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STOCHASTIC FLUCTUATIONS IN A POWER REACTOR

November 1958

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PITTSBURGH, PENNSYLVANIA**

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STOCHASTIC FLUCTUATIONS IN A POWER REACTOR

D. R. Harris

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STOCHASTIC FLUCTUATIONS IN A POWER REACTOR

D. R. Harris

ABSTRACT

The neutron and delayed neutron precursor contents and the temperature of a reactor are treated as random variables participating in a multiplicative stochastic process. It is shown that stationary fluctuations in these variables are moderated by the presence of delayed neutrons, a large neutron source, and a large negative temperature coefficient. Reactor instability with a positive temperature coefficient is discussed in terms of the non-existence of a stationary process.

I. INTRODUCTION

In an ordinary thermodynamic system which might be represented by a grand canonical ensemble, fluctuations in the number of component particles usually are negligible, and this is a direct result of the very large number of particles which are present. This observation does not apply to multiplying systems, and fluctuations in a power reactor need not be small merely because the number of neutrons is large. Similarly, fluctuations in an ordinary thermodynamic system can be large when the system is near a change of state, partially because multiplicative processes are important. On the other hand, operating experience with power reactors has not revealed sizable fluctuations. The present study was undertaken to explain the absence of sizable fluctuations in power reactors and to investigate some conditions under which such fluctuations would not be negligible. A relatively simple model of a power reactor is described, and this model is analyzed as a stochastic process. It is shown that delayed neutrons moderate the fluctuations, but that the principal moderating effects arise from use of large neutron sources and large negative temperature coefficients in power reactors. Throughout the study it is assumed that the fuel, structural material and coolant in the

reactor are not near a change of state, e.g., coolant boiling is excluded, so that the reactor's composition can be regarded as a simple function of the instantaneous temperature.

Mathematical techniques for treatment of multiplicative stochastic processes have been developed extensively (Ref 1). Statistical treatments of reactors with fixed composition also have been described (Refs 2 and 3). The technique employed here is described in Ref. 1, but it includes modifications required by the non-linear character of the problem. Ignoring delayed neutrons and measuring instruments for the moment, let $P(N, \theta_R; t) d\theta_R$ represent the probability that $N (= 0, 1, 2, \dots)$ neutrons are present in the reactor at time t , and simultaneously the reactor temperature is in the range θ_R to $\theta_R + d\theta_R$. The non-linearity arises because the instantaneous neutron multiplication factor depends on θ_R . A mathematical description of a probability function analogous to $P(N, \theta_R; t) d\theta_R$ is developed in Section III for a reactor model which is described in Section II. The probability function is investigated in detail in Sections IV and V for a stationary situation, viz, a situation where $P(N, \theta_R; t) d\theta_R$ is independent of time. It is assumed that the reactor is subcritical, and a desired power level is attained by multiplication of source neutrons. First and second population moments are obtained, and the existence of a stationary process is discussed. The results are summarized in Section VI.

The stationary probability distribution $P(N, \theta_R; t) d\theta_R$ can be interpreted in two ways. First, an ensemble of identical reactors is considered, and $P(N, \theta_R; t) d\theta_R$ represents the normalized number of reactors with neutron content N and temperature θ_R to $\theta_R + d\theta_R$ at time t . A second interpretation follows a form of ergodic hypothesis--if a particular reactor is observed at randomly chosen times, then the normalized number of times that the reactor's neutron content is found to be N and the temperature simultaneously is found to be θ_R to $\theta_R + d\theta_R$ approaches $P(N, \theta_R; t) d\theta_R$.

II. REACTOR TEMPERATURE, FISSION RATE, AND TEMPERATURE COEFFICIENT

The dynamic relationship between reactor temperature and fission rate generally depends in a complicated manner on details of reactor construction and mode of operation. A particular relationship is chosen here for simplicity.

Suppose the reactor consists of solid, fuel-bearing material and a liquid coolant. Let $\theta_R(t)$ represent the spatial average of the temperature of the solid portions of the reactor at a time t , when $N(t)$ is the number of neutrons present in the reactor. Let C_R represent the heat capacity of the solid part of the reactor, and let K represent a coefficient of heat transfer between the solid part of the reactor and a coolant. The spatial average of the temperature of the coolant within the reactor is θ_C . The probable number of fissions occurring in the reactor in the time interval t to $t + \delta t$ is represented by $\Lambda_f N(t) \delta t$, and when a fission occurs a quantity of heat, q , is produced in the solid portion of the reactor. Then if we ignore the mechanical work performed in the time δt on or by the solid portions of the reactor, the first law of thermodynamics requires

$$\delta [C_R \theta_R(t)] = q \Lambda_f N(t) \delta t - K [\theta_R(t) - \theta_C] \delta t \quad (\text{II},1)$$

In general, the parameters C_R , q , Λ_f , K and θ_C change when the solid material temperature and the neutron population change with time, for example, when the neutron population is redistributed in space or when the reactor composition is altered deliberately. The first basic assumption of this study is:

Assumption A. The parameters C_R , Λ_f , K , and θ_C may vary with time, but their time variation is not related to the time variation of temperature $\theta_R(t)$ and neutron number $N(t)$. The parameter q is independent of time; in particular, q does not fluctuate. The analysis of this section and the following Section III is not affected by such time variations of C_R , Λ_f , K , and θ_C , and the time

dependence of these parameters will not be indicated symbolically. Assumption A is especially crude in treating the coolant temperature θ_C as independent of the temperature $\theta_R(t)$ of the solid portions of the reactor; the flow of coolant through the reactor is considered to be so large that the coolant can be regarded as an infinite heat reservoir at the known temperature θ_C . It may be observed from the subsequent analysis that the assumption of fixed q is made for convenience and is not essential.

The temperature $\theta_R(t)$ and neutron number $N(t)$ will be regarded as random variables. The variable $N(t)$ can take on discrete values 0, 1, 2, ..., and it is convenient to replace $\theta_R(t)$ by a discrete variable $\theta(t)$ defined by

$$\theta(t) \equiv \frac{C_R \theta_R(t)}{q} ; \quad \theta(t) = 0, 1, 2, \dots \quad (\text{II}, 2)$$

The continuous variable $C_R \theta_R(t)/q$ is very large, e.g. 10^{19} when $C_R \theta_R$ is 3.2×10^5 kw-sec and q is 3.2×10^{-14} kw-sec, and the discrete variable $\theta(t)$ is a close approximation to this continuous variable. The discrete random variable

$\theta(t)$ can be regarded as describing the number of units of internal energy in the solid part of the reactor where each unit is of magnitude q . The treatment of internal energy as a collection of indistinguishable units, i.e., by particle statistics, distorts the ordinary thermodynamic fluctuation in internal energy. However when the number of energy units is very large, as in the present case, then the distortion is small compared with the fluctuations induced by the fluctuating number of fissions.

The deterministic Eq. (II,1) implies the following probabilistic statements:

If a fission occurs in δt , then θ certainly is increased by unity.

The probability that a particular energy unit is transferred from reactor to coolant in time δt is $K\delta t/C_R$. (II,3)

The probability that an energy unit is transferred from coolant to reactor in time δt is $K \theta_1 \delta t / C_R$, where $C_R \theta_0 / q$ is approximated by a known integer θ_1 .

We turn now to the reciprocal effect of reactor temperature on the neutron multiplication process. Let $\Lambda_C(t)N(t)\delta t$ represent the probable number of neutrons which are lost from the reactor in the time interval t to $t + \delta t$ by leakage or by non-productive capture. The second basic assumption of this study is:

Assumption B. The reactor temperature influences the multiplication process solely according to the relation

$$\Lambda_C(t) = \Lambda_{CO} - \bar{\nu} \Lambda_f \alpha [\theta_R(t) - \theta_{RO}] \quad (\text{II},4)$$

where α , θ_{RO} and $\bar{\nu}$ are independent of $N(t)$ and $\theta_R(t)$.

