RECEIVED BY DTIE SEP 22 1967

10

OAK RIDGE NATIONAL LABORATORY

operated by

UNION CARBIDE CORPORATION
NUCLEAR DIVISION

for the

U.S. ATOMIC ENERGY COMMISSION

ORNL - TM - 1977

MASTER

**EXACT EXPRESSIONS FOR THE COVARIANCES
BETWEEN PRODUCTS OF RANDOM VARIABLES**

John W. Wilkinson

NOTICE This document contains information of a preliminary nature and was prepared primarily for internal use at the Oak Ridge National Laboratory. It is subject to revision or correction and therefore does not represent a final report.

DISTRIBUTION OF THIS DOCUMENT IS UNLIMITED

DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency Thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

DISCLAIMER

Portions of this document may be illegible in electronic image products. Images are produced from the best available original document.

LEGAL NOTICE

This report was prepared as an account of Government sponsored work. Neither the United States, nor the Commission, nor any person acting on behalf of the Commission:

- A. Makes any warranty or representation, expressed or implied, with respect to the accuracy, completeness, or usefulness of the information contained in this report, or that the use of any information, apparatus, method, or process disclosed in this report may not infringe privately owned rights; or
- B. Assumes any liabilities with respect to the use of, or for damages resulting from the use of any information, apparatus, method, or process disclosed in this report.

As used in the above, "person acting on behalf of the Commission" includes any employee or contractor of the Commission, or employee of such contractor, to the extent that such employee or contractor of the Commission, or employee of such contractor prepares, disseminates, or provides access to, any information pursuant to his employment or contract with the Commission, or his employment with such contractor.

Contract No. W-7405-eng-26

COST PRICE

MATHEMATICS DIVISION

H.C. \$3.00; MN 65

EXACT EXPRESSIONS FOR THE COVARIANCES
BETWEEN PRODUCTS OF RANDOM VARIABLES

John W. Wilkinson

LEGAL NOTICE

This report was prepared as an account of Government sponsored work. Neither the United States, nor the Commission, nor any person acting on behalf of the Commission:

A. Makes any warranty or representation, expressed or implied, with respect to the accuracy, completeness, or usefulness of the information contained in this report, or that the use of any information, apparatus, method, or process disclosed in this report may not infringe privately owned rights; or

B. Assumes any liabilities with respect to the use of, or for damages resulting from the use of any information, apparatus, method, or process disclosed in this report.

As used in the above, "person acting on behalf of the Commission" includes any employee or contractor of the Commission, or employee of such contractor, to the extent that such employee or contractor of the Commission, or employee of such contractor prepared, disseminates, or provides access to, any information pursuant to his employment or contract with the Commission, or his employment with such contractor.

SEPTEMBER 1967

OAK RIDGE NATIONAL LABORATORY
Oak Ridge, Tennessee
operated by
UNION CARBIDE CORPORATION
for the
U. S. ATOMIC ENERGY COMMISSION

THIS PAGE
WAS INTENTIONALLY
LEFT BLANK

CONTENTS

	Page
Abstract	1
Introduction	1
The Case Where the Variables That Form the Products are Independent ..	2
The Case Where the Variables That Form the Products Need Not Be Independent	5
References	12

EXACT EXPRESSIONS FOR THE COVARIANCES
BETWEEN PRODUCTS OF RANDOM VARIABLES

John W. Wilkinson

ABSTRACT

In evaluating the variances of sums of products of random variables one needs the variances of products and the covariances between products of random variables. In this report, exact formulae are developed for these variances and covariances for products of k random variables. These expressions are compared with the usual approximate formulae.

1. Introduction

One often encounters sums of products of random variables in which the evaluation of the variances of such sums is of interest. In the expressions for the variances there are two types of terms; (i) variances of products, and (ii) covariances between products.

Discussion of (i) is handled in detail by Goodman in [1] and [2]. Using procedures and notation similar to his, we will devote attention to (ii). In a sense, (i) is a special case of (ii).

In Section 2 is discussed the case where the random variables that form the products are independent. For this case the exact formulae for the covariances are immediately usable since they involve only variances and covariances of the original random variables. However, for the case where the random variables forming the products need not be independent (discussed in Section 3) the exact covariance expressions are, in our opinion, (because of the involvement of higher order moments) of little practical use, except for gaining some appreciation of the character of the errors involved in using the approximate formulae.

