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UCRL - 73951
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RELATIVISTIC BEAM PLASMA INSTABILITY
IN STRONG MAGNETIC FIELD

H. L. Berk

May 22, 1972

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RELATIVISTIC BEAM PLASMA INSTABILITY
IN STRONG MAGNETIC FIELD*

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ABSTRACT

Stability criteria for a weak relativistic beam-plasma interaction in a strong magnetic field are found. Two beam modes occur, $\omega \approx k_z c$ and $\omega \approx k_z c - \omega_c$, where ω_c is the relativistic cyclotron frequency. The former mode is insensitive to thermal spread, but is stabilized if the beam thickness, L_p , is sufficiently small, viz. $L_p < (\pi \gamma c / \omega_p) \text{Min}[1, 1/2(n_c / \gamma n_b)^{1/4}]$, where γ is the usual relativistic parameter and n_c and n_b are the background and beam densities respectively. The latter mode is sensitive to thermal spread in γ , and stable regimes are found when the background collision frequency and the beam thermal spread are sufficiently large. These modes require a complete electromagnetic analysis over much of the parameter range of interest.

*Work performed under the auspices of the U. S. Atomic Energy Commission.

I. Introduction

Recently, there have been several suggestions for creating nearly force free relativistic toroidal beams to contain a plasma.^{1,2} The most evident instability in this configuration is the two stream instability. The relativistic two stream instability has been investigated theoretically in the limit of zero magnetic field,³ but there does not appear to be an analysis in the literature for this mode in a strong magnetic field.

In this work we will analyze the fluid equations for the beam and background electrons and delineate the stability parameters and growth rates. We find that electromagnetic effects are important over a wide range of parameters.

We assume that a relativistic beam propagates along a z-directed magnetic field, B_z . The self field of the beam is assumed to be much less than B_z and is neglected in this analysis. Similarly, toroidal effects are neglected.

We confine our study to the case where

$$\omega \text{ and } \omega_D \equiv \left(\frac{4\pi n_c e^2 \gamma^{1/2}}{m} \right) \ll \gamma \omega_c \equiv \frac{|e| B_z}{mc}$$

where n_c is the cold plasma density, $\gamma \omega_c$ the nonrelativistic cyclotron frequency (ω_c is the cyclotron frequency of the beam electrons),

$\gamma^2 = c^2/(c^2 - v_b^2)^{-1}$, v_b is the beam velocity, e is the electronic charge

and m the electronic rest mass. In this limit we can neglect the transverse

dynamics of the cold plasma. However, we will include the transverse dynamics of the beam, which is a consistent approximation if $\gamma \gg 1$.

If we consider wavelengths much less than the beam dimension, infinite medium theory can be used. For simplicity, we treat a system of finite transverse dimensions, L_p , by assuming that the perpendicular wavenumber is greater than the value $k_x = \pi/L_p$.

II. Derivation

Both the beam and the background plasma are described in the pressureless fluid limits where the beam speed is taken as v_b along the z direction while the background electron speed is zero. The perturbed equations are of the form,

$$\left(\frac{\partial}{\partial t} + v_b \frac{\partial}{\partial z} \right) \delta n_b + \nabla \cdot (n_b \delta \mathbf{v}_b) = 0 \quad (1)$$

$$\left(\frac{\partial}{\partial t} + v_b \frac{\partial}{\partial z} \right) (\gamma \delta \mathbf{y}_b) = \frac{e}{m} \left(\mathbf{E} + \frac{v_b}{c} \hat{z} \times \delta \mathbf{B} + \frac{\delta v_b}{c} \times \mathbf{B}_z \right) \quad (2)$$

$$\frac{\partial}{\partial t} \delta n_c + \frac{\partial}{\partial z} (n_c \delta v_{cz}) = 0 \quad (3)$$

$$\frac{\partial}{\partial t} \delta v_{cz} = \frac{e}{m} E_z \quad (4)$$

$$-\frac{\partial}{\partial t} \delta \underline{E} = \nabla \times \underline{E} \quad (5)$$

$$\nabla \times \delta \underline{B} = \frac{1}{c} \frac{\partial}{\partial t} \underline{E} + 4\pi e [(n_c \delta v_{cz} + \delta n_b v_b) \hat{z} + \delta n_b \delta v_y], \quad (6)$$

where the subscript b refers to the beam. Note that for the cold plasma we only treat the response along the field line.