Here θ_{RO} is some chosen reference temperature, and $\Lambda_{CO}N(t)\delta t$ is the probable number of neutrons lost in time δt when $\theta_R(t)$ happens to equal θ_{RO} . The expected or mean number of neutrons (prompt or delayed) emitted in a fission is $\bar{\nu}$, and α will be shown to represent a temperature coefficient of reactivity. It is convenient to regard the reactivity $\rho(t)$ as a random variable defined by:

$$\rho(t) \equiv 1 - \frac{\Lambda_f + \Lambda_C(t)}{\bar{\nu} \Lambda_f} \quad (\text{II},5)$$

Throughout this study expected values of random variables will be denoted by bars above the variables. Inserting Eq. (II,4) into Eq. (II,5) and taking expected values we note:

$$\alpha = \frac{\partial \bar{\rho}}{\partial \bar{\theta}_R} = \frac{\partial \bar{\rho}(t)}{\partial \bar{\theta}_R(t)} \quad (\text{II},6)$$

The parameter α is both the instantaneous and the average temperature coefficient of reactivity when the coolant temperature is fixed. Here α has been chosen

to be independent of the temperature $\theta_R(t)$ of the solid portions of the reactor; this choice greatly simplifies the analysis. Again we note that $\bar{v}$, α , θ_{RO} , and Λ_{CO} may be arbitrary, but known, functions of time, and that this time-dependence will not be indicated symbolically. In terms of the random variable $\theta(t)$ defined by Eq. (II,2), Eq. (II,4) can be expressed:

$$\Lambda_C(t) = \Lambda_{CO} - \gamma [\theta(t) - \theta_0] \quad (\text{II},7)$$

where θ_0 is an integer approximating $C_R \theta_{RO}/q$, and by definition

$$\gamma \equiv \bar{v} \Lambda_f \frac{q}{C_R} \alpha \quad (\text{II},8)$$

III. DESCRIPTION OF THE STOCHASTIC PROCESS

Let $C_1(t)$, $\dots$, $C_j(t)$, $\dots$, $C_J(t)$ represent the numbers of delayed neutron precursors of types 1, $\dots$, j, $\dots$, J which are present in the reactor at time t. Precursors are classified according to mean decay constant; thus a precursor of j-th type is associated with a decay constant λ_j . Also it is supposed that a fission counter is present such that if a fission occurs in the reactor, then a pulse may be formed^{*} in the counter with conditional probability ϵ . Let $Z(t)$ represent the number of pulses formed in the counter between some initial time and time t.

We are now in a position to state the final basic assumption of this study.

Assumption C. The state of the counter-reactor system is described completely by the discrete random variables $N(t)$, $C_j(t)$, $\theta(t)$, $Z(t)$, each of which describes the present number of a population of indistinguishable units. Consider, for example, the neutron population. In a real situation the neutrons are distinguishable by their position and velocity, and the probability that a particular neutron produces fission is dependent on these distinguishing characteristics. According to Assumption C the neutrons are regarded as indistinguishable, and the probability that any particular neutron produces fission in the time δt is taken to be a common value $\Lambda_f \delta t$, where $\Lambda_f \delta t$ was introduced in Section II as an average probability for the reactor as a whole. Thus,

$$\Lambda_f \delta t = \frac{\text{Reactor's fission content in the interval } t \text{ to } t + \delta t}{\text{Reactor's neutron content at time } t}$$

^{*}A pulse formed in the counter may not be recorded by the associated amplifier-discriminator-recorder system. Such count "losses" are ignored here.

Similarly, the loss probability $\Lambda_c(t)\delta t$ of Eq. (II,7) will be regarded as common^{ts} to any neutron. Equation (II,3) already has been stated in a manner which is consistent with Assumption C. The applicability of Assumption C to a real reactor is discussed in Section VI.

Let $P(N, C_1, \dots, C_J, \theta, Z; t)$ represent the probability that at time t the reactor's neutron content is a particular integer N , that the reactor's type 1 precursor content is a particular integer C_1 , etc. Define a probability generating function (abbreviated p.g.f.) for the system.

$$F(x, \xi_1, \dots, \xi_J, y, z, t) \quad (III,1)$$

$$= \sum_{N=0}^{\infty} \sum_{C_1=0}^{\infty} \dots \sum_{C_J=0}^{\infty} \sum_{\theta=0}^{\infty} \sum_{Z=0}^{\infty} P(N, C_1, \dots, C_J, \theta, Z; t) x^N \xi_1^{C_1} \dots \xi_J^{C_J} y^{\theta} z^Z$$

Here $x, \xi_1, \dots, \xi_J, y, z$ and t are continuous variables.

We wish to study the time variation of the entire set $P(N, C_1, \dots, C_J, \theta, Z; t)$. This probability set determines the shape of the p.g.f. F in the space of the continuous variables $x, \xi_1, \dots, \xi_J, y, z$; conversely, from the shape of F , properties of the probability set can be determined. It will be shown that the p.g.f. $F(x, \xi_1, \dots, \xi_J, y, z, t)$ satisfies a certain linear partial differential equation, a solution of which provides the desired information.

The conventional derivation (Ref 1) of the F-equation will be developed in an ad hoc manner. Consider a precursor of j -th type which is known to be present at time t ; the p.g.f. associated with this information is $0 \cdot \xi_j^0 + 1 \cdot \xi_j^1 + 0 \cdot \xi_j^2 + \dots$ or just ξ_j . At time $t + \delta t$ the conditional probability that the precursor is still present is $1 - \lambda_j \delta t$, while the conditional probability that the precursor has decayed with formation of a neutron is $\lambda_j \delta t$; the p.g.f. associated with this information at $t + \delta t$ is $[1 - \lambda_j \delta t] \xi_j^1$ plus

$\lambda_j \delta t \xi_j^0 x^1$. These observations can be abbreviated as follows:

$$\text{If } \xi_j \text{ at } t, \text{ then } \xi_j + \lambda_j [x - \xi_j] \delta t \text{ at } t + \delta t . \quad (\text{III}, 2)$$

Now consider a particular internal energy unit which is known to be present at time t ; the associated p.g.f. is y^1 . At time $t + \delta t$ the conditional probability that this energy unit has been transferred to the coolant is $K\delta t/C_R$, according to Eq. (II,3). The resulting p.g.f. at time $t + \delta t$ is $y^0 K\delta t/C_R$ plus $[1 - K\delta t/C_R] y^1$, or

$$\text{If } y \text{ at } t, \text{ then } y + \frac{K}{C_R} [1 - y] \delta t \text{ at } t + \delta t . \quad (\text{III}, 3)$$

The fate of a neutron which is known to be present at time t is more complicated. In the time interval t to $t + \delta t$ the neutron may produce a fission with conditional probability $\Lambda_f \delta t$, in which case a pulse may be formed in the fission counter with conditional probability ϵ , an energy unit is added in the reactor according to Eq. (II,3), and neutrons and precursors may be formed. Let $p(N, C_1, \dots, C_J)$ represent the probability that N prompt neutrons and C_j precursors of type j are formed in a fission. Define a p.g.f. $v(x, \xi_1, \dots, \xi_J)$ by

$$v(x, \xi_1, \dots, \xi_J) = \sum_{N=0}^{\infty} \sum_{C_1=0}^{\infty} \dots \sum_{C_J=0}^{\infty} p(N, C_1, \dots, C_J) x^N \xi_1^{C_1} \dots \xi_J^{C_J} . \quad (\text{III}, 4)$$

