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**ELECTROMAGNETIC SELF-FORCES OF A CHARGED
PARTICLE BEAM INTERACTING WITH A CYLINDRICAL CAVITY**

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January 9, 1975

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ELECTROMAGNETIC SELF-FORCES OF A CHARGED PARTICLE BEAM INTERACTING WITH A CYLINDRICAL CAVITY

INTRODUCTION

Any investigation of methods for accelerating high-current (multi kA) beams of electrons should consider the electromagnetic self-fields of the beam. The time and spatial dependence of these fields is determined by the geometry of the accelerating structure through which the beam passes, as well as the time dependence of the beam current. Obviously, the electric field applied to accelerate the beam must be greater than the axial (longitudinal) electric field generated by the beam. For the geometry and beam current treated in this work, the electric self-field decelerates particles in the forward portion of the beam and accelerates particles in the backward portion. This effect leads to an energy spread within the beam, which may be undesirable. Any upper limit on the percentage of energy spread will place even more stringent requirements on the applied accelerating field.

In this work, we take the accelerating structure to be a series of cylindrical pill-box cavities of radius b and axial extent h . We investigate theoretically the longitudinal component of the electric self-field generated by a beam passing through one such box. In practice, there must be a hole in the box for the beam to pass through, but we neglect the presence of the hole. This approximation is discussed and justified in more detail in the next section. We do not treat the electrostatic field arising from the charge in the beam. In practice, this field is primarily in the radial direction, terminating near the edge of the hole, and has no appreciable effect on the longitudinal motion of the particles.

We treat a beam length sufficiently short that the beam passes through the box before any signal is reflected back to the beam from the outer wall at radius b . The beam length is sufficiently long that we may neglect axial variations in the beam current within the axial extent, h , of the box. These two conditions make possible the calculation presented here.

In the concluding section, we calculate the amount of energy radiated into the box by integrating the product of the beam current and the electric self-field over the time interval required for the beam to pass through the box.

ELECTRIC SELF-FIELD

Neglecting the axial (z) dependence of the beam current density in the box, the current density $j(r, t)$ in the z direction is given by

$$j(r, t) = \frac{I_0 \exp(-r^2/2a^2)}{2\pi a^2} \begin{cases} \sin vt, & 0 < t < \pi/v, \\ 0, & \text{otherwise.} \end{cases} \quad (1)$$

The length L of the pulse equals $c\pi/v$ for relativistic particles where c is the speed of light. As stated before, we assume that $L \gg h$ and $L \lesssim 2b$. Any expression for current vs time may be made up of a series of terms, each varying as $\sin nvt$. The method used here allows treatment of the fields generated by the n th component as long as the condition $L/n \gg h$ is satisfied. If this condition is violated, the axial variation of the current within the box must be considered.

The walls of the box may be taken to be perfectly conducting, since losses in the walls will be negligible for the short time considered here. We expand the vector potential $\bar{A}(\bar{r}, t)$ in terms of the vector potentials $\bar{A}_\ell(\bar{r})$ of the normal modes of the box.¹ We have

$$\bar{A}(\bar{r}, t) = \sum_{\ell} q_{\ell}(t) \bar{A}_{\ell}(\bar{r}). \quad (2)$$

The time-dependent coefficients q_{ℓ} obey the equation

$$\ddot{q}_{\ell} + \omega_{\ell}^2 q_{\ell} = \frac{1}{\epsilon_0} \int \bar{j}(\bar{r}, t) \cdot \bar{A}_{\ell}(\bar{r}) d\mathbf{v}, \quad (3)$$

where ω_{ℓ} is the eigenfrequency of the ℓ th mode. The integral in Eq. (3) extends over the volume of the box. Rationalized MKS units are used in this work. We need consider only those modes with no axial or azimuthal variation. For these modes, we have

$$\bar{A}_{\ell}(\bar{r}) = N_{\ell} J_0(x_{\ell} r/a) \hat{z}, \quad (4)$$

in which x_{ℓ} is the ℓ th root of J_0 and N_{ℓ} is a normalization factor given by

$$N_{\ell}^2 = [\pi h b^2 J_1^2(x_{\ell})]^{-1}. \quad (5)$$

The quantities ω_{ℓ} and x_{ℓ} are related by

$$\omega_{\ell} \equiv c x_{\ell} / b. \quad (6)$$

As we shall subsequently take the limit $b \rightarrow \infty$, we extend the integration over r in Eq. (3) to infinity. Inserting Eqs. (1) and (4) into Eq. (3) and performing the integration, we obtain

$$\ddot{q}_{\ell} + \omega_{\ell}^2 q_{\ell} = (I_0 N_{\ell} h / \epsilon_0) \sin vt \exp[-(x_{\ell} a)^2 / 2b^2]. \quad (7)$$

1. E. U. Condon, J. Appl. Phys. 12, 129 (1941). (This reference uses Gaussian units, but in accordance with present international policy, we have adapted the formalism to rationalized MKS units.)