We consider the perturbations to be of the form, $\hat{f} \exp[-i(\omega t - k_z z - k_x x)]$. If we substitute this form into Eq. (5), we find that the perturbed magnetic force can be expressed in terms of $\underline{E}$,

$$\frac{v_b}{c} \hat{z} \times \delta \underline{B} = \frac{v_b}{\omega} [(k_x E_z - k_z E_x) \hat{x} - k_z E_y \hat{y}] \quad (7)$$

We then substitute Eq. (7) into Eq. (2), solve for δv_y , and find,

$$\delta v_z = \frac{ie}{m\gamma \tilde{\omega}} E_z \quad (8)$$

$$\delta v_x = \frac{ie}{m\gamma(\tilde{\omega}^2 - \omega_c^2)} \left[\frac{\tilde{\omega}^2}{\omega} E_x + \frac{i\tilde{\omega}\omega_c}{\omega} E_y + \frac{\tilde{\omega}}{\omega} k_x v_b E_z \right]$$

$$\delta v_y = \frac{ie}{m\gamma(\tilde{\omega}^2 - \omega_c^2)} \left[-\frac{i\tilde{\omega}\omega_c}{\omega} E_x + \frac{\tilde{\omega}^2}{\omega} E_y - \frac{i\omega_c}{\omega} k_x v_b E_z \right]$$

where $\tilde{\omega} = \omega - k_z v_b$. The perturbed beam density is then found by substituting Eq. (8) into the continuity equation, Eq. (1),

$$\delta n_b = \frac{in_b e}{m\gamma} \left[\frac{k_z E_z}{\omega^2 - \gamma^2} + \frac{k_x}{(\omega^2 - \omega_c^2)} \left(\frac{\omega}{\omega} E_x + \frac{i\omega_c}{\omega} E_y + \frac{k_x v_b}{\omega} E_z \right) \right] \quad (9)$$

Similarly, the solution of Eq. (4) yields the perturbed velocity of the cold electrons,

$$\delta v_{zc} = \frac{ieE_z}{m\omega} \quad (10)$$

By combining Eqs. (5) and (6) we obtain the wave equation,

$$\left(k^2 - \frac{\omega^2}{c^2} \right) \underline{E} - \underline{k} \underline{k} \cdot \underline{E} = \frac{i4\pi e\omega}{c^2} \left[\underline{\hat{z}} (n_c \delta v_z + \gamma n_b v_b) + n_b \delta v_{zb} \right] \quad (11)$$

and by substituting Eqs. (8), (9) and (10) into (11) we find,

$$\begin{aligned} (k_x^2 c^2 - \omega^2) E_z - k_x k_z c^2 E_x = & - \left(\omega_p^2 + \frac{\omega_b^2}{\gamma^2} \frac{\omega^2}{\omega^2} + \frac{\omega_b^2 k_x^2 v_b^2}{\omega^2 - \omega_c^2} \right) E_z \\ & - \frac{\omega_b^2 k_x v_b}{\omega^2 - \omega_c^2} (\omega E_x + i\omega_c E_y) \end{aligned} \quad (12)$$

$$(k_z^2 c^2 - \omega^2) E_x - k_x k_z c^2 E_z = - \frac{\omega_b^2}{\omega^2 - \omega_c^2} \left[\omega E_x + i\omega_c E_y + k_x v_b E_z \right]$$

$$(k^2 c^2 - \omega^2) E_y = - \frac{\omega_b^2}{\omega^2 - \omega_c^2} \left[-i\omega_c E_x + \omega E_y - \frac{ik_x v_b \omega_c}{\omega} E_z \right]$$

$$\text{where } \omega_b^2 = \frac{4\pi n_b e^2}{m\gamma}.$$

From Eq. (12) we obtain the following dispersion relation after some straight forward algebra,

$$\left[1 - \frac{\omega_p^2}{\omega^2} - \frac{\omega_b^2}{\gamma^2 \omega^2}\right] \left[\omega^2 - k_z^2 c^2 - \tilde{\omega}^2 \Lambda\right] - k_x^2 c^2 \left[1 - \frac{\Lambda}{\gamma^2}\right] = 0 \quad (13)$$

where

$$\Lambda = \frac{\frac{\omega_b^2}{\omega^2 - \omega_c^2}}{\frac{k_x^2 c^2 - \omega^2 + \frac{\omega_b^2}{\omega^2 - \omega_c^2}}{k_x^2 c^2 - \omega^2 + \frac{\omega_b^2}{\omega^2 - \omega_c^2}}}$$

III. Analysis of Instability

To analyze the stability of Eq. (13), we assume $\omega_b \ll \omega_p$ and ω_c . Therefore in Eq. (13) we need only keep terms linear in ω_b^2 . Hence we take

$$\Lambda = \frac{\omega_b^2}{\omega^2 - \omega_c^2}$$

The interesting regimes for stability then occur near the resonances where (A) $\tilde{\omega} = 0$, and (B) $\tilde{\omega}^2 = \omega_c^2$. For case (A) we can set $\Lambda = 0$ and for case (B) we can set $\omega_b^2 / \gamma^2 \omega^2 = 0$.