2. The Case Where the Variables That Form
The Products Are Independent.

Let (x_{11}, x_{21}) and $x_{12}, x_{22})$ be two sets of two independent random variables. That is, x_{1i} is independent of x_{2i} , $i = 1, 2$. Nothing is required concerning any independences among other combinations. In keeping with the notation introduced by Goodman, [1], [2], let us make the following definitions:

$$\begin{aligned}
 E(x_{ji}) &= X_{ji}, \\
 G(x_{ji}) &= V(x_{ji})/X_{ji}^2, \\
 \Delta x_{ji} &= x_{ji} - X_{ji}, \\
 \delta x_{ji} &= \Delta x_{ji}/X_{ji}, \\
 D(x_{1i}, x_{2i}) &= E(\delta x_{1i})(\delta x_{2i}) = \frac{\text{cov}(x_{1i}, x_{2i})}{X_{1i} X_{2i}},
 \end{aligned} \tag{2.1}$$

where E denotes expectation and V denotes variance.

(For simplicity we shall assume that all expectations differ from zero.)

Since

$$\begin{aligned}
 (x_{1i}x_{2i} - X_{1i}X_{2i}) &= X_{1i}X_{2i}[(\delta x_{1i} + 1)(\delta x_{2i} + 1) - 1] \\
 &= X_{1i}X_{2i}[\delta x_{1i} + \delta x_{2i} + \delta x_{1i}\delta x_{2i}], \quad i = 1, 2,
 \end{aligned} \tag{2.2}$$

it follows that

$$\begin{aligned}
 \text{cov}(x_{11}x_{21}, x_{12}x_{22}) &= E(x_{11}x_{21} - X_{11}X_{21})(x_{12}x_{22} - X_{12}X_{22}) \\
 &= X_{11}X_{21}X_{12}X_{22} E(\delta x_{11} + \delta x_{21} + \delta x_{11}\delta x_{21})(\delta x_{12} \\
 &\quad + \delta x_{22} + \delta x_{12}\delta x_{22}).
 \end{aligned} \tag{2.3}$$

Because of the independence of x_{1i} and x_{2i} , this simplifies to

$$\begin{aligned}
 \text{cov}(x_{11}x_{21}, x_{12}x_{22}) &= X_{11}X_{21}X_{12}X_{22}\{D(x_{21}, x_{22}) \\
 &\quad + D(x_{11}, x_{12}) + D(x_{11}, x_{12}) D(x_{21}, x_{22})\} \\
 &= X_{11}X_{12} \text{cov}(x_{21}, x_{22}) + X_{21}X_{22} \text{cov}(x_{11}, x_{12}) \\
 &\quad + \text{cov}(x_{11}, x_{12})\text{cov}(x_{21}, x_{22}).
 \end{aligned} \tag{2.4}$$

The usual approximate formula for this situation is

$$\begin{aligned}
 \text{cov}(x_{11}x_{21}, x_{12}x_{22}) &\cong X_{11}X_{12} \text{cov}(x_{21}, x_{22}) \\
 &\quad + X_{21}X_{22} \text{cov}(x_{11}, x_{12}).
 \end{aligned} \tag{2.5}$$

The expressions for the variance of the product of two independent random variables can be obtained directly from (2.4) and (2.5) by dropping the second subscripts. Doing this yields

$$V(x_1x_2) = X_1^2 V(x_2) + X_2^2 V(x_1) + V(x_1)V(x_2), \text{ and} \tag{2.6}$$

$$V(x_1x_2) \cong X_1^2 V(x_2) + X_2^2 V(x_1). \tag{2.7}$$

For the case concerning the product of k random variables we have

$$\left(\prod_{j=1}^k x_{ji} - \prod_{j=1}^k X_{ji} \right) = \prod_{j=1}^k X_{ji} \left[\prod_{j=1}^k (\delta x_{ji} + 1) - 1 \right], \quad i = 1, 2, \tag{2.8}$$

