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THERMAL STRESSES AND STRAINS IN A FINITE

CYLINDER WITH NO SURFACE FORCES

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THERMAL STRESSES AND STRAINS IN A FINITE CYLINDER WITH NO SURFACE FORCES

Introduction

Formulas for the stresses and strains far from the ends of a long circular cylinder were given in CP-1698, under the condition that the temperature T is a function of the radius alone. These were obtained under the assumption (1) of a longitudinal pressure $F(r) = \left[(E \alpha) / (1 - \nu) \right] \left[T(r) - \bar{T} \right]$ applied at the ends of the cylinder, which forces the latter into a state of plane strain which is easily analysed. Then (2) the problem of an unheated cylinder with a longitudinal tension $F(r)$ applied over the ends is solved. Superimposing these two solutions gives the results for the heated cylinder with free ends.

The solution of the second problem vanishes, by St. Venant's principle, far from the ends of a long cylinder; and so only the solution of the first problem is needed to find the results far from the ends of a heated cylinder, given in the report above.

The stresses and strains near the cylinder ends can be found only by a more detailed analysis of the second problem, and this is the purpose of the present report. For comparison, the problems of a disc (very short cylinder) and of a thin rectangular plate are also included.

In this report, E = Young's modulus, ν = Poisson's ratio, $\bar{T}$ = volume average of the temperature throughout the cylinder, α = linear

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coefficient of thermal expansion. Cylindrical coordinates (r, θ, z) are employed; tensile stresses are ($\sigma_r, \sigma_\theta, \sigma_z$); shear stresses are ($\tau_{ry}, \tau_{yz}, \tau_{zx}$), or τ_{rz} in the axially symmetrical problem, (u, v, w) are displacements.

Analysis of the Problem

General formulas covering this problem among others are given in a paper by J. Dougall, Trans. Royal Soc. of Edinburgh, vol. 49, p.895 (1914), but simpler formulas are found in Timoshenko, Theory of Elasticity, p. 309. In the series expansion for each stress function σ_r and τ_{rz} , each term will be made to satisfy the condition $\sigma_r = \tau_{rz} = 0$ at the lateral surface of the cylinder, $r = a$. At the ends of the cylinder the imposed conditions $\tau_{rz} = 0, \sigma_z = F(r)$ can only be satisfied approximately with the aid of several terms of the two series.

Let φ be a function satisfying the biharmonic equation $\nabla^4 \varphi = 0$ and such that

$$\sigma_r = \frac{\partial}{\partial z} \left[\nu \nabla^2 \varphi - \frac{\partial^2 \varphi}{\partial r^2} \right]$$

$$\sigma_\theta = \frac{\partial}{\partial z} \left[\nu \nabla^2 \varphi - \frac{1}{r} \frac{\partial \varphi}{\partial r} \right]$$

$$\sigma_z = \frac{\partial}{\partial z} \left[(2 - \nu) \nabla^2 \varphi - \frac{\partial^2 \varphi}{\partial z^2} \right]$$

$$\tau_{rz} = \frac{\partial}{\partial r} \left[(1 - \nu) \nabla^2 \varphi - \frac{\partial^2 \varphi}{\partial z^2} \right] .$$

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Then

$$s = \sigma_r + \sigma_\theta + \sigma_z = \frac{\partial}{\partial z} \left[(1 + \nu) \nabla^2 \varphi \right]$$

$$\nabla^2 \varphi = \frac{1}{1 + \nu} \int_{-\infty}^z s dz .$$

Let

$$s = \sum A_n J_0(\lambda_n r) e^{\lambda_n z} .$$

Then

$$\varphi = \frac{1}{2(1 + \nu)} \sum \frac{A_n}{\lambda_n^2} r J_1 e^{\lambda_n z} + \sum C_n J_0(\lambda_n r) e^{\lambda_n z}$$

and

$$\sigma_r = \sum \frac{A_n}{2(1 + \nu)} \left[(2\nu - 1) J_0 + r \lambda_n J_1 \right] e^{\lambda_n z} + \sum C_n \lambda_n^2 \left[\lambda_n J_0 - \frac{1}{r} J_1 \right] e^{\lambda_n z}$$

$$\tilde{\tau}_{rz} = \sum \frac{-A_n}{2(1 + \nu)} \left[2(1 - \nu) J_1 + r \lambda_n J_0 \right] e^{\lambda_n z} + \sum C_n \lambda_n^3 J_1 e^{\lambda_n z} .$$