Case (A) $\tilde{\omega} \neq 0$.

For this case Eq. (13) can be approximated as,

$$k_z^2 = \frac{\omega_p^2}{\epsilon^2} + \frac{\omega_b^2}{\gamma^2 (\epsilon - v_b)^2} + \frac{k_x^2 c^2}{\epsilon^2 - c^2} \quad (14)$$

where $\xi = \omega/k_z$. If $k_x^2 = 0$ (this is a purely electrostatic case), it is clear that Eq. (13) is unstable. If we choose $\omega = \omega_p + \delta\omega$, and $\omega_p/k_z = v_b$, we then find from Eq. (13), that the unstable root is given by

$$\begin{aligned}\delta\omega &= \frac{1}{2} \left(\frac{\omega_p^2}{2\gamma^2} \right)^{1/3} \left(-1 + i\sqrt{3} \right) \\ &= \frac{1}{2^{4/3}} \frac{\omega_p}{\gamma} \left(\frac{n_b}{n_c} \right)^{1/3} \left(-1 + i\sqrt{3} \right)\end{aligned}\quad (15)$$

Note that a thermal spread in v_b is relatively ineffective in quenching this instability (specifically in converting the two stream to a "bump-on-tail" instability) for large γ . The quenching condition is

$$\delta v_b > \omega_I/k_z = \frac{\omega_I}{\omega_p} v_b \quad (16)$$

where δv_b is the thermal spread in the beam and ω_I the growth rate.

Now,

$$\delta v_b = \delta \left[\frac{(\gamma^2 - 1)^{1/2}}{\gamma} c \right] = c \frac{1}{\gamma(\gamma^2 - 1)^{1/2}} \frac{\delta\gamma}{\gamma} \quad (17)$$

When we substitute Eqs. (15) and (17) into Eq. (16) we obtain, for the quenching condition,

$$\frac{\delta\gamma}{\gamma} > \frac{(\gamma^2 - 1)}{\gamma} \frac{3^{1/2}}{2^{4/3}} \left(\frac{n_b}{n_c} \right)^{1/3} \quad (18)$$

Hence, if $\gamma \geq (n_c/n_b)^{1/3}$, a thermal spread in γ of order γ itself is needed for quenching.

If k_x^2 is sufficiently large (or alternatively, if the beam column is sufficiently narrow) it is possible to show that Eq. (14) is stable. Let us consider a plot of the right hand side of Eq. (14) shown in Fig. 1. Typical curves are shown. For $0 < \xi < v_b$, two possibilities are evident. Either the minimum occurs for $k_z^2 > 0$ (solid curve) or for $k_z^2 < 0$ (dotted curve). The roots of Eq. (14) are the intersection of these curves with a horizontal line. When the minimum lies below the ξ axis it is evident that there are six roots for all $k_z^2 > 0$. However, if the minimum lies above the ξ axis, there are only four roots for sufficiently small k_z^2 . The other two roots are complex conjugates; one of them being unstable. Marginal stability corresponds to the minimum lying on the ξ axis.

To find the marginal stability let us assume $\omega_b \ll \omega_D$, and $\gamma \gg 1$. Therefore $\xi_{\min} \doteq v_b \doteq c$. We can then write Eq. (14), approximately as,

$$k_z^2 = \frac{\omega_D^2}{v_b^2} + \frac{\omega_b^2}{\gamma^2(\xi - v_b)^2} + \frac{k_x^2 c}{2(\xi - c)} \quad (19)$$

To find $\xi_{\min}$ we set the derivative of Eq. (19) to zero and find

$$\frac{(\xi_{\min} - v_b)^3}{(\xi_{\min} - c)^2} = - \frac{4\omega_b^2}{\gamma^2 k_x^2 c} \quad (20)$$

Two limits are easily investigated:

$$(a) \quad \frac{4\omega_b^2}{\gamma^2 k_x^2 c} \ll c - v_b, \quad (b) \quad \frac{4\omega_b^2}{\gamma^2 k_x^2 c} \gg c - v_b$$

