caused by special properties of amplitudes containing the axial vector current j_{λ}^5 .

Thus the application of the relationship (4) to study the behavior of single electron loop amplitudes for values of α near a root of $F_1(\alpha)$ appears quite uncertain. If particular amplitudes are assumed regular at $\alpha = \alpha_{\infty}$,

then equation (4) implies the validity of an interesting variant of the Federbush, Johnson, Jost, Schroer theorem. However, the same assumptions, if adopted for amplitudes containing an axial vector current j_{λ}^5 , are in conflict with the non-renormalization theorem of Adler and Bardeen.

V Conclusion

The conformal symmetry of Quantum Electrodynamics has been used to derive a simple expression, equation (4), for the single electron line vertex function $\Gamma_{\lambda}^B(x, y; z)$, in the special gauge $B = z$, in terms of the single electron loop vacuum polarization amplitude $\pi_{\mu\nu}(x, y)$ and two constants, E_1 and E_2 , appearing in the Wilson expansion for the product $\psi(x) \bar{\psi}(y)$. This relationship has been shown valid order by order in perturbation theory and verified by explicit calculation through lowest, non-trivial order. In addition to its application to higher order perturbation theory calculation of the quantity $F_1(\alpha)$ entering the vacuum polarization amplitude, this relation may also be used to study the behavior of single electron loop amplitudes in quantum electrodynamics for values of α near a root of $F_1(\alpha)$. However, as was seen in Section IV, possible interesting conclusions following from such a non-perturbative application depend on further assumptions which are not generally valid.

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FOOTNOTES

1. We will also evaluate these amplitudes in the Euclidean region by rotating the time component of all position four vectors clockwise through ninety degrees in the complex plane. Consequently a position x will be determined by four real coordinates x_μ , $\mu = 1, 2, 3, 4$ and $x^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2$.
2. For a review of conformal symmetry and its application to field theory, see J. Wess, *Nuovo Cimento* 18, 1086 (1960); G. Mack and A. Salam, *Ann. Phys.* 53, 174 (1969); D. Gross and J. Wess, *PR* D2, 753 (1970).
3. R. A. Abdellatif, University of Washington thesis (unpublished). S. Adler, *PR* D6, 3445 (1972).
4. K. Johnson and B. Zumino, *PRL* 3, 351 (1959); K. Johnson, M. Baker and R. Wiley, *PR* 136B, 1111 (1964); S. Adler and W. Bardeen, *PR* D4, 3045 (1971).
5. The constant F_1 equals $\frac{3\pi}{\alpha}$ times $f_1(\alpha)$, the single electron loop vacuum polarization amplitude introduced by K. Johnson and M. Baker, *PR* D3, 2541 (1971).
6. M. Gell-Mann and F. Low, *PR* 95, 1300 (1954); K. Johnson, M. Baker and R. Wiley, *PR* 163, 1699 (1967); M. Baker and K. Johnson *PR* D3, 2541 (1971).
7. S. Adler, *PR* D5, 3021 (1972).
8. Singular behavior at $\alpha = \alpha_\infty$ for coefficients appearing in the Wilson expansion for the operator product $j_\mu(x) j_\nu(y)$ has been suggested by the work of S. Adler, C. Callan, D. Gross and R. Jackiw, *PR* D6, 2982 (1972).
9. P. Federbush and K. Johnson, *PR* 120, 1926 (1960); R. Jost, *Lectures on Field Theory and the Many Body Problem*, edited by E. Caramello, Academic Press, New York (1961);

B. Schroer, Diplomarbeit, Hamburg (unpublished, 1958).

10. S. Adler and W. Bardeen, PR 182, 1517 (1969).

11. For a related application of conformal symmetry to quantum electrodynamics, see H. Schnitzer, PR D8, p. 385 (1973).

12. It may be helpful to recognize that the transformation (9) can be realized by compounding an inversion $x_\mu \rightarrow x_\mu/x^2$ with the translation $x_\mu/x^2 \rightarrow x_\mu/x^2 + b_\mu$ and then with a final inversion $x_\mu/x^2 + b_\mu \rightarrow \boxed{x_\mu/x^2 + b_\mu} / (x_\mu/x^2 + b_\mu)^2 = x'_\mu$. The full conformal group contains translations, Lorentz transformations and scale transformations in addition to the transformation (9) and therefore is a fifteen parameter group.