from which it follows that

$$\begin{aligned}
\text{cov}\left(\sum_{j=1}^k x_{j1}, \sum_{j=1}^k x_{j2}\right) &= \left(\sum_{j=1}^k x_{j1} x_{j2}\right) \left[\sum_{j=1}^k D(x_{j1}, x_{j2})\right. \\
&+ \sum_{j_1 < j_2} D(x_{j_1 1}, x_{j_1 2}) D(x_{j_2 1}, x_{j_2 2}) \\
&+ \sum_{j_1 < j_2 < j_3} D(x_{j_1 1}, x_{j_1 2}) D(x_{j_2 1}, x_{j_2 2}) D(x_{j_3 1}, x_{j_3 2}) \\
&+ \dots \\
&\left. + D(x_{11}, x_{12}) D(x_{21}, x_{22}) \dots D(x_{k1}, x_{k2})\right],
\end{aligned} \tag{2.9}$$

where $\sum_{j_1 < j_2 < \dots < j_s}$ denotes summation over all values of $j_1 < j_2 < \dots < j_s$ ranging over $1, 2, \dots, k$.

Specifically for $k = 3$, (2.9) reduces to

$$\begin{aligned}
\text{cov}(x_{11} x_{21} x_{31}, x_{12} x_{22} x_{32}) &= x_{11} x_{21} x_{31} x_{12} x_{22} x_{32} [D(x_{11}, x_{12}) \\
&+ D(x_{21}, x_{22}) + D(x_{31}, x_{32}) + D(x_{11}, x_{12}) D(x_{21}, x_{22}) \\
&+ D(x_{11}, x_{12}) D(x_{31}, x_{32}) + D(x_{21}, x_{22}) D(x_{31}, x_{32}) \\
&+ D(x_{11}, x_{12}) D(x_{21}, x_{22}) D(x_{31}, x_{32})].
\end{aligned} \tag{2.10}$$

The usual approximate formulae for (2.9) and (2.10) are, respectively,

$$\text{cov}\left(\sum_{j=1}^k x_{j1}, \sum_{j=1}^k x_{j2}\right) \cong \left[\sum_{j=1}^k x_{j1} x_{j2}\right] \left[\sum_{j=1}^k D(x_{j1}, x_{j2})\right], \text{ and} \tag{2.11}$$

$$\begin{aligned}
\text{cov}(x_{11} x_{21} x_{31}, x_{12} x_{22} x_{32}) &\cong x_{11} x_{21} x_{31} x_{12} x_{22} x_{32} [D(x_{11}, x_{12}) \\
&+ D(x_{21}, x_{22}) + D(x_{31}, x_{32})].
\end{aligned} \tag{2.12}$$

By dropping the second subscripts in (2.9) - (2.12) we obtain the expressions (exact and approximate, respectively) for the variance of the product of k independent random variables. These are, in corresponding order,

$$\begin{aligned}
 V\left(\prod_{j=1}^k x_j\right) &= \left(\prod_{j=1}^k x_j^2\right) \left[\sum_{j=1}^k G(x_j) + \sum_{j_1 < j_2} G(x_{j_1})G(x_{j_2}) \right. \\
 &\quad + \sum_{j_1 < j_2 < j_3} G(x_{j_1})G(x_{j_2})G(x_{j_3}) \\
 &\quad + \dots \\
 &\quad \left. + G(x_1)G(x_2)\dots G(x_k) \right], \tag{2.13}
 \end{aligned}$$

$$\begin{aligned}
 V(x_1 x_2 x_3) &= x_1^2 x_2^2 x_3^2 [G(x_1) + G(x_2) + G(x_3) \\
 &\quad + G(x_1)G(x_2) + G(x_1)G(x_3) + G(x_2)G(x_3) \\
 &\quad + G(x_1)G(x_2)G(x_3)], \tag{2.14}
 \end{aligned}$$

$$V\left(\prod_{j=1}^k x_j\right) \cong \left(\prod_{j=1}^k x_j^2\right) \left(\sum_{j=1}^k G(x_j)\right), \text{ and} \tag{2.15}$$

$$V(x_1 x_2 x_3) \cong x_1^2 x_2^2 x_3^2 [G(x_1) + G(x_2) + G(x_3)]. \tag{2.16}$$

