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ENERGY TRANSFER BETWEEN CHARGED
PARTICLES BY COULOMB COLLISIONS

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ENERGY TRANSFER BETWEEN CHARGED PARTICLES BY COULOMB COLLISIONS

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ABSTRACT

The theory of energy transfer between charged particles in a fully ionized gas is reviewed, and calculations are made of the rate at which energy is transferred from electrons at a temperature T to cooler ions (taken to be at rest). The interaction between ion and electron is assumed to be of the Debye form $(Ze/r) \exp(-r/\lambda)$, where λ is the Debye shielding length.

It is suggested that future improvements of the theory of Coulomb interactions in ionized gases are likely to be made in two ways:

- i) Through employing an effective (shielded) potential derived on the basis of quantum (Fermi) statistics for the electrons.
- ii) Through improvements in the two-body approximation to the many-body problem.

I. INTRODUCTION

According to classical mechanics two charged particles will each move in hyperbolic, parabolic, or elliptic orbits relative to their center of mass depending upon whether their total (c.m.) energy is positive, zero, or negative, respectively. If the particles are of like sign or if the particles are initially infinitely far apart, the total energy is nonnegative, and the possibility of closed orbits is ruled out.

According to quantum theory the negative energy states of the system are quantized (discrete spectrum) whereas the positive energy states are not (continuum). In speaking of a Coulomb collision between two particles

we specifically refer to a system of positive total energy so that the classical orbits are hyperbolic and the total energy is not quantized. We shall assume that the two particles are initially infinitely distant so that the condition of positive energy is fulfilled regardless of the signs of the charged particles.

According to classical electrodynamics, the acceleration of two interacting charged particles will give rise to the emission of electromagnetic radiation. Correspondingly in quantum electrodynamics, the interaction can result in the creation of one or more photons which will have a continuous spectrum under the condition of positive total energy. This radiation is called "bremsstrahlung" or "deceleration radiation." In calculations of the Coulomb interaction, either classical or quantum-mechanical, the radiation field is ignored and only the static Coulomb field taken into account. The calculation is said to be made to "zero-order" in the radiation field.

In summary then, a Coulomb collision is a radiationless encounter between two charged particles with positive total energy. The possibility of such a radiationless encounter exists in the quantum theory but not in the classical theory. The situation is analogous to the existence in quantum theory, but not in classical theory, of stationary (radiationless) states of the electrons in an atom.

So far, we have spoken only of the Coulomb interaction between two particles. In this case the calculations can be carried out exactly and classical and quantum mechanics lead to identical results (in the nonrelativistic limit). However, if we are interested in the slowing-down of a fast proton in a fully ionized gas, for example, we must solve the problem of the Coulomb interaction between many particles. Here we are confronted with the notoriously intractable many-body problem, and we must be content to employ approximate methods of solution.

The simplest approximate treatment of the behavior of a given charged particle (test particle) as it moves through a fully ionized gas (field particles) would be to assume that it undergoes a binary (two-body) encounter with each field particle as though there were no other particles present. This scheme is inapplicable, however, and leads to divergent results.

The next simplest approach is to assume that the test particle undergoes an independent binary encounter with each field particle, but also to

take account of the average effect of the other particles on the binary encounter. The average effect at the other particles is to screen the Coulomb fields of the two particles from each other thus decreasing their interaction. This shielding effect may be accounted for approximately by assuming that the Coulomb interaction between two particles vanishes when the distance between the particles exceeds some specified value called the "shielding distance." An alternative and somewhat less drastic modification of the Coulomb field is to assume that the Coulomb potential is replaced by a Yukawa-type potential:

$$V_{\text{Coul}} = \frac{Ze}{r} \rightarrow V_{\text{shielded}} = \left(\frac{Ze}{r} \right) \exp(-r/\lambda),$$

where the parameter λ would represent the shielding distance. It is this scheme of treating the interaction of a charged particle with a fully ionized gas as a sum of independent binary encounters, in which the Coulomb interaction is modified by the presence of the other particles, that is commonly employed.