Then the term of the p.g.f. at time $t + \delta t$ which is associated with production of a fission is $\Lambda_f \delta t y [1 - \epsilon + \epsilon z] v(x, \xi_1, \dots, \xi_J)$, a conclusion which is verified by writing out the terms of this p.g.f. If the number of energy units in the system at time t , $\theta(t)$, is a particular integer θ , then the probability that the original neutron has been lost by $t + \delta t$ and no fission has occurred is $[\Lambda_{co} - \gamma\theta + \gamma\theta_0] \delta t$ according to Eq. (II,7). Finally, the probability that

the original neutron is still present at time $t + \delta t$ is unity minus

$[\Lambda_f + \Lambda_{CO} - \gamma\theta + \gamma\theta_0] \delta t$. Combining the p.g.f.'s for all possibilities we find:

$$\begin{aligned} \text{If } \theta(t) = \theta, \text{ and } x \text{ at } t, \text{ then } x + [\Lambda_{CO} - \gamma\theta + \gamma\theta_0] [1 - x] \delta t \\ + \Lambda_f \left\{ y [1 - \epsilon + \epsilon z] v(x, \xi_1, \dots, \xi_J) - x \right\} \delta t \text{ at } t + \delta t \end{aligned} \quad (\text{III}, 5)$$

It will be assumed that a completely random neutron source is present; the probability that a neutron immigrates into the system in the time interval δt is $S\delta t$. Energy units also immigrate into the solid part of the reactor from the coolant with probability $K\theta_1\delta t/C_R$ in time δt , according to Eq. (II,3). The p.g.f.'s associated with these immigration processes in the time interval t to $t + \delta t$ are:

$$\begin{aligned} 1 + S [x - 1] \delta t \\ 1 + \frac{K\theta_1}{C_R} [y - 1] \delta t \end{aligned} \quad (\text{III}, 6)$$

In forming the p.g.f.'s of Eqs. (III,2), (III,3), (III,5), and (III,6) it has been assumed that δt is sufficiently small that one may ignore processes with probabilities of order $O[\delta t]^2$, e.g., immigration of two neutrons in δt .

We are now able to form the following basic relation:

$$\begin{aligned} \sum_{N=0}^{\infty} \sum_{C_1=0}^{\infty} \dots \sum_{C_J=0}^{\infty} \sum_{\theta=0}^{\infty} \sum_{Z=0}^{\infty} P(N, C_1, \dots, C_J, \theta, Z; t + \delta t) x^N \xi_1^{C_1} \dots \xi_J^{C_J} y^{\theta} z^Z = \\ \left\{ 1 + S [x - 1] \delta t \right\} \left\{ 1 + \frac{K\theta_1}{C_R} [y - 1] \delta t \right\} \sum_{N=0}^{\infty} \sum_{C_1=0}^{\infty} \dots \sum_{C_J=0}^{\infty} \sum_{\theta=0}^{\infty} \sum_{Z=0}^{\infty} \\ P(N, C_1, \dots, C_J, \theta, Z; t) \cdot \left\{ x + [\Lambda_{CO} - \gamma\theta + \gamma\theta_0] [1 - x] \delta t + \Lambda_f y [1 - \epsilon + \epsilon z] \right. \\ \left. v(x, \xi_1, \dots, \xi_J) \delta t - \Lambda_f x \delta t \right\}^N \cdot \left\{ \xi_1 + \lambda_1 [x - \xi_1] \delta t \right\}^{C_1} \dots \left\{ \xi_J + \lambda_J [x - \xi_J] \delta t \right\}^{C_J} \\ \left\{ y + \frac{K}{C_R} [1 - y] \delta t \right\}^{\theta} z^Z \end{aligned} \quad (\text{III}, 7)$$

The left side of Eq. (III,7) is the p.g.f. $F(x, \dots, z, t + \delta t)$. The summation in the right side of Eq. (III,7) has the functional form of the p.g.f. $F(x, \dots, z, t)$ but with the variables or p.g.f.'s $x, \xi_1, \dots, \xi_J, y$ replaced by the corresponding p.g.f.'s after the transition to $t + \delta t$ given by Eqs. (III,2), (III,3), and (III,5). The first two factors in the right side of Eq. (III,7) are the immigration p.g.f.'s of Eq. (III,6). Equation (III,7) can be demonstrated by considering singly the terms of the summation on the right and showing that the correct coefficient of $x^N \xi_1^{C_1} \dots \xi_J^{C_J} y^\theta z^Z$ is obtained on the left (Ref 1). When δt is very small the factors on the right of Eq. (III,7) can be expanded by the binomial theorem and multiplied. Ignoring terms of order $O[\delta t]^2$, we find:

$$\begin{aligned}
 F(x, \dots, z, t + \delta t) = & \sum_{N=0}^{\infty} \dots \sum_{Z=0}^{\infty} P(N, \dots, Z, t) x^N \xi_1^{C_1} \dots \xi_J^{C_J} y^\theta z^Z \\
 & \left\{ 1 + S [x - 1] \delta t + \frac{K\theta}{C_R} [y - 1] \delta t + \frac{N}{x} [\Lambda_{C_0} - \gamma\theta + \gamma\theta_0] [1 - x] \delta t \right. \\
 & + \frac{N}{x} \Lambda_{fY} [1 - \epsilon + \epsilon z] \cdot v(x, \dots, \xi_J) \delta t - N \Lambda_f \delta t \\
 & \left. + \frac{C_1}{\xi_1} \lambda_1 [x - \xi_1] \delta t + \dots + \frac{C_J}{\xi_J} \lambda_J [x - \xi_J] \delta t + \frac{\theta}{y} \frac{K}{C_R} [1 - y] \delta t \right\}.
 \end{aligned} \tag{III,8}$$

Returning to the definition of the p.g.f. F , Eq. (III,1), it is noted that F is unity at the unit point $x = 1, \xi_1 = 1, \dots, \xi_J = 1, y = 1, z = 1$ in the space of these variables. By the Weierstrass M-test the p.g.f. F is uniformly convergent for $|x| \leq 1, |\xi_1| \leq 1, \dots, |z| \leq 1$, and in this interval the series can be differentiated term by term. The left side of Eq. (III,8) is $F(x, \dots, z, t) + F_{,t}(x, \dots, z, t) \delta t + O[\delta t]^2$, where the subscript following a comma indicates a partial derivative. $F(x, \dots, z, t)$ also appears on the right side of Eq. (III,8) and is cancelled. The remainder of the terms on the right side of Eq. (III,8) involve F and its termwise partial derivatives, and dividing the result by δt we find as δt approaches zero:

$$\begin{aligned}
F_{,t} = & S [x - 1] F + \frac{K\theta_1}{C_R} [y - 1] F + [\Lambda_{CO} + \gamma\theta_0] [1 - x] F_{,x} \\
& + \Lambda_f \left\{ y [1 - \epsilon + \epsilon z] v(x, \dots, \xi_J) - x \right\} F_{,x} + \sum_{j=1}^J \lambda_j [x - \xi_j] F_{,\xi_j} \\
& + \frac{K}{C_R} [1 - y] F_{,y} - \gamma [1 - x] y F_{,xy} \quad . \quad (III,9)
\end{aligned}$$

The partial differential equation, Eq. (III,9), can be regarded as the fundamental equation describing the stochastic process. It is interesting to note that it is linear, although the process it describes usually is thought of as a non-linear process. The non-linearity affects the order of Eq. (III,9), not its degree. Thus if the temperature coefficient, α , were zero then the term in $F_{,xy}$ would not appear, while if α were proportional to temperature $\theta_R(t)$ then a term in $F_{,xyy}$ would appear (because a term in $N\theta^2$ would appear in the bracket on the right side of Eq. (III,8)). In the derivation of Eq. (III,9) no special assumptions have been required concerning the time dependence of S , K/C_R , θ_1 , Λ_{CO} , γ , and Λ_f , except that these parameters cannot depend on any random variable.