3. The Case Where the Variables That Form The Products Need Not Be Independent

Since

$$\begin{aligned}
 x_{1i} x_{2i} - E(x_{1i} x_{2i}) &= x_{1i} x_{2i} \left[(\delta x_{1i} + 1)(\delta x_{2i} + 1) - \frac{E(x_{1i} x_{2i})}{x_{1i} x_{2i}} \right] \\
 &= x_{1i} x_{2i} [\delta x_{1i} + \delta x_{2i} + \delta x_{1i} \delta x_{2i} - D(x_{1i}, x_{2i})], \tag{3.1}
 \end{aligned}$$

then

$$\begin{aligned}
 \text{cov}(x_{11}x_{21}, x_{12}x_{22}) &= E[x_{11}x_{21} - E(x_{11}x_{21})][x_{12}x_{22} - E(x_{12}x_{22})] \quad (3.2) \\
 &= X_{11}X_{21}X_{12}X_{22} E[\delta x_{11} + \delta x_{21} + \delta x_{11}\delta x_{21} \\
 &\quad - D(x_{11}, x_{21})][\delta x_{12} + \delta x_{22} + \delta x_{12}\delta x_{22} - D(x_{12}, x_{22})] \\
 &= X_{11}X_{21}X_{12}X_{22} [D(x_{11}, x_{12}) + D(x_{11}, x_{22}) \\
 &\quad + D(x_{21}, x_{12}) + D(x_{21}, x_{22}) + D(x_{11}, x_{12}, x_{22}) \\
 &\quad + D(x_{11}, x_{21}, x_{12}) + D(x_{21}, x_{12}, x_{22}) \\
 &\quad + D(x_{11}, x_{21}, x_{22}) + D(x_{11}, x_{21}, x_{12}, x_{22}) \\
 &\quad - D(x_{11}, x_{21})D(x_{12}, x_{22})],
 \end{aligned}$$

where

$$D(z_1, z_2, \dots, z_k) = E[(\delta z_1)(\delta z_2) \dots (\delta z_k)]. \quad (3.3)$$

The usual approximate formula is

$$\begin{aligned}
 \text{cov}(x_{11}x_{21}, x_{12}x_{22}) &\cong X_{11}X_{21}X_{12}X_{22} [D(x_{11}, x_{12}) + D(x_{11}, x_{22}) \\
 &\quad + D(x_{21}, x_{12}) + D(x_{21}, x_{22})]. \quad (3.4)
 \end{aligned}$$

The exact and approximate expressions for the variance of a product of two random variables can be obtained directly from (3.2) and (3.4), respectively, by dropping the second subscripts. Hence

$$\begin{aligned}
V(x_1 x_2) = & X_1^2 X_2^2 [G(x_1) + G(x_2) + 2D(x_1, x_2) + 2D(x_1, x_1, x_2) \\
& + 2D(x_1, x_2, x_2) + D(x_1, x_1, x_2, x_2) \\
& - \{D(x_1, x_2)\}^2], \text{ and}
\end{aligned} \quad (3.5)$$

$$V(x_1 x_2) \cong X_1^2 X_2^2 [G(x_1) + G(x_2) + 2D(x_1, x_2)]. \quad (3.6)$$

For the case concerning the product of k random variables we have

$$\left[\prod_{j=1}^k x_{ji} - E\left(\prod_{j=1}^k x_{ji}\right) \right] = \prod_{j=1}^k X_{ji} \left[\prod_{j=1}^k (\delta x_{ji} + 1) - \frac{E\left(\prod_{j=1}^k x_{ji}\right)}{\prod_{j=1}^k X_{ji}} \right], \quad (3.7)$$

$$i = 1, 2,$$

$$\begin{aligned}
= & \prod_{j=1}^k X_{ji} \left[\sum_{j=1}^k \delta x_{ji} + \sum_{j_1 < j_2} \delta x_{j_1 i} \delta x_{j_2 i} \right. \\
& + \sum_{j_1 < j_2 < j_3} \delta x_{j_1 i} \delta x_{j_2 i} \delta x_{j_3 i} + \dots \\
& \left. + \delta x_{1i} \delta x_{2i} \dots \delta x_{ki} - D(x_{1i}, x_{2i}, \dots, x_{ki}) \right].
\end{aligned}$$

Although Goodman in [2] has introduced a notation which permits a more compact presentation we have chosen not to use it because of the complexity of its introduction. Hence, we will proceed to obtain the covariance expression in terms of notation already established.