For case (a), $c - \xi \approx c - v_b$ and

$$\frac{(\xi_{\min} - v_b)^3}{(c - v_b)^3} = - \frac{4\omega_b^2}{\gamma^2 k_x^2 c (c - v_b)} \ll 1$$

To find marginal stability we substitute this result into Eq. (19), with $k_z^2 = 0$, and obtain,

$$\frac{\omega_p^2}{v_b^2} = \frac{k_x^2 c}{2(c - v_b)} \left[1 - \frac{1}{2} \left(\frac{4\omega_b^2}{\gamma^2 k_x^2 c (c - v_b)} \right)^{1/3} \right]$$

$$\approx \gamma^2 k_x^2 c.$$

Hence stability occurs for

$$k_x > \frac{\omega_p}{\gamma v_b} \approx \frac{\omega_p}{\gamma c} \quad (21)$$

By substituting k_x^2 into inequality (a) we find that Eq. (21) is valid if $\gamma^2 \omega_b^2 < \omega_p^2/8$.

For case (b) we find $v_b - \xi_{\min} = c - \xi_{\min} \gg c - v_b$, and therefore from Eq. (20)

$$(c - \xi_{\min}) = \frac{4\omega_b^2}{\gamma^2 k_x^2 c}.$$

When we substitute this expression into Eq. (19) we find for the stability condition,

$$k_x > 2 \frac{\omega_p}{\gamma^{3/4} c} \left(\frac{n_b}{n_c} \right)^{1/4} \quad (22)$$

which from inequality (b) is valid if $\gamma^2 \omega_b^2 \gg \omega_p^2/4$. For a beam of thickness L_p to be stable to this mode we need

$$L_p < \pi/k_x c.$$

When we combine Eqs. (21) and (22) we find,

$$L_p < \frac{\gamma \pi c}{\omega_p} \text{Min} \left[1, \frac{1}{2} \left(\frac{n_c}{\gamma n_b} \right)^{1/4} \right] \quad (23)$$

Case II. $\tilde{\omega} \neq \omega_c$, $\omega_b \ll \gamma \omega_c$

In this limit Eq. (13) can be approximated as,

$$\omega^2 - (\omega_p^2 + k^2 c^2) + k_z^2 c^2 \frac{\omega_p^2}{\omega^2} - \frac{\omega_b^2}{\tilde{\omega}^2 - \omega_c^2} \left[\tilde{\omega}^2 \left(1 - \frac{\omega_p^2}{\omega^2} \right) - \frac{k_x^2 c^2}{\gamma^2} \right] = 0 \quad (24)$$

Let us choose ω_0 as the solution of Eq. (24) when ω_b^2 is completely neglected. We then obtain,

$$\omega_0^2 = \frac{(k^2 c^2 + \omega_p^2)}{2} \pm \frac{1}{2} \left[(k^2 c^2 + \omega_p^2)^2 - 4 \omega_p^2 k_z^2 c^2 \right]^{1/2} \quad (25)$$

We now choose k_z such that $(\omega_0 - k_z v_D) = -\omega_c$ and $\delta\omega = \omega - \omega_0$.

If we expand Eq. (24) in $\delta\omega$, we then obtain,

$$\delta\omega^2 = - \frac{\omega_D^2 \omega_0^2 \left[\frac{\omega_0^2 k_z^2 c^2}{\gamma^2} + \omega_c^2 (\omega_p^2 - \omega_0^2) \right]}{4\omega_c \{ k_z^2 c^2 \omega_p^2 - \omega_0^4 \}} \quad (26)$$

If we substitute Eq. (25) into Eq. (26) we can find the parameters for instability. However, this leads to a fairly bulky relationship. To simplify the results we use the following approximate relations for ω_0 ,

$$\omega_0^2 = \frac{k_z^2 c^2 \omega_p^2}{\omega_p^2 + k_z^2 c^2} \quad (27a)$$

and

$$\omega_0^2 = \omega_p^2 + k_z^2 c^2 \quad (27b)$$

These solutions are asymptotic approximations if $(k_z^2 c^2 + \omega_p^2)^2 \gg k_z^2 c^2 \omega_p^2$, and they serve as qualitative interpolation formulae if the inequality does not apply.