The condition $\sigma_r(a) = \tilde{\tau}_{rz}(a) = 0$ gives the equation for each λ_n ,

$$\frac{J_0(\xi)}{J_1(\xi)} = \pm \sqrt{\frac{2(1 - \nu)}{\xi^2} - 1} \quad (\lambda_n a = \xi_n) .$$

If ξ_n is a solution, the conjugate imaginary $\bar{\xi}_n$ and $-\xi_n$ are also solutions; only those roots for which the real parts are positive are important near the end of the cylinder $z = 0$. Two smallest roots are

$$\xi_1 = 2.72 + i(1.35), \quad \xi_2 = 6.08 + i(1.65) .$$

For any value of ξ_n , there results the solution ($B_n = \frac{A_n \xi_n}{2(1 + \nu)}$)

$$\sigma_r = B_n e^{\lambda_n z} \left\{ \left[\frac{1}{\xi} + \gamma \right] J_0\left(\xi \frac{r}{a}\right) + \left[\frac{r}{a} - \frac{a}{\xi r} \left(\frac{2(1 - \nu)}{\xi} + \gamma \right) \right] J_1\left(\xi \frac{r}{a}\right) \right\}$$

$$\sigma_\theta = \frac{B_n e^{\lambda_n z}}{\xi} \left\{ (2\nu - 1) J_0\left(\xi \frac{r}{a}\right) + \frac{a}{r} \left[\frac{2(1 - \nu)}{\xi} + \gamma \right] J_1\left(\xi \frac{r}{a}\right) \right\}$$

$$\sigma_z = B_n e^{\lambda_n z} \left\{ \left[\frac{2}{\xi} - \gamma \right] J_0\left(\xi \frac{r}{a}\right) - \frac{r}{a} J_1\left(\xi \frac{r}{a}\right) \right\}$$

$$\tau_{rz} = -B_n e^{\lambda_n z} \left\{ \frac{r}{a} J_0\left(\xi \frac{r}{a}\right) - \gamma J_1\left(\xi \frac{r}{a}\right) \right\}$$

$$\gamma = \sqrt{\frac{2(1 - \nu)}{\xi^2} - 1}.$$

Numerical values were obtained with the real part of B_1 equal to 1, while the complex numbers B_1, B_2 were adjusted to satisfy approximately the condition $R(\tau_{rz}) = 0$, when 2 terms of the series only are included. As is seen from Fig. 1 the curve for $F_2(r)$ representing 2 terms of σ_z is only approximately parabolic, although the average value of each σ_z term on the end of the cylinder is zero.

For application to the actual end condition

$$\sigma_z = \frac{E\alpha}{1 - \nu} (T - \bar{T}) \quad (\text{or } F_2(r) \text{ representing 2 terms})$$

let f_z, f_r, f_θ be the σ values computed as indicated above, and assume $T - \bar{T} = 150$ at $r = 0$. It is assumed that $\nu = .3$, $\alpha = 12 \times 10^{-6}$. Then if

$$\sigma_z = E C f_z, \quad \sigma_r = E C f_r, \quad \sigma_\theta = E C f_\theta$$

there results $C = 5.65 \times 10^{-3}$.

The total extensions in the radial or longitudinal directions can be obtained by integration from the relations

$$E \frac{\partial u_r}{\partial r} = \sigma_r - \nu(\sigma_\theta + \sigma_z)$$

$$E \frac{\partial w}{\partial z} = \sigma_z - \nu(\sigma_r + \sigma_\theta) .$$

Computed values in mils are represented in Figures 2 and 3.