It should be stressed that the problem of the behavior of a charged particle in a fully ionized gas is fundamentally different from the problem of the interaction of particles with short-range forces such as billiard balls or particles with square-well potentials. The former is strictly a many-body problem in that the simultaneous interaction of many particles is an essential consideration, no matter how rarified the system. It is essential in that divergent results are obtained both classically and quantum-mechanically if the problem is treated as a two-body problem, so that many-body effects can hardly be regarded as a perturbation. On the other hand, the latter problem involving short-range forces can be quite accurately treated in terms of independent two-body encounters, provided only that the average distance between nearest neighbors is large compared with the (finite) range of the interaction.

II. THE CLASSICAL COULOMB COLLISION¹

Consider an encounter between two charged particles as shown below.

The distance b is the minimum distance of approach of the two particles in the absence of any interaction and is called the impact parameter. Inasmuch as we are neglecting radiation, both the angular momentum L and the total energy E of the system of particles are constants of the motion.

$$L = m_1 v b \quad (1)$$

$$E = (1/2)m_1 v^2 \quad (2)$$

Relations between the fractional energy change $\Delta E/E$ in the lab system, the scattering angle ψ in the lab system, the scattering angle θ in the c.m. system, and the impact parameter b of the incident particle are listed for reference below.

$$\frac{\Delta E}{E} = -\left(\frac{4\mu}{M}\right) \frac{B^2}{1+B^2} \quad (3)$$

$$= -\left(\frac{4\mu}{M}\right) \sin^2(\theta/2) \quad (4)$$

$$= -\left(\frac{2\mu}{M}\right) \left(1 + \epsilon \sin^2 \psi - \cos \psi \sqrt{1 - \epsilon \sin^2 \psi}\right) \quad (5)$$

$$\tan \psi = \frac{\sin \theta}{\cos \theta + \epsilon} = \frac{2B}{(1-B^2) + \epsilon(1+B^2)} \quad (6)$$

$$B = \frac{Z_1 Z_2 e^2}{\mu v^2 b} = \frac{V_b}{2T} = \tan(\theta/2) \quad (7)$$

$$\mu^{-1} = m_1^{-1} + m_2^{-1}, \quad M = m_1 + m_2, \quad \epsilon = m_1/m_2 \quad (8)$$

The quantity B is equal to one-half the ratio of the potential energy V_b of the two particles at a distance b apart to the initial kinetic energy T of the particles in the center-of-mass system.

In case the mass m_1 of the scattered particle is greater than or equal to that of the scattering particle m_2 , the laboratory scattering angle ψ cannot exceed the value:

$$\psi_{\max} = \sin^{-1}(m_2/m_1), \quad m_2 \leq m_1 \quad (9)$$

It follows from relation (7) that for $B > 1$ the scattering in the c.m. system will be in the backward direction. It will be convenient to define the backward scattering cross-section σ_B to be πb_1^2 where b_1 is that value of the impact parameter b for which $B = 1$. We have

$$\sigma_B = \pi \left(\frac{Z_1 Z_2 e^2}{\mu v^2} \right)^2 = \pi b^2 B^2 \quad (10)$$

In terms of the cross section σ_B , the differential scattering cross section (Rutherford cross section) is given by

$$\sigma(\theta) = \frac{\sigma_B}{4\pi} \left[\sin(\theta/2) \right]^{-4} \quad (11)$$

It will be worthwhile to illustrate the magnitude of the cross section σ_B by a specific example. Let us assume that the scattered particle is a 3.6 Mev α -particle and that the scattering particle is (i) a deuteron; and (ii) an electron. The cross section σ_B can be conveniently written

$$\sigma_B \approx \left(\frac{Z_1 Z_2 m_e c^2}{4T} \right)^2 \text{ barns,} \quad (12)$$

and we find that

$$\text{i) } \sigma_B = 4.5 \times 10^{-2} \text{ barns} \quad (\alpha \rightarrow d)$$

$$\text{ii) } \sigma_B = 2.7 \times 10^5 \text{ barns} \quad (\alpha \rightarrow e) .$$

For the fractional energy loss $(\Delta E/E)_H$ in a head-on collision we obtain

$$\text{i) } \left(\frac{\Delta E}{E} \right)_H = 8.9 \times 10^{-1} \quad (\alpha \rightarrow d)$$

$$\text{ii) } \left(\frac{\Delta E}{E} \right)_H = 5.5 \times 10^{-4} \quad (\alpha \rightarrow e) .$$