From the definition of covariance and (3.7) it follows that

$$\begin{aligned}
\text{cov}\left(\prod_{j=1}^k x_{j1}, \prod_{j=1}^k x_{j2}\right) = & \prod_{j=1}^k X_{j1} X_{j2} E\left[\prod_{j=1}^k \delta x_{j1} + \sum_{j_1 < j_2} \delta x_{j_1 1} \delta x_{j_2 1} \right. \\
& + \sum_{j_1 < j_2 < j_3} \delta x_{j_1 1} \delta x_{j_2 1} \delta x_{j_3 1} + \dots + \delta x_{11} \delta x_{21} \dots \delta x_{k1} - \\
& \left. \dots \right]
\end{aligned}$$

$$\begin{aligned}
& - D(x_{11}, x_{21}, \dots, x_{k1}) \left[\sum_{j=1}^k \delta x_{j2} + \sum_{j_1 < j_2} \delta x_{j_1 2} \delta x_{j_2 2} \right. \\
& + \sum_{j_1 < j_2 < j_3} \delta x_{j_1 2} \delta x_{j_2 2} \delta x_{j_3 2} + \dots + \delta x_{12} \delta x_{22} \dots \delta x_{k2} \\
& \left. - D(x_{12}, x_{22}, \dots, x_{k2}) \right] \\
= & \left(\sum_{j=1}^k x_{j1} x_{j2} \right) \left[\sum_{j=1}^k \left\{ \sum_{\ell=1}^k D(x_{j1}, x_{\ell 2}) \right. \right. \\
& + \sum_{j_1 < j_2} \left(D(x_{j1}, x_{j_1 2}, x_{j_2 2}) + D(x_{j_1 1}, x_{j_2 1}, x_{j_2 2}) \right) \\
& + \sum_{j_1 < j_2 < j_3} \left(D(x_{j1}, x_{j_1 2}, x_{j_2 2}, x_{j_3 2}) + D(x_{j_1 1}, x_{j_2 1}, x_{j_3 1}, x_{j_2 2}) \right) \\
& + \dots + D(x_{j1}, x_{12}, x_{22}, \dots, x_{k2}) + D(x_{11}, x_{21}, \dots, x_{k1}, x_{j2}) \} \\
& + \sum_{i_1 < i_2} \left\{ \sum_{j_1 < j_2} D(x_{i_1 1}, x_{i_2 1}, x_{j_1 2}, x_{j_2 2}) \right. \\
& + \sum_{j_1 < j_2 < j_3} \left(D(x_{i_1 1}, x_{i_2 1}, x_{j_1 2}, x_{j_2 2}, x_{j_3 2}) \right. \\
& + D(x_{j_1 1}, x_{j_2 1}, x_{j_3 1}, x_{i_1 2}, x_{i_2 2}) \left. \right) + \dots + D(x_{i_1 1}, x_{i_2 1}, x_{12}, x_{22}, \dots, x_{k2}) \\
& + D(x_{11}, x_{21}, \dots, x_{k1}, x_{i_1 2}, x_{i_2 2}) \} + \dots \\
& + D(x_{11}, x_{21}, \dots, x_{k1}, x_{12}, x_{22}, \dots, x_{k2}) \\
& - D(x_{12}, x_{22}, \dots, x_{k2}) \left\{ \sum_{j_1 < j_2} D(x_{j_1 1}, x_{j_2 1}) \right. \\
& + \sum_{j_1 < j_2 < j_3} D(x_{j_1 1}, x_{j_2 1}, x_{j_3 1}) + \dots + D(x_{11}, x_{21}, \dots, x_{k1}) \}
\end{aligned} \tag{3.8}$$

$$\begin{aligned}
& -D(x_{11}, x_{21}, \dots, x_{k1}) \left\{ \sum_{j_1 < j_2} D(x_{j_1 2}, x_{j_2 2}) + \sum_{j_1 < j_2 < j_3} D(x_{j_1 2}, x_{j_2 2}, x_{j_3 2}) \right. \\
& \left. + \dots + D(x_{j_1 2}, x_{j_2 2}, \dots, x_{k2}) \right\} + D(x_{11}, x_{21}, \dots, x_{k1}) D(x_{12}, x_{22}, \dots, x_{k2})]
\end{aligned}$$