We are excluding any externally applied electric or magnetic fields from the calculation, thus there are no fields present at $t = 0$, and the initial conditions for Eq. (7) are $q_\ell(0) = 0$ and $\dot{q}_\ell(0) = 0$. The desired solution is

$$q_\ell(t) = \frac{I_0 N_\ell h}{\omega_\ell \epsilon_0} Q_s \exp \left[-(x_\ell a)^2 / 2b^2 \right], \quad (8)$$

with

$$Q_s = \int_0^t \sin \nu t' \sin \omega_\ell (t - t') dt', \quad (9a)$$

$$= (\nu \sin \omega_\ell t - \omega_\ell \sin \nu t) / (\nu^2 - \omega_\ell^2). \quad (9b)$$

The electric field of the ℓ th mode is in the axial direction with magnitude given by

$$E_\ell(r, t) = -\dot{q}_\ell N_\ell J_0(x_\ell r/b), \quad (10)$$

and the total electric field generated by the beam is

$$E(r, t) = \sum_\ell E_\ell(r, t). \quad (11)$$

We now average the electric field over the current distribution, and introduce the spatially averaged electric field $\overline{E}(t)$ by the definition

$$\overline{E}(t) = \int j(r, t) E(r, t) dv / h I_0 \sin \nu t. \quad (12)$$

Using the above expressions, we obtain

$$\overline{E}(t) = -\frac{I_0 \nu}{\pi \epsilon_0 b^2} \sum_\ell \frac{(\cos \omega_\ell t - \cos \nu t)}{J_1^2(x_\ell)(\nu^2 - \omega_\ell^2)} \exp \left[-(x_\ell a/b)^2 \right]. \quad (13)$$

If the beam passes through the box in a time shorter than $2b/c$, we may evaluate the sum in Eq. (13) by taking the limit $b \rightarrow \infty$ and converting the sum into an integral.

This procedure is not strictly consistent with the Gaussian distribution of current we have used, but so long as the actual beam radius is very much less than the radius of the box, the treatment should be quite valid.

In order to convert the sum into an integral, we need to know the values of the roots, x_ℓ . Using the asymptotic expression for J_0 , namely

$$J_0(z) = (2/\pi z)^{1/2} \cos(z - \pi/4), \quad (14)$$

and see that the roots x_ℓ are given by

$$x_\ell = (2\ell + 1) \frac{\pi}{2} + \frac{\pi}{4} \quad (15)$$

with ℓ an integer. In the asymptotic limit, we have $J_1^2(x_\ell) = 2/\pi x_\ell$. We shall use ω_ℓ as the variable of integration. In the limit $b \rightarrow \infty$, we set $\omega_\ell = \ell c\pi/b$, drop the subscript ℓ , and write the following integral form for Eq. (13):

$$\overline{E(t)} = \frac{I_0 \nu}{2\pi \epsilon_0 c^2} \int_0^\infty \exp \left[-(\omega a/c)^2 \right] \frac{(\cos \nu t - \cos \omega t) \omega d\omega}{(\nu^2 - \omega^2)} . \quad (16)$$

In the limit $a \rightarrow 0$, this result may be obtained by a different technique. We consider an infinitely long current with time dependence $\sin \nu t$ and no z dependence. A straightforward integration of the retarded potential function yields Eq. (16) with $a = 0$. However, with $a = 0$, the integral in Eq. (16) does not converge, and for a beam with finite radius, the method becomes quite laborious. Even though laborious, the applicability of the method lends justification to an earlier approximation, namely, neglecting the presence of the hole in the box. Conducting plates at constant z do not affect the electric and magnetic fields of the current in this model, because the electric field is in the z direction and the magnetic field is in the azimuthal direction. Thus the result will be independent of the radius of any hole cut in such plates, and our results should be quite accurate if the accelerating structure consists of a series of closed cylinders with common ends. The results become somewhat less accurate when applied to a series of cylinders connected by a beam pipe of radius r_p ($a < r_p < b$) because of the boundary condition $E_z = 0$ at $r = r_p$ between the boxes.