If we define the energy-loss cross section σ_{BE} for backward scattering as

$$\sigma_{BE} = \left(\frac{4\mu}{M} \right) \sigma_B = \left(\frac{\Delta E}{E} \right)_H \sigma_B , \quad (13)$$

we have

$$\text{i) } \sigma_{BE} = 4.0 \times 10^{-2} \text{ barns} \quad (\alpha \rightarrow d)$$

$$\text{ii) } \sigma_{BE} = 1.5 \times 10^2 \text{ barns} \quad (\alpha \rightarrow e) .$$

The average fractional energy loss in a backward scattering collision $(\Delta E/E)_B$ is approximately 69% of that in a head-on collision. Specifically, the relation is

$$\left(\frac{\Delta E}{E} \right)_B = (\ln 2) \left(\frac{\Delta E}{E} \right)_H \quad (14)$$

We see that the backward scattering cross section σ_B is small in the case of $(\alpha \rightarrow d)$ and enormous in the case $(\alpha \rightarrow e)$. We also note that even though the fractional energy loss is much smaller for scattering by an

electron than by a deuteron, the energy-loss cross section σ_{BE} for ($\alpha \rightarrow e$) is still nearly 4000 times as great as for ($\alpha \rightarrow d$).

In contrast with backward scattering encounters (i. e., close encounters), we have the following approximate relations for distant encounters ($\psi \ll 1$)

$$\psi \sim \frac{m_2}{M} \theta \sim \frac{V_b}{E} \quad (15)$$

$$\Delta E/E \sim - \frac{m_1}{m_2} \psi^2 \sim - \frac{\mu}{M} \theta^2. \quad (16)$$

Equations (4), (5), (6, left side), (9), (15, left side), and (16) are based on conservation of energy and momentum alone and are therefore independent of the nature of the interaction between the particles.

In Eqs. (1)-(16) above we have listed the properties of the classical two-body Coulomb collision that we shall need in subsequent discussion. Let us now consider the question of the slowing-down of a fast charged particle in a relatively "cool" fully ionized gas. That is, we shall assume that the velocity of the fast test particle v is large compared with the rms velocity of the field particles so that the latter can be assumed to be at rest. Assuming further that the charged particle slows down as the result of the totality of independent binary Coulomb encounters with the field particles, we have

$$\frac{dE}{ds} = 2\pi N_2 \int_0^R \Delta E_b b db \quad (17)$$

$$= - \left(\frac{4\mu}{M} \right) EN_2 \sigma_B \ln \left(1 + \frac{\pi R^2}{\sigma_B} \right), \quad (18)$$

where N_2 is the number of field particles per cm^3 and s is the arc-length along the trajectory of the test particle.

In obtaining the result (18) we have included only those binary encounters for which the impact parameter b is less than R . We see that if we let R tend to infinity the expression for dE/ds diverges logarithmically. Evidently, our assumption that the energy loss may be treated as a sum of independent binary Coulomb encounters is wrong.

The trouble is not hard to find. We see from Eqs. (3) and (7) that the energy loss in a binary encounter is independent of whether the two particles

are of like or opposite sign. However, in a many-body encounter the question of like or unlike sign enters in an essential way. For example, let us suppose that we have two charged field particles confined in a small volume δV and our test particle interacts with them with an impact parameter b which is large compared to the dimensions of the volume. If the two field particles are of like sign, the test particle will be deflected by the monopole field of the particles and the energy loss (for a distant encounter) will be approximately four times that for a single particle. On the other hand, if the particles are of opposite sign, the test particle will be deflected by a much weaker dipole field and the energy loss will be correspondingly less. Carrying this argument to the limit as the number of particles in δV increases without bound (being electrically neutral as a whole), we may expect the effective potential of a given field particle to asymptotically decrease faster than any finite inverse power of the distance away from it (infinite order multipole). We shall see in the next section that this is indeed the case.

It follows that, if we wish to retain the idea that the slowing-down of a fast particle can be treated as the result of independent binary encounters, we must give up the assumption that the interaction is a monopole Coulomb interaction. That is, we must replace the Coulomb potential by an appropriate modification to account for multipole or shielding effects.