The approximate formula is a delight to look at after the above exact formula. It is

$$\text{cov} \left(\sum_{j=1}^k x_{j1}, \sum_{j=1}^k x_{j2} \right) \cong \left(\sum_{j=1}^k x_{j1} x_{j2} \right) \left(\sum_{j=1}^k D(x_{j1}, x_{j2}) \right) \quad (3.9)$$

The corresponding expressions for the variances, obtained by dropping the second subscripts are, respectively,

$$\begin{aligned}
V \left(\sum_{j=1}^k x_j \right) &= \left(\sum_{j=1}^k x_j^2 \right) \left[\sum_{j=1}^k G(x_j) + 2 \sum_{j_1 < j_2} D(x_{j_1}, x_{j_2}) \right] \quad (3.10) \\
&+ 2 \sum_{j=1}^k \left\{ \sum_{j_1 < j_2} D(x_j, x_{j_1}, x_{j_2}) + \sum_{j_1 < j_2 < j_3} D(x_j, x_{j_1}, x_{j_2}, x_{j_3}) + \dots \right. \\
&+ D(x_j, x_1, x_2, \dots, x_k) \left. \right\} \\
&+ 2 \sum_{i_1 < i_2} \left\{ \frac{1}{2} \sum_{j_1 < j_2} D(x_{i_1}, x_{i_2}, x_{j_1}, x_{j_2}) + \sum_{j_1 < j_2 < j_3} D(x_{i_1}, x_{i_2}, x_{j_1}, x_{j_2}, x_{j_3}) \right. \\
&+ \dots + D(x_{i_1}, x_{i_2}, x_1, x_2, \dots, x_k) \left. \right\} \\
&+ \dots + D(x_1, x_2, \dots, x_k, x_1, x_2, \dots, x_k) \\
&- 2D(x_1, x_2, \dots, x_k) \left\{ \sum_{j_1 < j_2} D(x_{j_1}, x_{j_2}) + \sum_{j_1 < j_2 < j_3} D(x_{j_1}, x_{j_2}, x_{j_3}) \right. \\
&+ \dots + D(x_1, x_2, \dots, x_k) \left. \right\} \\
&+ \{D(x_1, x_2, \dots, x_k)\}^2, \text{ and}
\end{aligned}$$

$$V(\prod_{j=1}^k x_j) \cong (\prod_{j=1}^k x_j^2) [\sum_{j=1}^k G(x_j) + 2 \sum_{j_1 < j_2} D(x_{j_1}, x_{j_2})]. \quad (3.11)$$

One will note that in (3.10), all the terms with more than two entries for the arguments of the D's involve higher order moments of both sets of variables.

For example

$$\begin{aligned} D(x_1, x_2, x_1, x_2, x_3) &= E(\delta x_1)^2 (\delta x_2)^2 \delta(x_3) \\ &= \frac{E(x_1 - X_1)^2 (x_2 - X_2)^2 (x_3 - X_3)}{x_1^2 x_2^2 x_3}. \end{aligned}$$

It is for this reason that we made the earlier comment that the use of the exact formulae for variance and covariance of products is impractical when the variables forming the products are dependent.

To help to appreciate the form of (3.8), (3.9), (3.10) and (3.11) the rewriting of these expressions for $k = 3$ may help. The corresponding formulae become, respectively,