No attempt will be made to determine $F(x, \dots, z, t)$ as a solution of Eq. (III,9). Instead, equations relating expected values and moments of the random variables are derived from Eq. (III,9). First it is noted that population moments can be calculated directly from the p.g.f. F . For example, the expected number of neutrons at time t , $\bar{N}(t)$, the variance minus the expected value of the number of energy units at time t , $\mu_{\theta\theta}(t)$, and the covariance of the neutron number with the type 1 precursor number at time t , $\mu_{NC_1}(t)$, are defined and related to F by:

$$\begin{aligned}
\bar{N}(t) &\equiv \sum_{N=0}^{\infty} \cdots \sum_{R=0}^{\infty} P(N, \dots, R; t) N = F_{,x}(1, 1, \dots, 1, t) \\
\mu_{\theta\theta}(t) &\equiv \sum_{N=0}^{\infty} \cdots \sum_{R=0}^{\infty} P(N, \dots, R; t) [\theta - \bar{\theta}(t)]^2 - \bar{\theta}(t) \\
&= F_{,yy}(1, 1, \dots, t) - \bar{\theta}^2(t) \quad (\text{III},10) \\
\mu_{NC_1}(t) &\equiv \sum_{N=0}^{\infty} \cdots \sum_{R=0}^{\infty} P(N, \dots, R; t) [N - \bar{N}(t)] [c_1 - \bar{c}_1(t)] \\
&= F_{,x\xi_1}(1, 1, \dots, 1, t) - \bar{N}(t)\bar{c}_1(t) .
\end{aligned}$$

Equations (III,10) can be verified by reference to the definition of the p.g.f. F , Eq. (III,1). The unusual moments μ_{NN} , $\mu_{\theta\theta}$ and $\mu_{c_1c_1}$, which represent variances minus means, are introduced for convenience. Conversely, if Eq. (III,9) is operated on by $\partial/\partial x$, or $\partial^2/\partial x \partial y$, or some other differential operator, and then evaluated at the unit point $x = 1$, $\xi_1 = 1$, $\dots$, $z = 1$, the result can be expressed in terms of expected values $\bar{N}(t)$, $\bar{c}_j(t)$, $\bar{\theta}(t)$, $\bar{z}(t)$ and moments $\mu_{NN}(t)$, $\mu_{NC_j}(t)$, $\mu_{c_jc_{j'}}(t)$, $\mu_{\theta\theta}(t)$, etc. Before stating the results of such operations it is necessary to define factorial moments of the populations emitted in fission.

$$\bar{v}_0 \equiv \sum_{N=0}^{\infty} \cdots \sum_{C_J=0}^{\infty} p(N, \dots, C_J) N = v_{,x}(1, \dots, 1) \quad (\text{III},11)$$

$$\bar{v}_j \equiv \sum_{N=0}^{\infty} \cdots \sum_{C_J=0}^{\infty} p(N, \dots, C_J) C_J = v_{,\xi_j}(1, \dots, 1); j = 1, 2, \dots, J$$

$$v_{00} \equiv \sum_{N=0}^{\infty} \cdots \sum_{C_J=0}^{\infty} p(N, \dots, C_J) N [N - 1] = v_{,xx}(1, \dots, 1)$$

$$v_{jj'} \equiv \sum_{N=0}^{\infty} \cdots \sum_{C_J=0}^{\infty} p(N, \dots, C_J) [C_{j'} - \delta_{jj'}] = v_{,\xi_j \xi_{j'}}(1, \dots, 1) ,$$

where $j, j' = 1, 2, \dots, J$.

Equations (III,11) can be verified by reference to the definition of the p.g.f. $v(x, \dots, \xi_j)$, Eq. (III,4).

When Eq. (III,9) is operated on by first order operators $\partial/\partial x$, or $\partial/\partial \xi_j$, or $\partial/\partial y$, or $\partial/\partial z$, and evaluated at the unit point, we find:

$$\begin{aligned}\bar{N}_{,t} &= \left\{ \bar{v}_0 \Lambda_f - \Lambda_f - \Lambda_{CO} + \gamma [\bar{\theta} - \theta_0] \right\} \bar{N} + \sum_{j=1}^J \lambda_j \bar{C}_j + S + \gamma \mu_{N\theta} \\ \bar{\theta}_{,t} &= \Lambda_f \bar{N} - \frac{K}{C_R} [\bar{\theta} - \theta_1] \\ \bar{C}_{j,t} &= \bar{v}_j \Lambda_f \bar{N} - \lambda_j \bar{C}_j \\ \bar{Z}_{,t} &= \epsilon \Lambda_f \bar{N}\end{aligned}\tag{III,12}$$

When Eq. (III,9) is operated on by second order operators $\partial^2/\partial x^2$, $\partial^2/\partial x \partial \xi_j$, etc., and evaluated at the unit point, we find:

$$\begin{aligned}\mu_{NN,t} &= 2 \left\{ \bar{v}_0 \Lambda_f - \Lambda_f - \Lambda_{CO} + \gamma [\bar{\theta} - \theta_0] \right\} \mu_{NN} + 2 \sum_{j=1}^J \lambda_j \mu_{NC_j} \\ &\quad + v_{00} \Lambda_f \bar{N} + 2\gamma [\bar{N} \mu_{N\theta} + \mu_{NN\theta}] \\ \mu_{C_j C_{j'},t} &= \bar{v}_j \Lambda_f \mu_{NC_{j'}} + \bar{v}_{j'} \Lambda_f \mu_{NC_j} - [\lambda_j + \lambda_{j'}] \mu_{C_j C_{j'}} + v_{jj'} \Lambda_f \bar{N} \\ \mu_{\theta\theta,t} &= 2 \Lambda_f \mu_{N\theta} - 2 \frac{K}{C_R} \mu_{\theta\theta}\end{aligned}\tag{III,13}$$

$$\begin{aligned}\mu_{NC_j,t} &= \left\{ \bar{v}_0 \Lambda_f - \Lambda_f - \Lambda_{CO} + \gamma [\bar{\theta} - \theta_0] - \lambda_j \right\} \mu_{NC_j} + \bar{v}_j \Lambda_f \mu_{NN} \\ &\quad + \sum_{j'=1}^J \lambda_{j'} \mu_{C_j C_{j'}} + v_{0j} \Lambda_f \bar{N} + \gamma [\bar{N} \mu_{\theta C_j} + \mu_{N\theta C_j}]\end{aligned}$$

$$\mu_{N\theta,t} = \left\{ \bar{v}_0 \Lambda_f - \Lambda_f - \Lambda_{CO} + \gamma [\bar{\theta} - \theta_0] \right\} \mu_{N\theta} + \Lambda_f \mu_{NN} \\ + \sum_{j=1}^J \lambda_j \mu_{\theta C_j} + \bar{v}_0 \Lambda_f \bar{N} + \gamma [\bar{N} \mu_{\theta\theta} + \bar{N} \bar{\theta} + \mu_{N\theta} + \mu_{N\theta\theta}]$$

$$\mu_{\theta C_j,t} = - \left[\lambda_j + \frac{K}{C_R} \right] \mu_{\theta C_j} + \bar{v}_j \Lambda_f \mu_{N\theta} + \Lambda_f \mu_{NC_j} + \bar{v}_j \Lambda_f \bar{N}$$

and:

$$\mu_{ZZ,t} = 2\epsilon \Lambda_f \mu_{NZ}$$

$$\mu_{NZ,t} = \left\{ \bar{v}_0 \Lambda_f - \Lambda_f - \Lambda_{CO} + \gamma [\bar{\theta} - \theta_0] \right\} \mu_{NZ} + \epsilon \Lambda_f \mu_{NN} \\ + \sum_{j=1}^J \lambda_j \mu_{ZC_j} + \epsilon \bar{v}_0 \Lambda_f \bar{N} + \gamma [\bar{N} \mu_{Z\theta} + \mu_{N\theta Z}]$$

(III,14)

$$\mu_{ZC_j,t} = - \lambda_j \mu_{ZC_j} + \bar{v}_j \Lambda_f \mu_{NZ} + \epsilon \Lambda_f \mu_{NC_j} + \epsilon \Lambda_f \bar{v}_j \bar{N}$$

$$\mu_{\theta Z,t} = - \frac{K}{C_R} \mu_{\theta Z} + \epsilon \Lambda_f \mu_{N\theta} + \Lambda_f \mu_{NZ} + \epsilon \Lambda_f \bar{N}$$

The third order moments appearing in Eqs. (III,13) and (III,14) represent central moments, e.g., $\mu_{N\theta C_1}$ is the expected value of $[N - \bar{N}] [\theta - \bar{\theta}] [C_1 - \bar{C}_1]$.