First let us use ω_0 determined from Eq. (27a). In this case both the bracketed terms in the numerator and denominator of Eq. (26) are positive:

$$\frac{\omega_0^2 k_z^2 c^2}{\gamma^2} + \omega_c^2 (\omega_p^2 - \omega_0^2) = \frac{\omega_0^2 k_z^2 c^2}{\gamma^2} + \frac{\omega_c^2 \omega_D^2 (\omega_p^2 + k_z^2 c^2)}{\omega_p^2 + k_z^2 c^2} > 0,$$

$$k_z^2 c^2 \omega_p^2 - \omega_0^4 = \frac{\omega_p^2 k_z^2 c^2}{(\omega_p^2 + k_z^2 c^2)^2} \left[\omega_p^4 + \omega_p^2 (k_z^2 c^2 + k_z^2 c^2) + k_z^4 c^4 \right] > 0.$$

Hence, instability occurs if $\omega_0 > 0$. The only proviso for instability is that parameters exist for which $k_x^2 > 0$. By using,

$$k_z v_b = \omega_0 + \omega_c, \quad \omega_0 = \frac{k_z c \omega_p}{(\omega_p^2 + k^2 c^2)^{1/2}},$$

we find,

$$k_z = \frac{\omega_c (\omega_p^2 + k^2 c^2)^{1/2}}{v_b (\omega_p^2 + k^2 c^2)^{1/2} - c \omega_p}, \quad (28)$$

$$k_x^2 = k^2 - \frac{\omega_c^2 (\omega_p^2 + k^2 c^2)}{\omega_p^2 (c^2 + v_b^2) + k^2 c^2 v_b^2 - 2 \omega_p c v_b (\omega_p^2 + k^2 c^2)^{1/2}} > 0. \quad (29)$$

To simplify our analysis, we study two limits $\omega_D \ll \omega_c$ and $\omega_c \ll \omega_p \ll \omega_{cY}$.

In the limit $\omega_p \ll \omega_c$ it is clear that $\omega_0 \ll \omega_c$, and therefore

$k_z v_b \doteq \omega_c$. Since, $k c > k_z v_b \doteq \omega_c \gg \omega_p$, we have,

$$\omega_0 = k_z \frac{\omega_D}{k}$$

which is an electrostatic oscillation. We also find that Eqs. (28) and (29) become,

$$k_z = \frac{\omega_c}{v_b}, \quad k_x^2 = k^2 - \frac{\omega_c^2}{v_b^2} \quad (30)$$

Therefore, instability can be found for $k^2 > \omega_c^2/v_b^2$. If we substitute ω_0 and $k_z c$ into Eq. (26), we obtain,

$$\delta\omega^2 = -\frac{\omega_c^2 \omega_p}{3\omega} \frac{\omega_c}{kv_b} \left(1 - \frac{\omega_c^2}{k^2 v_b^2} \right) \quad (31)$$

The maximum value of the right hand side occurs when $kv_b/\omega_c = \sqrt{3}$, and therefore the maximum growth rate, ω_I , is

$$\omega_I = \frac{1}{3^{1/4}} \left(\frac{1}{6} \right)^{1/2} \omega_b \left(\frac{\omega_p}{\omega_c} \right)^{1/2} \quad (32)$$

Now let us investigate the other limit, $\omega_c < \omega_0 < \gamma\omega_c$. We first find the regime where $k_x^2 > 0$. From Eq. (29) it is clear that $k_x^2 > 0$ for sufficiently large k , and negative for sufficiently small k . Let us assume that the cross-over point occurs for $kc \ll \omega_p$. Then Eq. (29) can be approximated as,

$$k_x^2 = k^2 - \frac{4\omega_c^2}{c^2 \left[\frac{k^2 c^2}{\omega_p^2} - \frac{1}{\gamma^2} \right]^2} > 0 \quad (33)$$

If we assume $\frac{kc}{\omega_p} > \frac{1}{\gamma}$, then k satisfies the relation

$$k^6 > \frac{4\omega_c^2 \omega_p^4}{c^6} \quad (34)$$

Since we have assumed $kc/\omega_p > 1/\gamma$, we see that Eq. (34) is valid if $\gamma > (\omega_p/\omega_c)^{1/3}$.