III. THE DEBYE SHIELDING DISTANCE²

We wish to calculate the potential $\phi(r)$ due to a fixed charge (Z, e) located at the origin ($r = 0$) and immersed in a neutral sea of electrons and ions. A relation between the potential ϕ and the charge density ρ is given by the Poisson equation

$$\nabla^2 \phi = \frac{1}{r} \frac{d^2}{dr^2} (r\phi) = -4\pi\rho \quad (1)$$

If n^+ and n^- are the particle densities (No./cm³) of ions (charge Ze) and electrons, respectively, we have according to the Boltzmann principle

$$\begin{aligned} n^+ &= n^+(\infty) \exp(-Ze\phi/kT_e) \\ n^- &= n^-(\infty) \exp(e\phi/kT_e), \end{aligned} \quad (2)$$

so that a second relation between ϕ and ρ is

$$\begin{aligned}\rho &= \rho^+ - \rho^- = e(Zn^+ - n^-) \\ &= en_e \left[\exp(-Ze\phi/kT_e) - \exp(e\phi/kT_e) \right].\end{aligned}\quad (3)$$

Eliminating ρ between Eqs. (1) and (3) we obtain the equation

$$\frac{Z+1}{x} \frac{d^2}{dx^2} (xy) = \exp(y) - \exp(-Zy), \quad (4)$$

where

$$y = e\phi/kT_e, \quad x = r/\lambda \quad (5)$$

$$\lambda = \sqrt{\frac{kT_e}{4\pi e^2 n_e (Z+1)}}. \quad (6)$$

We look for solutions of (4) that vanish at $r = \infty$. For r sufficiently large so that $Zy \ll 1$, Eq. (4) becomes, approximately

$$\frac{d^2}{dx^2} (xy) \approx xy, \quad Zy \ll 1 \quad (7)$$

The solutions of (7) that vanish at ∞ are

$$y = \frac{c}{x} e^{-x},$$

so that we obtain

$$\phi(r) \sim \frac{c}{r} e^{-r/\lambda}, \quad (8)$$

where c denotes an arbitrary constant. Equation (8) gives the form of the potential for large r . For small r , the potential must reduce to that of the fixed charge ($Z_1 e$) at the origin. Hence, the potential must be of the form

$$\phi(r) = \left[\frac{Z_1 e}{r} \exp(-r/\lambda) \right] F(r) , \quad (9)$$

where $F(r)$ is an undetermined smooth function with the properties

$$\begin{aligned} F(r) &\xrightarrow{r \rightarrow 0} 1 \\ \frac{d}{dr} F(r) &\xrightarrow{r \rightarrow \infty} 0 \end{aligned} \quad (10)$$

It should be noted that the differential equation (4) does not admit solutions of the required form ($\sim 1/r$) at the origin. The differential equation is incorrect when r is small, because the Boltzmann relation (2) cannot be applied in the immediate neighborhood of the fixed charge at the origin. We see from (3) that if the potential is to have a simple pole at the origin the Boltzmann relation predicts an essential singularity in the negative charge density, a physically impossible result.

The simplest assumption consistent with the requirements (10) that can be made concerning the undetermined function $F(r)$ is that it is a constant, and therefore identically equal to unity. The potential ϕ is then

$$\phi(r) = \frac{Z_1 e}{r} \exp(-r/\lambda) . \quad (11)$$

This result will be employed to represent the potential of a charged particle immersed in a plasma. It is asymptotically correct for small r and has the correct functional behavior for large r . Beyond having these properties, however, it is not strictly a derived result and this fact should be kept in mind.

The distance parameter λ given by Eq. (6) and appearing in expression (11) above is called the Debye shielding distance. It was originally obtained by Debye in his work on the theory of strong electrolytes. The value of λ given by (6) applies to the shielding effects in the steady state and includes the shielding produced by both electrons and positive ions. In employing the shielding distance in nonsteady problems such as the slowing down of fast particles in a plasma, the shielding by positive ions is usually neglected and the shielding distance given as

$$\lambda = \sqrt{\frac{kT_e}{4\pi e^2 n_e}} \quad (12)$$

Substituting the potential given by (11) into the Poisson equation (1), we find that the charge density ρ is

$$\rho = - \frac{Z_1 e}{4\pi \lambda^2 r} \exp(-r/\lambda) \quad (13)$$

The charge density has a simple pole at the origin. Integrating the charge density over all space, we obtain

$$4\pi \int_0^\infty \rho r^2 dr = -Z_1 e \quad (14)$$

Adding the point charge ($Z_1 e$) at the origin to the distributed charge ($-Z_1 e$) given by (14) we see that the system as a whole is electrically neutral as required.