In the remainder of this study the moment equations, Eqs. (III,12), (III,13), and (III,14) are investigated, and the fundamental Eq. (III,9) is ignored. This is unfortunate from two points of view. First, Eq. (III,9) contains much more information than can be provided by first and second order moments. Second, when the temperature coefficient is not zero the moment equations, Eqs. (III,12), (III,13), and (III,14), cannot be solved exactly, because with the appearance of

third order central moments $\mu_{NN\theta}$, $\mu_{N\theta\theta}$, etc., there are more unknowns than equations. Similarly, the equations yielding the time derivatives of third order central moments involve fourth order central moments, etc. This difficulty is circumvented in Section V by assuming that the third order moments can be ignored in Eqs. (III,13) and (III,14).

It is interesting to note that the covariance $\mu_{N\theta}$ appears in Eq. (III,12). Consider a particular reactor with neutron number $N(t)$, temperature $\theta_R(t)$, and precursor number $C_j(t)$. Balancing first order probabilities of neutron gain or loss in the time interval t to $t + \delta t$:

$$N(t + \delta t) - N(t) = \left\{ \bar{v}_0 \Lambda_f - \Lambda_f - \Lambda_{CO} + \bar{v} \Lambda_f \alpha [\theta_R(t) - \theta_{R0}] \right\} N(t) \delta t \\ + \sum_{j=1}^J \lambda_j C_j(t) \delta t + S \delta t$$

Now suppose the time average of this relation is obtained. The time average $\overline{\theta_R(t)N(t)}$ appears, and this average can be expressed as

$$\overline{\theta_R(t)N(t)} = \overline{\theta_R(t)} \overline{N(t)} + [\overline{\theta_R(t) - \overline{\theta_R(t)}}] [\overline{N(t) - \overline{N(t)}}]$$

By the ergodic hypothesis discussed in Section I the second term on the right is to be identified with $\mu_{N\theta}(t)$. It is inferred that the appearance of second order moments in first order moment equations such as Eq. (III,12) is a natural consequence of the non-linear character of the process.

IV. STATIONARY REACTOR WHEN THE TEMPERATURE COEFFICIENT IS ZERO

The reactor will be said to be stationary if the probability that N neutrons, C_j precursors of j -th type, and θ energy units are present is independent of time, i.e.,

$$\frac{\partial}{\partial t} \sum_{Z=0}^{\infty} P(N, C_1, \dots, C_J, \theta, Z; t) = 0. \quad (\text{IV},1)$$

The probability that Z pulses have been formed in the counter since some initial time changes with time; thus the entire counter-reactor system is not stationary. A necessary, though not sufficient, condition that Eq. (IV,1) is satisfied is that $\bar{N}$, $\bar{C}_j$, θ , μ_{NN} , $\mu_{C_j C_j}$, $\mu_{\theta\theta}$, μ_{NC_j} , $\mu_{N\theta}$ and $\mu_{C_j\theta}$ are independent of time. Henceforth we shall be concerned solely with stationary reactors. It will be assumed that all parameters other than moments and γ which appear in Eqs. (III,12), (III,13), and (III,14) are time-independent and non-negative;* equivalently, the reactor composition is fixed, the coolant temperature is fixed, and the neutron source is steady. For simplicity it is assumed that only one type of precursor is present.

Now suppose that the temperature coefficient is zero, i.e., α and γ vanish. Eliminating $\bar{N}$ from the second of Eqs. (III,12) by means of the first and third of these equations, we find that if the reactor is stationary, then

$$\bar{\theta} = \theta_1 + \frac{C_R}{K} \Lambda_f \bar{N} = \theta_1 - \frac{C_R}{K} \Lambda_f \frac{S}{\bar{v} \Lambda_f - \Lambda_f - \Lambda_{CO}}. \quad (\text{IV},2)$$

Here $\bar{v}$ represents $\bar{v}_0 + \bar{v}_1$, and $\Lambda_{CO} \delta t$ is chosen to be the neutron capture probability at the temperature θ_R corresponding to $\bar{\theta}$, i.e., θ_0 is chosen to be $\bar{\theta}$. Assume that $\bar{\theta}$ is the solution of Eq. (IV,2), and suppose that $\bar{v} \Lambda_f - \Lambda_f - \Lambda_{CO}$

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*Factorial moments are always non-negative; thus v_{00} , v_{01} , v_{11} are non-negative.

is negative because $\bar{N}$ must be positive. Then from Eqs. (III,12)

$$\bar{N} = \frac{S}{\bar{v} \Lambda_f [-\bar{\rho}]}$$

$$\bar{\theta} = \theta_1 + \frac{C_R}{K} \Lambda_f \bar{N} \quad (\text{IV},3)$$

$$\bar{C}_1 = \Lambda_f \frac{\bar{v}_1}{\lambda_1} \bar{N}$$

where the expected reactivity $\bar{\rho}$, defined as $1 - [\Lambda_f + \Lambda_{CO}] / \bar{v} \Lambda_f$, has been introduced from Eq. (II,5). Similarly, if the reactor is stationary the algebraic Eqs. (III,13) can be solved for the constant second moments.

$$\mu_{NN} = \frac{\frac{v_{00}}{2} - \frac{\lambda_1}{\Lambda_f \bar{v} \bar{\rho}} \left[\frac{v_{00}}{2} + v_{01} + \frac{v_{11}}{2} \right]}{\bar{v}_1 - \bar{v} \bar{\rho} + \lambda_1 / \Lambda_f} \bar{N}$$

$$\mu_{NC_1} = \frac{\Lambda_f}{\lambda_1} \left\{ [\bar{v}_1 - \bar{v} \bar{\rho}] \mu_{NN} - \frac{v_{00}}{2} \bar{N} \right\}$$

$$\mu_{C_1 C_1} = \frac{\Lambda_f^2}{\lambda_1^2} \bar{v}_1 [\bar{v}_1 - \bar{v} \bar{\rho}] \mu_{NN} + \frac{\Lambda_f}{\lambda_1} \left[v_{11} - \frac{\Lambda_f}{\lambda_1} \bar{v}_1 v_{00} \right] \bar{N} \quad (\text{IV},4)$$

$$\mu_{\theta\theta} = \frac{\left\{ \frac{1}{\bar{v}_1} - \bar{v} \bar{\rho} + \frac{\lambda_1}{\Lambda_f} + \frac{K}{C_R \Lambda_f} \right\} \mu_{NN} + \left\{ -\frac{v_{00}}{2} + \bar{v}_0 \frac{K}{C_R \Lambda_f} + \bar{v} \frac{\lambda_1}{\Lambda_f} \right\} \bar{N}}{\frac{K}{C_R \Lambda_f} \left\{ \frac{K}{C_R \Lambda_f} [\bar{v}_1 - \bar{v} \bar{\rho}] - \bar{v} \bar{\rho} \frac{\lambda_1}{\Lambda_f} \right\}}$$

$$\mu_{N\theta} = \frac{K}{C_R \Lambda_f} \mu_{\theta\theta}$$

$$\mu_{\theta C_1} = [\bar{v}_1 - \bar{v} \bar{\rho}] \frac{K}{C_R \lambda_1} \mu_{\theta\theta} - \frac{\Lambda_f}{\lambda_1} \mu_{NN} - \bar{v}_0 \frac{\Lambda_f}{\lambda_1} \bar{N}$$

These equations can be simplified when the parameters C_R , K , $\bar{\rho}$, Λ_f , etc., are chosen to be appropriate for a power reactor. Table I illustrates a possible set of parameters for a 32 Mw, U^{235} fueled reactor.