If $\gamma < (\omega_p/\omega_c)^{1/3}$, we seek to satisfy the inequality near the vanishing of the denominator in Eq. (33). Hence, we obtain $k = (\omega_p/c\gamma) + \delta k$, and we find,

$$k_x^2 = \frac{\omega_p^2}{\gamma^2 c^2} - \frac{\gamma^2 \omega_c^2 \omega_0^2}{\delta k^2 c^4} > 0$$

or

$$\delta k^2 > \frac{\gamma^4 \omega_c^2}{c^2} \quad (35)$$

As $(\gamma^4 \omega_c^2/c^2) < (\omega_p^2/\gamma^2 c^2)$ if $\gamma < (\omega_p/\omega_c)^{1/3}$, Eq. (35) is the correct solution in this limit. When we combine the results of Eqs. (34) and (35) we find that instability exists when

$$k > \text{Max} \left[\frac{\omega_p}{c} \left(\frac{2\omega_c}{\omega_p} \right)^{1/3}, \frac{\omega_p}{\gamma c} \right] \quad (36)$$

To find the maximum growth rate, we will assume that $v_b \approx c$, $\omega_0^2 (k_x^2 c^2/\gamma^2) \ll \omega_c^2 \omega_p^2$, $k_z c \sim \omega_0 \ll \omega_p$, but consider $kc \sim \omega_p$. We use Eq. (28) for k_z , set

$$\omega_0 = \frac{\omega_p \omega_c}{[(\omega_p^2 + k^2 c^2)^{1/2} - \omega_p]},$$

and substitute these quantities into Eq. (26), to obtain

$$\delta \omega^2 = - \frac{\omega_b^2 \omega_p \omega_c}{4k_z^2 c^2} = - \frac{\omega_b^2 \omega_p [(\omega_p^2 + k^2 c^2)^{1/2} - \omega_p]}{4(\omega_p^2 + k^2 c^2)} \quad (37)$$

The maximum of $\delta\omega^2$ occurs when $[1 + (k^2 c^2 / \omega_p^2)] = 4$, and the maximum growth rate, ω_I , is then,

$$\omega_I = \frac{3^{1/2}}{4} \omega_b \quad (38)$$

One can readily verify that all our assumptions are satisfied. If we combine Eq. (32) and Eq. (38) we find that the maximum growth rate is given by,

$$\frac{\omega_I}{\omega_b} = \text{Min} \left[\frac{1}{3^{1/4} \cdot 6^{1/2}} \left(\frac{\omega_p}{\omega_c} \right)^{1/2}, \frac{3^{1/2}}{4} \right]. \quad (39)$$

This mode is sensitive to thermal spread in γ . In order to weaken the mode to a "bump-on-tail" instability, we need to have the thermal spread in $\omega_c^2 \equiv \omega_{co}^2 / \gamma^2$ (ω_{co} is the rest cyclotron frequency) exceed $2\omega_c \omega_I$. This yields,

$$\frac{\delta\gamma}{\gamma} > \frac{\omega_I}{\omega_c} = \frac{\omega_b}{\omega_c} \text{Min} \left[\frac{1}{3^{1/4}} \left(\frac{v_b}{c} \right) \left(\frac{\omega_p}{\omega_c} \right)^{1/2}, \frac{3^{1/2}}{4} \right] \quad (40)$$

We now examine the spreading effect quantitatively to exhibit more precise stability conditions. To obtain actual stabilization criteria we introduce the effect of collisions on the background species, by altering Eq. (4) to be of the form,

$$\frac{\partial}{\partial t} \delta v_{cz} = \frac{e}{m} E_z - \nu \delta v_{cz} \quad (4b)$$

where ν is the collision frequency. With a thermal spread in the beam particles one finds from the Vlasov equation (we omit the derivation in this work) that Eq. (24) can be written as

$$\omega^2 \left[1 - \frac{\omega_p^2}{\omega(\omega + i\nu)} \left(1 - \frac{k_z^2 c^2}{\omega^2} \right) \right] + k^2 c^2 - \omega_b^2 \int \frac{d\gamma F(\gamma)}{\omega^2 - \frac{\omega_p^2}{\omega_0^2} \gamma^2} \left[\tilde{\omega}^2 \left(1 - \frac{\omega_p^2}{\omega^2} \right) - \frac{k_x^2 c^2}{\gamma_0^2} \right] = 0 \quad (41)$$

where $\int F(\gamma) d\gamma = 1$ and $F(\gamma)$ is sufficiently peaked about $\gamma = \gamma_0$ so that the thermal spread in ω_b^2 is neglected. We have also assumed that γ_0 is sufficiently large so that we can approximate $\tilde{\omega}$ as $\omega - k_z c$, and $(k_x v_\perp / \omega_c) \ll 1$, where $v_\perp$ is the particle velocity perpendicular to the magnetic field. To zeroth order in ν and ω_b^2 , we have seen that the potentially unstable root is given by

$$\omega_0^2 = \frac{\omega_p^2 k_z^2 c^2}{(\omega_p^2 + k^2 c^2)} \quad \text{and} \quad \omega_0 - k_z c = -\frac{\omega_{co}}{\gamma_0} \equiv \omega_c.$$