In the integral in Eq. (16), we shall retain the exponential factor in the first term only. By simple manipulation, we obtain

$$\int_0^\infty \exp \left[-(\omega a/c)^2 \right] \frac{\omega d\omega}{\nu^2 - \omega^2} = -\frac{\exp \left[-(\nu a/c)^2 \right]}{2} \int_{-(\nu a/c)}^\infty e^{-y} \frac{dy}{y} , \quad (17)$$

in which the bar on the integral sign indicates the principal value. For small values of $\nu a/c$, corresponding to a beam with a radius very much less than its length, we may use the small argument approximation and write

$$\int_{-q}^\infty e^{-y} \frac{dy}{y} \approx -\ln(q) . \quad (18)$$

For $\nu a/c \ll 1$, we may set $\exp \left[-(\nu a/c)^2 \right] = 1$ in Eq. (17) and finally obtain

$$\int_0^\infty \exp \left[-(\omega a/c)^2 \right] \frac{\omega d\omega}{\nu^2 - \omega^2} = \ln(\nu a/c) . \quad (19)$$

We note that if we set the exponential factor in the integrand equal to unity and replace the upper limit by c/a , we obtain $-\frac{1}{2} \ln[(c/\nu a)^2 - 1]$, which is identical to the result in Eq. (20) in the limit $\nu a/c \ll 1$.

In the second term of the integrand in Eq. (16), we set $a = 0$ and obtain

$$\int_0^{\infty} \frac{\cos \omega t \omega d\omega}{(\nu^2 - \omega^2)^2} = \text{Ci}(\nu t) \cos \nu t + \text{Si}(\nu t) \sin \nu t, \quad (20)$$

in which Ci and Si are cosine and sine integrals, respectively, and are defined by the relations

$$\text{Ci}(y) = -\int_y^{\infty} \cos x \frac{dx}{x}, \quad (21a)$$

$$\text{Si}(y) = \int_0^y \sin x \frac{dx}{x}. \quad (21b)$$

We note that $\text{Ci}(\nu t)$ is logarithmically infinite at $t = 0$, so our neglect of the exponential factor in this term is not valid for very short times. However, we know that $\overline{E}(t)$ is zero for $t = 0$ on physical grounds as well as by inspection of Eq. (16).

Inserting Eqs. (19) and (20) into Eq. (16), we obtain

$$\overline{E}(t) = -\frac{I_0 \nu}{2\pi \epsilon_0 c^2} f(\nu t), \quad (22a)$$

in which

$$f(\nu t) = [\text{Ci}(\nu t) + \ln(c/\nu a)] \cos \nu t + \text{Si}(\nu t) \sin \nu t. \quad (22b)$$

The function $f(\nu t)$ is plotted over the interval $0 < \nu t < \pi$ in Fig. 1 for $\ln(c/\nu a) = 4.5$ ($L/a = 300$) and $\ln(c/\nu a) = 3$ ($L/a = 63$). Using the relation $\nu/c = \pi/L$, we have $\overline{E}(t) = -(I_0/2\pi\epsilon_0 c)(\pi/L)f(\nu t)$. Recalling that $(4\pi\epsilon_0)^{-1} = 9 \times 10^9$, we have $\overline{E}(t) = 60\pi I_0/L$, where $\overline{E}(t)$ is in volts/cm, I_0 in A, and L in cm. For $I_0 = 10^4$ A and $L = 300$ cm, we have $\overline{E}(t) = 2\pi f(\nu t)$ in kV/cm. Referring to Fig. 1, we see that for $L/a = 300$ ($a = 1$ cm), the decelerating electric field reaches a value of 8π kV/cm near the front of the beam and the accelerating electric field becomes 9π kV/cm at the tail of the beam. For $L/a = 63$ ($a = 4.75$ cm), the fields are respectively 5π kV/cm and 6π kV/cm.

Our calculation indicates that self-induced electric fields of the order of a few tens of kV/cm act to decelerate particles at the front of the beam and accelerate particles at the tail. Our results are slightly pessimistic in that the radial extent of the beam has been neglected in portions of the treatment. However, the calculation is more than slightly optimistic in that the current used is a smoothly varying function of time. Such a distribution is hard to achieve in practice. If we follow the above procedure to calculate the electric field from a current varying as $\sin n\nu t$, we arrive at a result given by Eqs. (22a) and (22b) with ν replaced by $n\nu$. The logarithmic variation of f with n is small compared to the factor n in Eq. (22a). Thus rapid

Fig. 1. Graph of the function $f(vt)$ as defined by Eq. (22b) for $\ln(c/va) = 3$ and 4.5.

variations of current will lead to larger (as well as far more complicated) induced electric fields.