TABLE I
PARAMETERS OF A TYPICAL POWER REACTOR

$q = 3.2 \times 10^{-14}$ kw-sec/fission	$\bar{v}_0 = 2.47$ prompt neutrons/fission
$C_R = 10^3$ kw-sec/ $^{\circ}\text{C}$	$\bar{v}_1 = 0.017$ precursors/fission
$K = 10^3$ kw/ $^{\circ}\text{C}$	$\lambda_1 = 0.05 \text{ sec}^{-1}$
$\Lambda_f = 10^4 \text{ sec}^{-1}$	$v_{00} = 4.85$
$\Lambda_f \bar{N} = 10^{18}$ fissions/sec	$v_{01} = 0.042$
	$v_{11} = 0$

The single delayed neutron group at our disposal has been given average characteristics of all delayed neutrons emitted in fission, e.g., the mean decay time is 20 sec. The prompt neutron dispersion parameter v_{00} has been measured (Ref 4), but v_{01} and v_{11} are chosen arbitrarily. The choice $v_{01} = \bar{v}_0 \bar{v}_1 = 0.042$ corresponds to an assumption that there is no correlation between prompt neutron and precursor formation, while the choice $v_{11} = 0$ assumes that not more than one precursor is emitted per fission. Large neutron sources are present ordinarily in power reactors. Suppose, however, that $S \leq 2.49 \times 10^8$ neutrons/sec. Then from Eq. (IV,3) and the fission rate of Table I:

$$0 > \bar{\rho} > -10^{-10}. \quad (\text{IV},5)$$

For such values of reactivity and for parameters of order of magnitude of those given in Table I,

$$\bar{v} \bar{\rho} \ll \lambda_1 / \Lambda_f \ll \bar{v}_1 \quad (\text{IV},6)$$

$$\frac{K}{C_R \Lambda_f} \ll \bar{v}_1$$

Then from Eq. (IV,4), $\mu_{NN} \gg \bar{N}$, and combining Eqs. (IV,3) and (IV,4):

$$\frac{\mu_{NN}}{\bar{N}^2} \cong \frac{\mu_{C_1 C_1}}{\bar{C}_1^2} \cong \frac{\mu_{\theta\theta}}{[\bar{\theta} - \theta_1]^2} \cong \frac{\mu_{\theta_R \theta_R}}{[\bar{\theta}_R - \theta_C]^2} \cong \frac{\lambda_1}{\bar{v}_1} \left[\frac{v_{00}}{2} + v_{01} + \frac{v_{11}}{2} \right] \frac{1}{S}. \quad (\text{IV},7)$$

Here the real temperature θ_R of the solid portion of the reactor has been reintroduced from Eq. (II,2). Strictly speaking, the moments μ_{NN} , $\mu_{C_1 C_1}$ and $\mu_{\theta\theta}$ represent population variances minus expected values, but because $\mu_{NN} \gg \bar{N}$, $\mu_{C_1 C_1} \gg \bar{C}_1$, $\mu_{\theta\theta} \gg \bar{\theta}$, these moments closely approximate variances. Thus Eq. (IV,7) relates relative variances with $\lambda_1/\bar{v}_1$, v_{00} , v_{01} , v_{11} and the important parameter, the source strength S . It is suggested that the source strength appears as a measure of the reactivity required to attain a given power level. For the parameters illustrated in Table I, the relative standard deviations of neutron number, precursor number and temperature are approximately

$$\frac{\mu_{NN}^{1/2}}{\bar{N}} \cong \frac{\mu_{C_1 C_1}^{1/2}}{\bar{C}_1} \cong \frac{\mu_{\theta_R \theta_R}^{1/2}}{\bar{\theta}_R - \theta_C} \cong \frac{2.7}{S^{1/2}}. \quad (\text{IV},8)$$

According to Eq. (IV,7) the relative variance is sensitive to the choice of delayed neutron parameters $\bar{v}_1$ and λ_1 , and we may question the adequacy of the single precursor type model. On the other hand, if J precursor types are included then the solution of the $(J+1)(J+2)/2$ algebraic Eqs. (III,13) in as many unknowns presents a formidable problem. If $\bar{v}_1$ is held constant in Eq. (IV,7) while λ_1 is allowed to range over the entire time spectrum of precursor decay then the correct relative variance with J precursor types must be included in the resulting range of relative variances. Thus when λ_1 ranges from 4 sec^{-1} to 0.01 sec^{-1} , the relative standard deviation of Eq. (IV,8) ranges from $24/S^{1/2}$ to $1.2/S^{1/2}$, and the correct result is in this range. Thus Eq. (IV,8) is correct

as to order of magnitude of the fluctuations. It is noted that if delayed neutrons are assumed to be emitted promptly a much different result is obtained; in this case

$$\frac{\mu_{NN}^{1/2}}{\bar{N}} = \sqrt{\Lambda_f \frac{v_{00}}{2} \frac{1}{S}} = \frac{156}{S^{1/2}}$$

(IV,9)

$$\frac{\mu_{\theta\theta}^{1/2}}{\bar{\theta} - \theta_1} = \sqrt{\frac{K}{C_R} \frac{1}{\bar{v}_0} \left[1 + \frac{v_{00}}{2\bar{v}_0^2 [-\bar{\rho}_0]} \right] \frac{1}{S}} \approx \frac{10^5}{S^{1/2}}$$

These expressions can be obtained from Eqs. (IV,4) by letting $\bar{v}_1$, v_{01} and v_{11} vanish. It is evident that delayed neutrons moderate the fluctuations.

In the remainder of this section we consider the observability of the fluctuations. Suppose a thermometer with a very short time constant were in contact with the solid portion of the reactor. The thermometer response would fluctuate with $\theta_R(t)$. From Eqs. (IV,3) and (II,2), the data of Table I indicates a mean temperature difference of 32°C between coolant and fuel plates. Then, from Eq. (IV,8)

$$\mu_{\theta_R \theta_R}^{1/2} \cong 100/S^{1/2} \text{ } ^\circ\text{C} \quad . \quad (\text{IV},10)$$

The usual large reactor source strength of about 10^8 neutrons per second implies a small temperature fluctuation with standard deviation about 0.01°C, and such a fluctuation probably would not be observed. If for some reason the artificial source were removed from a U^{235} -fueled reactor, then background neutrons from spontaneous fission, cosmic rays, etc., would yield a source strength not much larger than 10^3 neutrons per second; in this case temperature fluctuations would be observable.