When we expand to first order we obtain

$$\frac{2\delta\omega}{\omega_0} \left[\omega_0^2 - \frac{k_z^2 c^2 \omega_p^2}{\omega_0^2} \right] + \frac{i\nu\omega_p^2}{\omega} \left(1 - \frac{k_z^2 c^2}{\omega_0^2} \right) + \frac{\omega_b^2}{2} \int \frac{d\gamma F(\gamma)}{-\omega_c \delta\omega + \omega_c \delta\gamma / \gamma_0} \left[\frac{\omega_c^2 (k_x^2 c^2 + \omega_p^2)}{k_z^2 c^2} + \frac{k_x^2 c^2}{\gamma^2} \right] = 0 \quad (42)$$

where $\delta\gamma = \gamma - \gamma_0$.

In order to evaluate the integral in γ , we choose for simplicity,

$$F(\gamma) = \frac{1}{2\Delta\gamma}, \quad \gamma_0 - \Delta\gamma < \gamma < \gamma_0 + \Delta\gamma$$

$$F(\gamma) = 0, \text{ otherwise,}$$

Eq. (42) then becomes,

$$\begin{aligned} & -2 \frac{\delta\omega\omega_0}{k_z^2 c^2 \omega_p^2} \left[\omega_p^4 + (k_x^2 + k_z^2) c^2 \omega_p^2 + k^4 c^4 \right] \\ & - i\nu \frac{k_z^2 c^2}{k_z^2 \omega_p^2} \left(\omega_p^2 + k^2 c^2 \right)^{1/2} + \frac{\omega_b^2 \gamma_0}{2\omega_c \Delta\gamma} \ln \left(\frac{\delta\omega - \frac{\Delta\gamma}{\gamma_0} \omega_c}{\delta\omega + \frac{\Delta\gamma}{\gamma_0} \omega_c} \right) \\ & \cdot \left[k_x^2 c^2 \left(\frac{1}{\gamma^2} + \frac{\omega_c^2}{k_z^2 c^2} \right) + \frac{\omega_b^2 \omega_c^2}{k_z^2 c^2} \right] = 0 \end{aligned} \quad (43)$$

If $\delta\omega \ll \frac{\Delta\gamma}{\gamma_0} \omega_c$, then $\ln \left(\frac{\delta\omega - \frac{\Delta\gamma}{\gamma_0} \omega_c}{\delta\omega + \frac{\Delta\gamma}{\gamma_0} \omega_c} \right) \doteq i\pi$.

The stability condition of Eq. (43) is that its imaginary part is negative, which yields,

$$\nu > \frac{\pi k_z \omega_p \omega_b^2}{2\omega_c^2 k^2 c^2 (\omega_p^2 + k^2 c^2)^{1/2}} \frac{\gamma_0}{\Delta\gamma} \left[k_x^2 c^2 \left(\frac{1}{\gamma^2} + \frac{\omega_c^2}{k_z^2 c^2} \right) + \frac{\omega_b^2 \omega_c^2}{k_z^2 c^2} \right] \quad (44)$$

If, $\omega_p < \omega_{co}/\gamma_0$, we have seen that $k_z c \approx \omega_c$ and $k_z c \gg \omega_p$. Thus (44) becomes,

$$\nu > \frac{\pi}{2} \frac{\omega_b^2 \omega_p}{\omega_c^3} \frac{\gamma_0}{\Delta Y} [k_z^2 c^2 - \omega_c^2] \quad (45)$$

The maximum occurs when $kc/\omega_c = 3^{1/2}$, hence for stability we need,

$$\nu > \frac{\pi}{3\sqrt{3}} \frac{\omega_b^2 \omega_p}{\omega_c^2} \frac{\gamma_0}{\Delta Y} (\omega_p < \omega_{co}/\gamma) \quad (46)$$

In the limit $(\omega_{co}/\gamma_0) < \omega_p < \omega_{co}$, we have found that the interesting regime is $\omega_c \ll \omega_0 - k_z c \ll \omega_p \sim kc$. Eq. (44) then takes the form, (with $k_z c = (\omega_p^2 + k^2 c^2)^{1/2} \omega_c / [(\omega_p^2 + k^2 c^2)^{1/2} - \omega_p]$)

$$\nu > \frac{\pi \omega_b^2 \omega_p}{2 \omega_c} \frac{\gamma_0}{\Delta Y} \left[(k^2 c^2 + \omega_p^2)^{1/2} + \omega_p \right]^{-1} \quad (47)$$