ENERGY RADIATED

We now calculate the amount of energy per unit length, W , radiated by the beam. This is the energy left at the time the tail of the beam passes, (i. e., $\nu t = \pi$) and excludes that energy transferred from one group of particles to another within the beam. This calculation is valid if the electric field induced by the beam is the only electric field acting on the beam. If there is also an applied electric field, the fields add linearly but the energy does not. We have

$$W = -\frac{1}{2} \int_0^{\pi/\nu} \overline{E(t)} \sin \nu t \, dt. \quad (23)$$

It is convenient to employ Eq. (16) for $\overline{E(t)}$ rather than Eqs. (22a) and (22b). Inserting Eq. (16) into Eq. (23) and interchanging the order of integration, we see that the term in Eq. (16) containing $\cos \nu t$ does not contribute to the integral in Eq. (23).

The divergence that appears in the integral in Eq. (16) if $a = 0$ is a manifestation of the infinite magnetic energy associated with a beam of vanishingly small radius. When the beam has passed, this divergence is no longer a problem, and we may set $a = 0$ in the present calculation. We thus have

$$W = \frac{\mu_0 I_0^2}{2\pi} \int_0^{\pi/\nu} \sin \nu t \, dt \int_0^\infty \frac{\omega \cos \omega t \, d\omega}{(\nu^2 - \omega^2)}. \quad (24)$$

Interchanging the order of integration, performing the integration over t , and introducing the variable $u \equiv \omega/\nu$, we obtain

$$W = \frac{\mu_0 I_0^2}{2\pi} \int_0^\infty u \frac{(\cos \pi u + 1) \, du}{(u^2 - 1)^2}. \quad (25)$$

Integration by parts yields:

$$W = -\frac{\mu_0 I_0^2}{2\pi} \left\{ 1 + \frac{\pi}{2} \int_0^\infty \frac{\sin \pi u \, du}{u^2 - 1} \right\}. \quad (26)$$

The integral in Eq. (26) is equal to $-\text{Si}(\pi) = -1.852$. Inserting this value, we obtain

$$W = 3.8 \times 10^{-7} I_0^2 \text{ (J/m)}. \quad (27)$$

Notice that this quantity is independent of ν . For $I_0 = 10^4$ A, we find $W = 38$ J/m. To obtain the amount of energy radiated into a box of length h , multiply Eq. (27) by h . We introduce a time-average electric field $\langle E \rangle$ by the definition below. (We have long

since averaged the electric field over the cross-section of the current.)

$$\langle \bar{E} \rangle = -W \left[\int_0^{\pi/\nu} I_0 \sin \nu t dt \right]^{-1}, \quad (28a)$$

$$\begin{aligned} &= 1.9 \times 10^{-7} \nu I_0 \\ &= 57 \pi I_0 / L. \end{aligned} \quad (28b)$$

Again for $L = 300$ cm and $I_0 = 10^4$ A, we find that the time-average decelerating electric field is 6 kV/cm. Comparing with the results of the last section, we see that the time-average decelerating field is a factor of 3 or 4 times smaller than the peak electric field responsible for transferring energy from one group of particles to another within the beam.

If particles enter the accelerator with relativistic velocity, ($v \approx c$), they do not change their relative axial position within the beam. A particle will be accelerated or decelerated by the electric field $\bar{E}$ depending on its position in the beam. The maximum energy imparted to a beam particle after a length ℓ of accelerating structure is $e\ell(E_a + \bar{E}_{\max})$, where E_a is the applied accelerating field. The minimum energy is $e\ell(E_a + \bar{E}_{\min})$. The average energy is $e\ell(E_a + \langle \bar{E} \rangle)$. For $\bar{E}_{\min} < 0$, the fractional energy spread in the beam, F , is thus given by:

$$F = \frac{\bar{E}_{\max} - \bar{E}_{\min}}{E_a + \langle \bar{E} \rangle}. \quad (29)$$

We have shown that the numerator of this expression can be expected to be of the order of several tens of kV/cm. An applied electric field of several hundred kV/cm is therefore necessary to reduce the energy spread in the beam to a value of the order of 10%. Admittedly, this very intense electric field need be maintained only for a time interval of the order of 10 ns, but the technology of accomplishing this is not established.

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