In Section III it was assumed that a fission counter was present. Suppose that at time zero the counter is turned on, i.e., $\bar{Z}(0)$, $\mu_{NZ}(0)$, $\mu_{C_1Z}(0)$, and $\mu_{ZZ}(0)$ are zero. From the last of Eqs. (III,12),

$$\bar{Z}(t) = \epsilon \Lambda_f \bar{N} t \quad (IV,11)$$

When the temperature coefficient is zero, Eqs. (III,14) can be solved readily by the method of Laplace transforms. Let $L(\mu_{ZZ})$ represent the Laplace transform of $\mu_{ZZ}(t)$ with respect to p . Then

$$L(\mu_{ZZ}) = \frac{2\epsilon^2 \Lambda_f^2}{p^2} \frac{[\mu_{NN} + \bar{v}_0 \bar{N}] [p + \lambda_1] + [\mu_{NC_1} + \bar{v}_1 \bar{N}] \lambda_1}{[p + \bar{v}_1 \Lambda_f - \bar{v} \bar{\rho} \Lambda_f] [p + \lambda_1] - \bar{v}_1 \lambda_1 \Lambda_f} \quad (IV,12)$$

Let p_1 and p_2 represent the real, negative (for $\bar{\rho} < 0$) roots of the equation

$$p^2 + p [\lambda_1 + \bar{v}_1 \Lambda_f - \bar{v} \bar{\rho} \Lambda_f] - \bar{v} \bar{\rho} \lambda_1 \Lambda_f = 0 \quad (IV,13)$$

Then inverting the Laplace transform of Eq. (IV,12)

$$\begin{aligned} \mu_{ZZ}(t) = 2\epsilon^2 \Lambda_f^2 [\mu_{NN} + \bar{v}_0 \bar{N}] & \left\{ \frac{t\lambda_1}{p_1 p_2} + \frac{[e^{p_1 t} - 1] [p_1 + \lambda_1]}{p_1^2 [p_1 - p_2]} + \frac{[e^{p_2 t} - 1] [p_2 + \lambda_1]}{p_2^2 [p_2 - p_1]} \right\} \\ & + 2\epsilon^2 \Lambda_f^2 [\mu_{NC_1} + \bar{v}_1 \bar{N}] \left\{ \frac{t\lambda_1}{p_1 p_2} + \frac{[e^{p_1 t} - 1] \lambda_1}{p_1^2 [p_1 - p_2]} + \frac{[e^{p_2 t} - 1] \lambda_1}{p_2^2 [p_2 - p_1]} \right\} \quad (IV,14) \end{aligned}$$

Define

$$B(pt) \equiv 1 + \frac{1 - e^{pt}}{pt} \quad (IV,15)$$

In terms of the B-function Eq. (IV,14) can be expressed

$$\begin{aligned}
\frac{\mu_{ZZ}(t)}{[\bar{Z}(t)]^2} = \frac{2}{\bar{N}t} \left[\frac{\mu_{NN}}{\bar{N}} + \bar{v}_0 \right] & \left\{ \frac{p_1 + \lambda_1}{p_1 [p_2 - p_1]} B(p_1 t) + \frac{p_2 + \lambda_1}{p_2 [p_1 - p_2]} B(p_2 t) \right\} \\
& + \frac{2}{\bar{N}t} \left[\frac{\mu_{NC1}}{\bar{N}} + \bar{v}_1 \right] \left\{ \frac{\lambda_1}{p_1 [p_2 - p_1]} B(p_1 t) + \frac{\lambda_1}{p_2 [p_1 - p_2]} B(p_2 t) \right\} . \quad (IV,16)
\end{aligned}$$

When the approximations of Eq. (IV,6) are assumed, the roots of Eq. (IV,13) can be approximated closely by

$$\begin{aligned}
p_1 &\cong - [\lambda_1 + \bar{v}_1 \Lambda_f - \bar{v} \bar{\rho} \Lambda_f] \quad - p_2 \cong - \bar{v}_1 \Lambda_f \\
p_2 &\cong \frac{\bar{v} \bar{\rho} \lambda_1 \Lambda_f}{\lambda_1 + \bar{v}_1 \Lambda_f - \bar{v} \bar{\rho} \Lambda_f} \cong \bar{v} \bar{\rho} \frac{\lambda_1}{\bar{v}_1} . \quad (IV,17)
\end{aligned}$$

Here p_1 and p_2 are negative, so the B-functions are monotone, increasing functions of t which approach unity for $t \gg |p|^{-1}$, while for $0 \leq t \ll |p|^{-1}$, the B-functions can be approximated by $-pt$. Now $|p_2|^{-1}$ is 10^{10} sec or longer, so for all practical purposes $B(p_2 t)$ can be approximated by $-p_2 t$. When the approximate values of p_1 and p_2 are substituted into Eq. (IV,16), and $\bar{N}$, μ_{NN} , μ_{NC} , are evaluated in terms of Eqs. (IV,3) and (IV,4) subject to the approximations of Eq. (IV,6), a simple result is obtained.

$$\frac{\mu_{ZZ}(t)}{[\bar{Z}(t)]^2} \cong \frac{\lambda_1}{\bar{v}_1} \frac{1}{S} [v_{00} + 2v_{01} + v_{11}] \left[1 - \frac{\bar{v} \bar{\rho} \lambda_1}{\bar{v}_1^2 \Lambda_f} \frac{B(-\bar{v}_1 \Lambda_f t)}{\bar{v}_1 \Lambda_f t} \right] . \quad (IV,18)$$

The time dependent part of Eq. (IV,18) is negligible for any time t . It may be recalled that the moment $\mu_{ZZ}(t)$ represents the variance $\sigma_{ZZ}^2(t)$ minus the mean

$\bar{Z}(t)$ of the count population. Thus:

$$\frac{\sigma_{ZZ}^2(t)}{[\bar{Z}(t)]^2} \cong \frac{1}{\bar{Z}(t)} + \frac{\lambda_1}{\bar{v}_1} \frac{1}{S} [v_{00} + 2v_{01} + v_{11}] \quad (IV,19)$$

The counter was turned on at time zero; suppose the counter is turned off at time t and the number $Z(t)$ of accumulated counts is recorded. If this process is repeated again and again a sample of accumulated counts is obtained. According to the ergodic hypothesis the unbiased estimate of variance which can be calculated from the sample, and the sample mean are related by Eq. (IV,19). When the mean number of accumulated counts $\bar{Z}(t)$ is sufficiently large the relative standard deviation is independent of counting time t and counter efficiency ϵ . Comparing Eq. (IV,19) with previous results we conclude that fluctuations in accumulated counts will be small unless the reactor source is weak; also, when count fluctuations are small then reactor temperature fluctuations are smaller.

V. STATIONARY REACTOR WHEN THE TEMPERATURE COEFFICIENT IS FINITE

In the preceding Section IV the stationary situation with a zero temperature coefficient was solved with some degree of completeness. When the temperature coefficient is not zero, complete solutions cannot be obtained from the moment equations, Eqs. (III,12) - (III,14), even for the stationary reactor condition. The following program will be developed. First, the moment equations will be "linearized" by the double assumption of small fluctuations and small temperature coefficient; the solution of these "linearized" equations is found to be valid for a power reactor only when the temperature coefficient is exceedingly small. The nature of the solution and its rapid variation with temperature coefficient suggest: (1) For sufficiently large, positive values of the temperature coefficient no stationary condition exists; (2) For sufficiently large, negative values of the temperature coefficient, fluctuations are drastically moderated. The first conclusion then is demonstrated without invoking the "linearizing" assumptions. The second conclusion is shown to be plausible, but it is not demonstrated. Usual values of power reactor temperature coefficients, $-10^{-4} \text{ }^{\circ}\text{C}^{-1}$ to $-10^{-6} \text{ }^{\circ}\text{C}^{-1}$, are found to be large and negative in the sense discussed above.