Let us now reconsider the slowing down of a fast particle in a relatively cool plasma. In the preceding section it was noted that the expression for dE/ds diverged logarithmically as the upper limit R of the integral (II, 17) over all impact parameters was allowed to tend to infinity. The fact that the potential ϕ decreases much faster than $1/r$, according to (11) when $r > \lambda$, suggests that we restrict the integration to impact parameters less than the Debye shielding distance λ . We have then

$$\frac{dE}{ds} = -N_2 \sigma_{BE} E \ln \left[1 + \Lambda^2 \right] \quad (15)$$

where

$$\Lambda^2 = \frac{\pi \lambda^2}{C_B} = \frac{kT_e}{4\pi n_e} \left(\frac{\mu v^2}{Z_1 Z_2 e^3} \right)^2 \quad (16)$$

The energy change per unit path length $(dE/ds)_B$ due to (unshielded) backward scatters alone is given by

$$\left(\frac{dE}{ds}\right)_B = -N_2 \sigma_{BE} E \ln 2, \quad (17)$$

so we can interpret the quantity $\left[\ln(1 + \Lambda^2)\right]/\ln 2$ as the ratio of the rate of energy loss to the rate of energy loss that would result from (unshielded) backward scattering collisions alone. The parameter Λ is usually large compared to unity so that Eq. (15) becomes

$$\frac{dE}{ds} \approx -2N_2 \sigma_{BE} E \ln \Lambda, \quad \Lambda \gg 1. \quad (18)$$

It is this "large Λ " approximation to the energy loss equation (15) that is often quoted in the literature.

IV. QUANTUM EFFECTS³

It was remarked earlier that the quantum-mechanical treatment of Coulomb scattering led to the same results as the classical treatment in the non-relativistic limit. This is no longer true when the Coulomb potential is replaced by the shielded Coulomb potential (III, 11) obtained in the preceding section.

Let us calculate the scattering cross section, in first Born approximation, for the case of the shielded Coulomb potential (Yukawa potential). We have in the center-of-mass system

$$\sigma(\theta) = \left| \frac{2\mu}{\hbar^2} \int_0^\infty \frac{\sin Kr}{Kr} \phi(r) r^2 dr \right|^2, \quad (1)$$

where

$$K = (2p/\hbar) \sin(\theta/2). \quad (2)$$

The quantity $p = \mu v$ is the invariant relative momentum of the two particles and $\hbar/p$ is the corresponding de Broglie wavelength λ .

Substituting the expression (III, 11) into (1) we obtain by an elementary integration

$$\sigma(\theta) = \frac{\sigma_B}{4\pi} \left[\sin^2(\theta/2) + (1/a)^2 \right]^{-2}, \quad (3)$$

where

$$a = 2(\lambda/\chi) = 2\mu v\lambda/\hbar. \quad (4)$$

We observe that as the shielding distance λ tends to infinity, we obtain the classical Rutherford cross section. The total cross section (σ_T) is easily found to be

$$\sigma_T = \sigma_B \left(\frac{a^4}{1+a^2} \right). \quad (5)$$

To determine the rate of energy loss we must evaluate:

$$\frac{dE}{ds} = N_2 E \int_{4\pi} \left(\frac{\Delta E}{E} \right) \sigma(\theta) d\Omega, \quad (6)$$

where $(\Delta E/E)$ is the fractional energy loss in the lab system, and is given in terms of (θ) by (II, 4). Evaluating the integral (6), we obtain

$$\frac{dE}{ds} = -N_2 \sigma_B E \left[\ln(1+a^2) - \frac{a^2}{1+a^2} \right] \quad (7)$$