13. The factor $\Lambda^2 B^4 / (v_i - B)^2 (v_j - B)^2$ can be treated as a constant when v_i and v_j approach the position of a third vertex v_k only if $v_k \neq B$. If the gauge four vector B_μ is distinct from the position z_μ of the external electromagnetic vertex, then the region $(v_k - B)^2 < \epsilon$ will make a vanishing contribution to the integral over v_k as ϵ tends to zero. Thus the limit $B_\mu \rightarrow z_\mu$ considered in Section III is a particularly delicate one and can be taken only after the limit $\Lambda^2 \rightarrow \infty$.

14. We now consider only amplitudes without photon self energy insertions and define any conditionally convergent Feynman integrals by softening the short distance behavior of all photon propagators as in equations (20) and (22).

15. In addition these functions must satisfy a set of coupled partial differential equations implied by current conservation. We will not make use of these constraints in this paper.

16. K. Wilson, Cornell Report No. LNS - 64-15, 1964 (unpublished); Phys. Rev. 179, 1499 (1969). R. Brandt, Ann. Phys. (N.Y.) 44, 221 (1967). W. Zimmermann in Lectures on Elementary Particles and Quantum Field Theory, edited by S. Deser et al. (MIT Press, Cambridge, Mass., 1971), Vol I, p. 397.
17. K. Johnson (private communication).
18. R. Jost and J. Luttinger, Helv. Phys. Acta 23, 201 (1950).
19. R. Abdelatif, op. cit.; K. Johnson (private communication).
20. It should be emphasized that equation (4) has been shown valid only order by order in perturbation theory in the limit of zero electron mass. Since the exchange of the perturbation theory sum and the zero mass limit has not been justified, sections II and III do not provide a rigorous justification for non-perturbative applications of the relation (4).
21. S. Adler, C. Callan, D. Gross and R. Jackiw, op. cit.
22. This relationship between the vanishing of the single electron loop amplitudes $\pi_{\mu\nu}(x, y)$ and $\pi_{\lambda_1, \dots, \lambda_{2n}}(z_1, \dots, z_{2n})$ is identical to the Federbush, Johnson, Jost, Schroer theorem (reference 9) obeyed by the complete amplitudes, including all numbers of electron loops.
23. Adler and Bardeen, PR, 182, 1517 (1969), show that higher order contributions to $M_{\mu\nu\lambda}(x, y; z)$ must obey all the appropriate Ward identities. Schrier, PR D3, 982 (1971), using conformal symmetry, shows that any gauge invariant set of higher order graphs corresponds to an amplitude proportional to the zeroth order amplitude, in the limit of vanishing electron mass. However, Adler, PR 177, 2426 (1969) shows that the zeroth order amplitude cannot satisfy all of these Ward

identities. Thus the above constant of proportionality must be zero and consequently all contributions to $M_{\mu\nu\lambda}(x, y; z)$ above zeroth order must vanish in the limit of vanishing electron mass.

24. The author thanks R. Crewther for suggesting this possibility.

FIGURE CAPTIONS

- Figure 1 Diagrams which contribute to the single electron line electromagnetic vertex $\Gamma_{\lambda}^B(x, y; z)$ computed in the generalized Landau gauge. The crosses on the Fermion lines represent the bare electromagnetic vertex γ_{μ} .
- Figure 2 A diagram illustrating the separation of a complete one electron line amplitude $F_{\lambda}^B(z)$ containing an external electromagnetic vertex at z into two factors: i) the improper vertex function $\Gamma_{\lambda}^B(x, y; z)$ and ii) a second two-electron irreducible amplitude $H^B(x, y)$.
- Figure 3 The two triangle diagrams which give the sole contribution to the single electron loop, zero mass amplitude $M_{\mu\nu\lambda}(x, y; z)$ when computed to arbitrary order in perturbation theory.

Figure 1

Figure 2

Figure 3