From the stress-strain relations

$$\frac{\partial u}{\partial x} = \frac{1}{E} \left[\sigma_x - \nu(\sigma_y + \sigma_z) \right] + \alpha T$$

$$\frac{\partial v}{\partial y} = \frac{1}{E} \left[\sigma_y - \nu(\sigma_x + \sigma_z) \right] + \alpha T$$

$$\frac{\partial w}{\partial z} = \frac{1}{E} \left[\sigma_z - \nu(\sigma_x + \sigma_y) \right] + \alpha T$$

$$\Delta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{1 - 2\nu}{E} s + 3\alpha T$$

$$s = \sigma_x + \sigma_y + \sigma_z .$$

If (X,Y,Z) is the external force per unit volume, (X_s,Y_s,Z_s) the force per unit area of the surface, the fundamental compatibility relations give

$$\iiint_V s \, dx \, dy \, dz = \iiint_V (xX + yY + zZ) \, dx \, dy \, dz + \iint_S (xX_s + yY_s + zZ_s) \, dS .$$

For similar relations see Love, Elasticity, p. 175, 4th ed., (1944). If both volume and surface forces vanish, it results from the integration of Δ that $\delta V = 3\alpha \bar{T} V$ represents the change in total volume when the temperature is raised from 0 to $T(x,y,z)$ and $\bar{T}$ is the volume average temperature.

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In our problem only forces on the ends may be different from zero owing to the approximation; at $z = \text{constant}$, $\bar{\sigma}_z = 0$, leaving $xX_s + yY_s = \pm r\tilde{\tau}_{rz}$ which may not vanish on the average. Calculation gives

$$\iiint_V dx dy dz = \sum_{z=\text{const}} \pm \iint r^2 \tilde{\tau}_{rz} dr d\theta$$

and on one end

$$\iint_{z=0} r^2 \tilde{\tau}_{rz} dr d\theta = 4\pi(1+\nu)a^3 R \left\{ \sum_n \frac{B_n}{n^3} J_1(\zeta_n) \right\}.$$

The total possible error in the total volume increase is therefore $2(1-2\nu)/E$ multiplied by the right-hand side above.

Case of the Thin Circular Disc

The stress-strain relations in cylindrical coordinates, with (u,w) the radial and longitudinal displacements, respectively, are

$$\frac{du}{dr} = \frac{1}{E} [\sigma_r - \nu(\sigma_\theta + \sigma_z)] + \alpha T$$

$$\frac{u}{r} = \frac{1}{E} [\sigma_\theta - \nu(\sigma_r + \sigma_z)] + \alpha T \quad \frac{d}{dr}(r\sigma_r) - \sigma_\theta = 0$$

$$\frac{dw}{dz} = \frac{1}{E} [\sigma_z - \nu(\sigma_r + \sigma_\theta)] + \alpha T.$$

If all quantities are functions of r alone, and primed quantities are derivatives with respect to r , from these equations

$$r \left\{ \sigma_\theta - \nu(\sigma_z + \sigma_r) \right\}' + (1+\nu)(\sigma_\theta - \sigma_r) + \alpha E r T' = 0.$$

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Let $\sigma_z = 0$ for the thin disc; then

$$r \left\{ \sigma_{\theta} - \nu \sigma_r \right\}' + (1 + \nu) r \sigma_r' + \alpha E r T' = 0$$

and

$$r(\sigma_{\theta}' + \sigma_r') = -\alpha E r T'.$$

Integrating,

$$\sigma_r + \sigma_{\theta} = A - \alpha E T = \sigma_r + \frac{d}{dr} (r \sigma_r)$$

from which

$$\frac{d}{dr} (r^2 \sigma_r) = r(A - \alpha E T).$$

The outer boundary condition requires that $\sigma_r = 0$, $r = a$. Then $A = \alpha E \bar{T}$