The condition under which the Born approximation used in obtaining the cross section $\sigma(\theta)$ is valid is

$$f^2 = (aZ_1 Z_2 c/v)^2 = \sigma_B / \pi \chi^2 \ll 1, \quad (8)$$

where $(a = e^2/\hbar c = 1/137)$ is the fine structure constant. The relation between a and Λ is simply expressed in terms of f

$$\Lambda f = a/2 = \lambda/\chi. \quad (9)$$

When the condition (8) is fulfilled, we should use the quantum result (7). On the other hand, when the de Broglie wavelength χ is small enough

that $\sigma_B / \pi \lambda^2 \gg 1$, we should use the classical result. To summarize, we have

$$\frac{dE}{ds} = \begin{cases} - \left(\frac{4\mu}{M} \right) N_2 \sigma_B E \left[\ln(1+a^2) - \frac{a^2}{1+a^2} \right], & \sigma_B \ll \pi \lambda^2 \\ - \left(\frac{4\mu}{M} \right) N_2 \sigma_B E \ln(1 + \Lambda^2), & \sigma_B \gg \pi \lambda^2 \end{cases} \quad (10)$$

We observe that the two expressions are similar in form except for the presence of the extra term in the quantum-mechanical expression. However, it should be recalled that the quantum result was derived from the Yukawa potential, while the classical result was derived from the Coulomb potential by limiting the impact parameter to values smaller than the shielding distance λ .

In the limit of small velocity v the classical result must be employed ($f \gg 1$) and we have

$$\begin{aligned} \frac{dE}{ds} &= - \left(\frac{4\mu}{M} \right) N_2 \sigma_B E \ln \left(1 + \frac{\pi \lambda^2}{\sigma_B} \right) \\ &\xrightarrow{v \rightarrow 0} - \pi \lambda^2 \left(\frac{4\mu}{M} \right) N_2 E, \end{aligned} \quad (11)$$

so that the energy loss is due to head-on collisions with the cross section $\pi \lambda^2$.

While considering quantum effects, let us reconsider the derivation of the Debye potential given earlier. This derivation ran into trouble because the classical Boltzmann statistics led to an essential singularity at the origin in the negative charge density. However, if Fermi statistics are used to relate the negative (electron) charge density to the potential, this trouble disappears and solutions with the correct form at the origin (simple pole) are admitted.⁴ The exclusion principle prevents the catastrophic accumulation of negative charge around the fixed positive charge at the origin.

The effective potential ϕ of the fixed point charge is obtained by employing the Poisson equation (III, 1) as before, together with the relations

$$\rho = \rho^+ - \rho^- = n e_e \exp(-Ze\phi/kT) - \rho^- \quad (12)$$

$$\rho^- = \frac{8\pi e}{h^3} \int_0^\infty \frac{p^2 dp}{1 + \exp \left[(p^2/2m_e - e\phi - \chi) / kT \right]}, \quad (13)$$

where χ is determined so that

$$\rho^-(\infty) = en_e = \frac{8\pi e}{h^3} \int_0^\infty \frac{p^2 dp}{1 + \exp \left[(p^2/2m_e - \chi) / kT \right]}, \quad (14)$$

If the temperature and density are such that degeneracy of the electron gas is negligible at $r = \infty$, the quantity χ becomes

$$e\chi/kT = \frac{n_e}{2} \left(\frac{h^2}{2\pi m_e kT} \right)^{3/2}. \quad (15)$$

We find from (13) that as r tends to zero (and ϕ tends to + infinity)

$$\rho^- \sim \frac{8\pi e}{3h^3} (2m_e e\phi)^{3/2} \sim \frac{8\pi e^4}{3h^3} \left(\frac{2m_e Z_1}{r} \right)^{3/2}, \quad (16)$$

so that

$$Q^-(r) = 4\pi \int_0^r \rho^- r^2 dr \sim \frac{64\pi^2 e^4}{9h^3} (2m_e Z_1 r)^{3/2}, \quad (17)$$

and therefore

$$\frac{1}{r} Q^-(r) \xrightarrow{r \rightarrow 0} 0. \quad (18)$$

Hence, although the negative charge density ρ^- becomes infinite at the origin, the singularity is sufficiently weak that the potential is asymptotically determined in the neighborhood of the origin by the fixed positive charge alone. That is, it is now possible to satisfy the boundary conditions,

$$\begin{aligned} r\phi &\xrightarrow{r \rightarrow 0} Z_1 e \\ \phi &\xrightarrow{r \rightarrow \infty} 0, \end{aligned} \quad (19)$$

the first of which could not be satisfied using classical Boltzmann statistics for the electrons. The actual determination of ϕ is rather difficult but can be accomplished numerically by machine computation. The asymptotic result (16) shows that the electron gas becomes completely degenerate in the neighborhood of the fixed positive charge, so it is easy to understand the failure of classical statistics in this region.