The maximum occurs as $(kc/\omega_p) \rightarrow 0$, and therefore for stability we need

$$\nu > \frac{\pi \omega_b^2 \gamma_0}{4 \omega_c \Delta Y} \quad (48)$$

By combining Eqs. (46) and (48) we find that the stabilization criteria is,

$$\nu > \frac{\pi \omega_b^2 \gamma_0}{\omega_c \Delta Y} \text{Min} \left[\frac{1}{3\sqrt{3}} \frac{\omega_p}{\omega_c}, \frac{1}{4} \right] \quad (49)$$

where ΔY is larger than the value determined in Eq. (40).

Now let us investigate if instability is possible when

$$\omega_0 = \pm(\omega_p^2 + k^2 c^2)^{1/2}. \text{ In this case}$$

$$k_z = \frac{\omega_c \pm (\omega_p^2 + k^2 c^2)^{1/2}}{v_b}$$

and

$$k_x^2 = k^2 - k_z^2 = -\frac{1}{v_b^2} \left[\frac{k^2 c^2}{\gamma^2} + \omega_0^2 + \omega_c^2 \mp 2\omega_c (\omega_p^2 + k^2 c^2)^{1/2} \right] \quad (50)$$

Note that $k_x^2 < 0$, if $\omega_0 > 0$ and therefore no mode exists for this case.

If $\omega_0 < 0$, then $k_x^2 > 0$ is possible. Since, $\omega_0^2 = \omega_p^2 + k^2 c^2$, it is evident from Eq. (26) that the denominator is negative, and instability can occur only if the bracket term in the numerator is greater than zero. Specifically, we need for instability,

$$\frac{1}{\gamma^2} \frac{c^2}{v_b^2} \left[2\omega_c (\omega_p^2 + k^2 c^2)^{1/2} - \omega_p^2 - \omega_c^2 - \frac{k^2 c^2}{\gamma^2} \right] > \frac{\omega_c^2 k^2 c^2}{\omega_p^2 + k^2 c^2} \quad (51)$$

We now show that this inequality cannot be satisfied simultaneously with the basic approximations $\omega_p < \gamma \omega_c$ and $\omega_0^2 \equiv \omega_p^2 + k^2 c^2 < \gamma^2 \omega_c^2$. First let us consider $kc < \omega_p$. Eq. (51) then becomes (set $\frac{c}{v_b} = 1$)

$$\frac{1}{\gamma^2} \left[-(\omega_p - \omega_c)^2 + k^2 c^2 \left(\frac{\omega_c}{\omega_p} - \frac{1}{\gamma^2} \right) \right] > \frac{\omega_c^2}{\omega_p^2} k^2 c^2 \quad (52)$$

To satisfy this inequality we need

$$\frac{\omega_p}{\omega_c \gamma^2} \geq 1$$

which violates our assumption that $\omega_p < \gamma \omega_c$ when $\gamma \gg 1$.

When $kc = \omega_p$ the left hand side is negative unless $\omega_0 = \omega_c$. In this case the order of magnitude of the left hand side is ω_0^2/γ^2 , while the order of magnitude of the right hand side is ω_0^2 . Hence, the inequality cannot be satisfied. If $kc \gg \omega_p$, then Eq. (51) becomes,

$$\frac{1}{\gamma^2} \left[2\omega_c kc - \omega_p^2 - \omega_c^2 - \frac{kc^2}{\gamma^2} \right] > \omega_c^2$$

or

$$\left(kc - \omega_c \gamma^2 \right)^2 < -\gamma^2 (\omega_p^2 + \omega_c^2) \quad (53)$$

This inequality can be satisfied when $kc \approx \omega_c \gamma^2$, but this again violates the fundamental inequality

$$\omega_0^2 \equiv \omega_p^2 + k^2 c^2 < \gamma^2 \omega_c^2.$$

Hence, we have shown that the mode with frequency, $\omega_0 = -(\omega_0^2 + k^2 c^2)^{1/2}$, is stable when a beam is present in the region where our theory is valid.

Acknowledgement: The author would like to thank L. D. Pearlstein for several relevant comments and suggestions

Figure Caption: Structure of Dispersion Curves

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