$$\begin{aligned} \text{cov}(x_{11}x_{21}x_{31}, x_{12}x_{22}x_{32}) &= (x_{11}x_{21}x_{31}x_{12}x_{22}x_{32}) [\sum_{j=1}^3 \{ \sum_{\ell=1}^3 D(x_{j1}, x_{\ell 2}) \} \quad (3.12) \\ &+ \sum_{j_1 < j_2} (D(x_{j_1 1}, x_{j_1 2}, x_{j_2 2}, x_{j_2 2}) + D(x_{j_1 1}, x_{j_2 1}, x_{j_2 2})) \\ &+ D(x_{j_1 1}, x_{j_2 1}, x_{j_2 2}, x_{j_2 2}) + D(x_{11}, x_{21}, x_{31}, x_{j_2 2}) \} \\ &+ \sum_{i_1 < i_2} \{ \sum_{j_1 < j_2} D(x_{i_1 1}, x_{i_2 1}, x_{j_1 2}, x_{j_2 2}) \\ &+ D(x_{i_1 1}, x_{i_2 1}, x_{j_2 2}, x_{j_2 2}) + D(x_{11}, x_{21}, x_{31}, x_{i_1 2}, x_{i_2 2}) \} + \end{aligned}$$

$$\begin{aligned}
& + D(x_{11}, x_{21}, x_{31}, x_{12}, x_{22}, x_{32}) \\
& - D(x_{11}, x_{21}, x_{31}) \{D(x_{12}, x_{22}) + D(x_{12}, x_{32}) + D(x_{22}, x_{32}) \\
& + D(x_{12}, x_{22}, x_{32})\} \\
& - D(x_{12}, x_{22}, x_{32}) \{D(x_{11}, x_{21}) + D(x_{11}, x_{31}) + D(x_{21}, x_{31}) \\
& + D(x_{11}, x_{21}, x_{31})\} \\
& + D(x_{11}, x_{21}, x_{31}) D(x_{12}, x_{22}, x_{32})],
\end{aligned}$$

$$\text{cov}(x_{11}x_{21}x_{31}, x_{12}x_{22}x_{32}) \cong (x_{11}x_{21}x_{31}x_{12}x_{22}x_{32}) \left[\sum_{j, \ell=1}^3 D(x_{j1}, x_{\ell 2}) \right], \quad (3.13)$$

$$V(x_1 x_2 x_3) = x_1^2 x_2^2 x_3^2 [G(x_1) + G(x_2) + G(x_3) + 2D(x_1, x_2) \quad (3.14)$$

$$+ 2D(x_1, x_3) + 2D(x_2, x_3) + 2 \sum_{j=1}^3 \sum_{j_1 < j_2} D(x_j, x_{j_1}, x_{j_2})$$

$$+ 4 \sum_{j=1}^3 D(x_j, x_1, x_2, x_3) + \sum_{j_1 < j_2} D(x_{j_1}, x_{j_1}, x_{j_2}, x_{j_2})$$

$$+ 2 \sum_{j_1 < j_2} D(x_{j_1}, x_{j_2}, x_1, x_2, x_3)$$

$$+ D(x_1, x_1, x_2, x_2, x_3, x_3)$$

$$- 2D(x_1, x_2, x_3) \{D(x_1, x_2) + D(x_1, x_3) + D(x_2, x_3)$$

$$+ D(x_1, x_2, x_3)\} + \{D(x_1, x_2, x_3)\}^2],$$

$$V(x_1 x_2 x_3) \cong (x_1 x_2 x_3)^2 [G(x_1) + G(x_2) + G(x_3) \quad (3.15)$$

$$+ 2D(x_1, x_2) + 2D(x_1, x_3) + 2D(x_2, x_3)].$$

REFERENCES

Goodman, L. A., "On the exact variance of products," JASA, 55 (1960), pp. 708-713.

Goodman, L. A., "The variance of the product of K random variables," JASA, 57 (1962), pp. 54-60.

INTERNAL DISTRIBUTION

- 1- 2. Central Research Library
- 3. Document Reference Section
- 4- 8. Laboratory Records
- 9. Laboratory Records - Record Copy
- 10- 24. Division of Technical Information
- 25. ORNL Patent Office
- 26. John J. Beauchamp
- 27. W. J. Blot
- 28- 37. D. A. Gardiner
- 38. David G. Gosslee
- 39. T. L. Hebble
- 40. A. S. Householder
- 41. M. A. Kastenbaum
- 42. Claudia S. Lever
- 43. W. E. Lever
- 44. T. J. Mitchell
- 45. V. R. R. Uppuluri
- 46- 70. John W. Wilkinson

EXTERNAL DISTRIBUTION

- 71. K. O. Bowman (ORGDP)
- 72. J. P. Reavis (Y-12)
- 73. Laboratory and University Div., CORO