V. AVERAGES OVER THE MAXWELL-BOLTZMANN DISTRIBUTION

Let us determine the instantaneous rate of energy increase of a charged particle (1) at rest in a sea of electrons at a temperature T_e . We have

$$\frac{dE}{dt} = \left(\frac{4\mu}{M}\right) N_e \int_0^\infty \left[v E \sigma_B \ln(1 + \Lambda^2) \right] P(E) dE, \quad (1)$$

where

$$P(E) = \frac{2}{\sqrt{\pi}} \frac{\sqrt{E}}{(kT_e)^{3/2}} e^{-E/kT_e} \quad (2)$$

$$\Lambda = \left(\frac{\lambda}{Z_1 e^2} \right) \mu v^2 = \left(\frac{2\lambda}{Z_1 e^2} \right) \left(\frac{\mu}{m_e} \right) E. \quad (3)$$

The integral (1) can be written

$$\frac{dE}{dt} = \frac{8N_e Z_1^2 e^4}{m_1 \langle v_e \rangle} \int_0^\infty e^{-x} \ln \left[1 + \frac{x^2}{\beta^2} \right] dx, \quad (4)$$

where $\langle v_e \rangle$ is the average speed of the electrons

$$\langle v_e \rangle = \sqrt{8kT_e / \pi m_e} \quad (5)$$

$$\beta^2 = \pi N_e Z_1^2 e^6 m_e^2 / \mu^2 (kT_e)^3. \quad (6)$$

The integral in Eq. (4) can be evaluated in terms of the Integral Sine and Cosine Functions (si) and (Ci), and we obtain:

$$I = -2 [\sin(\beta) \operatorname{si}(\beta) + \cos(\beta) \operatorname{Ci}(\beta)] \quad (7)$$

If β is small, as is commonly the case,

$$I \sim 2 \ln(1/\gamma\beta), \quad \beta \ll 1, \quad (8)$$

where $\gamma = 1.7811$. Hence we have finally

$$\frac{dE}{dt} \approx \frac{16N_e Z_1^2 e^4}{m_1 \langle v_e \rangle} \ln \Lambda_T \quad (9)$$

$$\Lambda_T = \frac{\mu}{\gamma Z_1 m_e c^3} \left(\frac{k^3 T_e^3}{\pi N_e} \right)^{1/2} \gg 1 \quad (10)$$

On the other hand, we can determine dE/dt using the quantum result of (IV, 10). In this case we obtain

$$\frac{dE}{dt} = - \frac{8N_e Z_1^2 e^4}{m_1 \langle v_e \rangle} \left[(1 + \zeta) e^\zeta \operatorname{Ei}(-\zeta) + 1 \right], \quad (11)$$

where Ei denotes the Exponential Integral Function and

$$\zeta = \frac{m_e \hbar^2}{8\mu^2 \lambda^2 k T_e} = \frac{1}{2} \left(\frac{\beta}{Z_1 a} \right)^2 \left(\frac{k T_e}{m_e c^2} \right). \quad (12)$$

If ζ is small, (11) becomes

$$\frac{dE}{dt} = \frac{8N_e Z_1^2 e^4}{m_1 \langle v_e \rangle} \left[\ln(1/\gamma\zeta) - 1 \right] \quad (13)$$

$$= \frac{16N_e Z_1^2 e^4}{m_1 \langle v_e \rangle} \left[\ln a_T - 1/2 \right], \quad (14)$$

where

$$a_T = 1/\sqrt{\gamma\zeta} = f_T \Lambda_T \quad (15)$$

$$f_T = Z_1 a \sqrt{\frac{2\gamma m_e c^2}{k T_e}} \quad (16)$$

Hence, we may summarize our results as follows

$$\frac{dE}{dt} \approx \frac{16N_e Z_1^2 e^4}{m_1 \langle v_e \rangle} \begin{cases} \ln a_T, & f_T \leq 1 \\ \ln \Lambda_T, & f_T \geq 1 \end{cases} \quad (17)$$

(Note: Spitzer's Λ is equal to $(3\gamma/2) \Lambda_T$ and his quantum correction factor is equal to $(2/3\gamma)^{1/2} f_T$).⁵