and

$$r^2 \sigma_r = \alpha E \int_0^r r(\bar{T} - T) dr$$

$$\sigma_r + \sigma_{\theta} = \alpha E(\bar{T} - T).$$

Substituting above,

$$\frac{dw}{dz} = \alpha \bar{T} + \alpha(1 + \nu)(T - \bar{T})$$

and the change in thickness becomes, if h is the original thickness,

$$\Delta w = h \left[\alpha \bar{T} + \alpha(1 + \nu)(T - \bar{T}) \right].$$

From preceding formulas

$$\frac{du}{dr} = \alpha T + \frac{1}{E} \left[\sigma_r - \nu \frac{d}{dr} (r \sigma_r) \right]$$

$$u(r) = \alpha \int_0^r T dr + \frac{1}{E} \int_0^r \sigma_r dr - \frac{\nu}{E} r \sigma_r$$

$$u(a) = \alpha a \bar{T}.$$

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Computed values of σ_r , σ_θ for the circular disc are denoted by σ_r^t , σ_θ^t in figures 4 and 5.

Formulas for corresponding stresses and strains far from the ends of a long cylinder, and in a thin disc of the same radius are as follows:

	<u>Cylinder</u>	<u>Disc</u>
$r^2 \sigma_r$	$\frac{\alpha E}{1 - \nu} \int_0^r r(\bar{T} - T) dr$	$\alpha E \int_0^r r(\bar{T} - T) dr$
$\sigma_r + \sigma_\theta$	$\frac{\alpha E}{1 - \nu} (\bar{T} - T)$	$\alpha E (\bar{T} - T)$
σ_z	$\frac{\alpha E}{1 - \nu} (\bar{T} - T)$	0
$\frac{u}{a}$	$\propto \bar{T}$	$\propto \bar{T}$
$\frac{dw}{dz}$	$\propto \bar{T}$	$\propto \bar{T} + \alpha(1 + \nu)(\bar{T} - T)$
$\frac{d\bar{w}}{dz}$	$\propto \bar{T}$	$\propto \bar{T}$
Total volume change	$3 \propto V \bar{T}$	$3 \propto V \bar{T}$

The Thin Rectangular Plate

If $\sigma_z = 0$, and e_x , e_y are the principal strain components,

$$e_x = \frac{1}{E} [\sigma_x - \nu \sigma_y] + \alpha T$$

$$e_y = \frac{1}{E} [\sigma_y - \nu \sigma_x] + \alpha T$$

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The equations of equilibrium give

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tilde{\tau}_{xy}}{\partial y} = 0$$

$$\frac{\partial \tilde{\tau}_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0 .$$

As in the case of the cylinder, end loadings are first imposed such that all quantities are independent of the x coordinate. Then

$$\frac{\partial \tilde{\tau}_{xy}}{\partial y} = 0,, \quad \tilde{\tau}_{xy} = 0, \quad y = \pm b$$

$$\tilde{\tau}_{xy} = 0 .$$

$$\frac{\partial \sigma_y}{\partial y} = 0, \quad \sigma_y = 0, \quad y = \pm b$$

$$\sigma_y = 0 .$$

The condition is imposed that the strain in the x direction is constant.

Hence

$$e_x = \frac{\partial u}{\partial x} = \frac{\sigma_x}{E} + \alpha T; \quad \bar{e}_x = \frac{\bar{\sigma}_x}{E} + \alpha \bar{T}$$

$$\sigma_x = \bar{\sigma}_x + E\alpha(\bar{T} - T) .$$

The remaining quantities may be computed from σ_x ; the constant term $\bar{\sigma}_x$ corresponds to a state of simple strain, and for convenience this term is assumed to be zero.

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In the second part of the problem forces are applied at the ends $x = \pm a$ to reduce σ_x to zero here; the only end force required is

$$\sigma_x' = E\alpha(T - \bar{T}) .$$

Except for the constant term $-E\alpha\bar{T}$ producing simple strain, the problem is similar to one treated by Timoshenko (Elasticity, pp.151-154). Further calculations are made with the aid of his results, with $E\alpha T = A(1 - y^2/b^2)$. The resulting end stress σ_y is plotted in Fig. 6 for two cases, (1) the square, and (2) a rectangle of length twice the width. Corresponding lateral displacements of the sides are plotted in Fig. 7.