So far we have considered only the rate of energy gain of a charged particle initially at rest. If the particle is moving, the calculations are the same in principle but the trigonometry and algebra get rather involved. We will not attempt to present the derivation, but will simply quote the approximate result obtained by Chandrasekhar.⁶ Namely,

$$\frac{dE}{dt} \approx - \frac{4\pi e^4 N_2 Z_1^2 Z_2^2 \ln \Lambda_T}{m_2 v_1} \left[\Phi(y) - \left(\frac{m_2}{\mu} \right) y \Phi'(y) \right], \quad (18)$$

where

$$\Phi(y) = \text{erf}(y) = \frac{2}{\sqrt{\pi}} \int_0^y e^{-x^2} dx \quad (19)$$

$$y^2 = \frac{m_2 v_1^2}{2kT} = \frac{m_2}{m_1} \frac{E}{kT} = \frac{3}{2} \frac{v_1^2}{\langle v_2^2 \rangle} \quad (20)$$

For $y \gg 1$, (18) becomes

$$\frac{dE}{dt} = - \left(\frac{4\mu}{M} \right) N_2 \sigma_B E v_1 \ln \Lambda_T^2, \quad (21)$$

which is identical with the classical result (IV, 10) except that $1 + \Lambda^2$ has been replaced by Λ_T^2 .

For $y \ll 1$, (18) becomes

$$\frac{dE}{dt} = \frac{16N_2 Z_1^2 Z_2^2 e^4}{m_1 \langle v_2 \rangle} \ln \Lambda_T \left(1 - \frac{2E}{3kT} \right), \quad (22)$$

which reduces to our result (9) when the particle (1) is at rest ($E = 0$).

It is clear that the logarithmic factor Λ_T must be large compared with unity in order that our results be reasonably accurate. We see that this is necessary in order that the integral (7) be reasonably approximated by the expression (8). Also in the quantum result (17) we have retained only the log term and discarded the term $-1/2$ so that, again, the log term must be large compared with unity. If the log term is large compared with unity, this means that only a small fraction of the energy loss is due to backward scattering events in the c.m. system. That is, most of the energy loss is due to distant small angle encounters.

VI. CONCLUSIONS

It appears that future improvement in the theory of Coulomb interactions in fully ionized gases is likely to be made in two ways:

i) Through employing an effective (shielded) potential derived on the basis of quantum (Fermi) statistics for the electrons. The equations defining this potential have been given in Section IV and can be solved by machine computation.

ii) Through improvements in the two-body approximation to the many-body problem.

The former improvement (i) can readily be carried out on the basis of existing techniques. The latter (ii) will depend on basic advances in the approximate treatment of the many-body problem with long-range forces. Advances in this direction can probably be made in the classical case by treating the many-body problem by means of "particle following" methods using computing machines with large memory capacity. Unfortunately, in many problems of interest in plasma physics quantum effects need to be taken into account. Under these circumstances the method of "particle following," depending as it does on the classical notion of particles with well-defined coordinates and momenta, breaks down. What is needed is a workable quantum theoretic method for treating the many-body problem with long-range forces. In this regard, it would be worthwhile to see if the formalism recently developed by Brueckner for dealing with the nuclear many-body problem can be applied to the study of plasmas.⁷

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³ A brief discussion of scattering in a screened Coulomb field (in Born approximation) is given in Quantum Mechanics, L. I. Schiff, McGraw-Hill, N. Y. (1949), pp. 168-169.

⁴ The use of Fermi statistics in determining the Debye potential is discussed in a Doctoral Dissertation, Debye-Huckel; Fermi-Thomas Theory of Plasmas and Liquid Metals, Richard Plock, Yale Univ. (1957).

⁵ Lyman Spitzer, Jr., Physics of Fully Ionized Gases, Interscience Pub. Inc., N. Y. (1956), pp. 72-73. Chapter 5 of this book gives a general review of encounters between charged particles in a plasma. Many results are quoted in a form convenient for calculation.

⁶ S. Chandrasekhar, Astrophys. J., 93, 285 (1941), and 97, 255 (1943). These papers give the first correct treatment of the integration over scattering angles, velocities, etc.

⁷ An account of Brueckner's method is given in Annals of Physics 2, 199 (1957).