Discussion

From Fig. 1 it is clear that the curve $F_2(r)$ deviates from a parabola by remaining flatter near the cylindrical surface $r = a$, leading to a slightly smaller value for $|\sigma_\theta|$ in this neighborhood; in Fig. 4 σ_θ^s decreases in fact. Considering $F_2(r)$ as a sufficiently good representation for σ_z far from the ends, we have $\sigma_\theta^s/\sigma_\theta^o = 1.1$. This represents an increase of 10 per cent in the hoop stress close to the cylindrical surface, as compared to an increase of 20 percent for a thin cylinder at the end, computed from data of Timoshenko; but the 2 term function $F_2(r)$ may lead to errors near the boundary.

From Fig. 3 it is seen that the radial expansion of the cylinder becomes negligible for $|z/a| > 1.6$, which represents about the distance from the end at which the end effects disappear. The contraction of the

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cylinder which sets in a short distance from the end is similar to that of the rectangle.

From Figs. 4 and 5 it is clear that the temperature distribution in a finite cylinder produces positive stresses σ_r , σ_θ near the central region of the ends, in place of the negative stresses near the axis far from the ends.

For the same temperature distribution in the thin disc, the stress components σ_r , σ_θ differ from those of the cylinder far from the ends by a factor of 0.7, and are plotted in Figs. 4 and 5.

From Fig. 6 it is seen that the stress component σ_y is positive in the rectangular plate near the end, giving an increased probability of cracks close to the ends in the central region. This component is almost the same for a square plate and one of length twice the width.

In Fig. 7 the lateral displacements of the sides of the two plates produced by the end forces alone are plotted; the sides are contracted near the center, and rapidly expanding near the ends, becoming almost equal. The small contraction for the longer plate, compared to the contraction for the square near the center, indicates a marked end effect here for the square.

The shapes of the cylinder and rectangular plates which result from the application of parabolic end forces with zero averages are as follows:

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Fig 2 Longitudinal Expansion of Cylinder

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in mils, $(T - T_{h.o.}) = 150$

$\alpha = 12 \times 10^{-6}$

resulting from end force $T_2(G)$

w ↑
mils

r/a →

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Fig 3 - Radial Expansion near cylinder end

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in mils, $(T-7), E=150, \alpha=12 \times 10^{-6}$

resulting from end force $F_a(r)$

$u(r)$
(mils)

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Fig 4 Total Hoop Stresses for a cylinder
with $\sigma = F_2(r)$

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$\times 10^3$

0.1

0

-1×10^{-3}

.1

.2

.3

.4

.5

.6

.7

.8

.9

.9

1.0

r/a

σ_{θ}^s (end of cylinder)

σ_{θ}^s (at $r/a = 1$)

σ_{θ}^s (at $r/a = 0$)

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Fig. 5 Total Radial Stresses in a Cylinder

σ_r^* far from end of cylinder

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σ_r^* end of long cylinder

σ_r^* thin disc

+1
 $\times 10^{-3}$

σ_r

0

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$\times 10^{-3}$

σ_r^*

σ_r^*

σ_r^*

.1

.2

.3

.4

$r/a.5$

.6

.7

.8

.9

1.0

Fig 7 - Lateral Extension of a rectangle
with end force $\bar{x} = \pm (1 - y^2 - \frac{2}{3})$ at $x = \pm a$

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- (1) $a = b = 1$
- (2) $a = 2b = 2$

+ .3

+ .2

+ .1

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- .1

$\uparrow E L_y$

.1

.2

.3

.4

.5

.6

.7

.8

.9

1.0

$x/a \rightarrow$

Fig. 8 Thermal Stresses in a rectangle

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$T = T(y)$, Boundaries $x = \pm a$; $y = \pm b$

$\sigma_x = \pm (1 - y^2/b^2) \sigma_0$, $x = \pm a$

(1) σ_y for $a = b = 1$, at $x = a$.

(2) σ_y for $a = 2b$, $b = 1$ at $x = a$.

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