Consider an array of reactors with different temperature coefficients, but with identical fixed compositions, coolant temperatures and source strengths. One of these reactors has a zero temperature coefficient, and when $\bar{v} \Lambda_f - \Lambda_f - \Lambda_{CO}$ is negative then the reactor can be in a stationary condition with population moments $\bar{N}$, $\bar{\theta}$, $\bar{C}_1$, μ_{NN} , μ_N , etc. Assume that a reactor with a particular finite temperature coefficient α can be stationary with population moments $\bar{N} + \Delta\bar{N}$, $\bar{C}_1 + \Delta\bar{C}_1$, $\bar{\theta} + \Delta\bar{\theta}$, $\mu_{NN} + \Delta\mu_{NN}$, $\mu_{N\theta} + \Delta\mu_{N\theta}$, etc. In either case the population moments must satisfy Eqs. (III,12) and (III,13). Subtracting the equations

for the $\alpha = 0 = \gamma$ reactor from those for the reactor with finite temperature coefficient, and letting $\bar{\rho}$ represent the expected reactivity of the $\alpha = 0 = \gamma$ reactor, we find

$$0 = [\bar{v} \bar{\rho} - \bar{v}_1] \Lambda_f \Delta \bar{N} + \lambda_1 \Delta \bar{C}_1 + \gamma [\bar{N} + \Delta \bar{N}] \Delta \bar{\theta} + \gamma [\mu_{N\theta} + \Delta \mu_{N\theta}]$$

$$0 = \Lambda_f \Delta \bar{N} - K \Delta \bar{\theta} / c_R$$

$$0 = \bar{v}_1 \Lambda_f \Delta \bar{N} - \lambda_1 \Delta \bar{C}_1$$

$$0 = [\bar{v} \bar{\rho} - \bar{v}_1] \Lambda_f \Delta \mu_{NN} + \lambda_1 \Delta \mu_{NC_1} + \frac{v_{00}}{2} \Lambda_f \Delta \bar{N} \\ + \gamma \left\{ [\bar{N} + \Delta \bar{N}] [\mu_{N\theta} + \Delta \mu_{N\theta}] + [\mu_{NN} + \Delta \mu_{NN}] \Delta \bar{\theta} + \mu_{NN\theta} + \Delta \mu_{NN\theta} \right\}$$

$$0 = \bar{v}_1 \Lambda_f \Delta \mu_{NC_1} - \lambda_1 \Delta \mu_{C_1 C_1} + \frac{v_{11}}{2} \Lambda_f \Delta \bar{N}$$

$$0 = \Lambda_f \Delta \mu_{N\theta} - \frac{K}{c_R} \Delta \mu_{\theta\theta} \quad (V,1)$$

$$0 = [\bar{v} \bar{\rho} \Lambda_f - \bar{v}_1 \Lambda_f - \lambda_1] \Delta \mu_{NC_1} + \bar{v}_1 \Lambda_f \Delta \mu_{NN} + \lambda_1 \Delta \mu_{C_1 C_1} + v_{01} \Lambda_f \Delta \bar{N} \\ + \gamma \left\{ [\bar{N} + \Delta \bar{N}] [\mu_{\theta C_1} + \Delta \mu_{\theta C_1}] + [\mu_{NC_1} + \Delta \mu_{NC_1}] \Delta \theta + \mu_{N\theta C_1} + \Delta \mu_{N\theta C_1} \right\}$$

$$0 = [\bar{v} \bar{\rho} - \bar{v}_1] \Lambda_f \Delta \mu_{N\theta} + \Lambda_f \Delta \mu_{NN} + \lambda_1 \Delta \mu_{\theta C_1} + \bar{v}_0 \Lambda_f \Delta \bar{N} \\ + \gamma \left\{ [\bar{N} + \Delta \bar{N}] [\mu_{\theta\theta} + \Delta \mu_{\theta\theta} + \bar{\theta} + \Delta \bar{\theta}] + [\mu_{N\theta} + \Delta \mu_{N\theta}] + \mu_{N\theta} + \Delta \mu_{N\theta} + \mu_{N\theta\theta} + \Delta \mu_{N\theta\theta} \right\}$$

$$0 = - \left[\lambda_1 + \frac{K}{c_R} \right] \Delta \mu_{\theta C_1} + \bar{v}_1 \Lambda_f \Delta \mu_{N\theta} + \Lambda_f \Delta \mu_{NC_1} + \bar{v}_1 \Lambda_f \Delta \bar{N}$$

Now suppose that fluctuations are not large in the $\alpha = 0 = \gamma$ reactor in the sense that

$$\mu_{NN\theta} \ll \bar{N} \mu_{N\theta}, \mu_{N\theta C_1} \ll \bar{N} \mu_{\theta C_1}, \mu_{N\theta\theta} \ll \bar{N} \mu_{\theta\theta}, \mu_{N\theta} \ll \bar{N} \mu_{\theta\theta}, \quad (V,2)$$

and that the temperature coefficient γ is sufficiently small that changes in the moments are small compared with the moments themselves, i.e.,

$$\Delta \bar{N} \ll \bar{N}, \Delta \mu_{N\theta} \ll \mu_{N\theta}, \Delta \mu_{N\theta C_1} \ll \mu_{N\theta C_1}, \text{ etc.} \quad (V,3)$$

In this case the coefficients of γ in the 1st, 4th, 7th and 8th of Eqs. (V,1) are approximately $\bar{N} \Delta \bar{\theta} + \mu_{N\theta}$, $\bar{N} \mu_{N\theta}$, $N \mu_{\theta C_1}$ and $\bar{N} [\mu_{\theta\theta} + \bar{\theta}]$, respectively. Solutions to Eqs. (V,1) are presented below, first with no further approximations, and second with the approximations of Eq. (IV,6) which are appropriate for a power reactor.

$$\frac{\Delta \bar{N}}{\bar{N}} = \frac{\mu_{\theta\theta}}{[\bar{\theta} - \theta_1]^2 \left[\frac{SK}{\bar{N}^2 \Lambda_f C_R} \frac{1}{\gamma} - 1 \right]} \cong \frac{\lambda_1}{\bar{v}_1} \left[\frac{v_{00}}{2} + v_{01} + \frac{v_{11}}{2} \right] \frac{1}{S} \left/ \left[\frac{\alpha_C}{\alpha} - 1 \right] \right.$$

$$\frac{\Delta \bar{\theta}}{[\bar{\theta} - \theta_1]} = \frac{\Delta \bar{N}}{\bar{N}} \quad (V,4)$$

$$\frac{\Delta \mu_{NN}}{\mu_{NN}} = \frac{\Delta \bar{N}}{\bar{N}} + \frac{\gamma}{S} \frac{\bar{N}^2}{\mu_{NN}} \frac{\mu_{N\theta} \left[\frac{\lambda_1}{\Lambda_f} - \frac{\bar{v}}{\bar{\rho}} \right] + \frac{\lambda_1}{\Lambda_f} \mu_{\theta C_1}}{\bar{v}_1 + \frac{\lambda_1}{\Lambda_f} - \frac{\bar{v}}{\bar{\rho}}} \cong \frac{\Delta \bar{N}}{\bar{N}} + \frac{\alpha}{\alpha_C}$$

$$\frac{\Delta \mu_{\theta\theta}}{\mu_{\theta\theta}} = \frac{\Delta \mu_{N\theta}}{\mu_{N\theta}} \cong \frac{\Delta \mu_{NN}}{\mu_{NN}}$$

where by definition

$$\alpha_C \equiv \frac{KS}{\bar{v}_q [\bar{N} \Lambda_f]^2} \text{ } ^\circ C^{-1} \quad (V,5)$$

According to Eqs. (V,4), relative changes in moments are small when $|\alpha| \ll \alpha_C$, and only for such small temperature coefficients is the approximation valid. Regarding this limitation for the moment, Eqs. (V,4) suggest that: (1) As α increases from zero $\Delta\bar{N}/\bar{N}$, $\Delta\mu_{NN}/\mu_{NN}$, etc., increase without limit as α approaches α_C ; (2) As α decreases from zero $\Delta\bar{N}/\bar{N}$ and $\Delta\theta/[\bar{\theta} - \theta_1]$ remain small, of order $-S^{-1}$, while $\Delta\mu_{NN}/\mu_{NN}$, $\Delta\mu_{\theta\theta}/\mu_{\theta\theta}$ and $\Delta\mu_{N\theta}/\mu_{N\theta}$ approach -1 as α approaches $-\alpha_C$.

We now show that the first suggestion is valid in general. The first three of Eqs. (V,1) can be combined and rearranged to yield

$$0 = [\Delta\bar{N}]^2 + \Delta\bar{N} \bar{N} \left[1 - \frac{\alpha_C}{\alpha} \right] + \frac{K}{\Lambda_{fC_R}} [\mu_{N\theta} + \Delta\mu_{N\theta}] \quad (V,6)$$

When α is zero, Eqs. (IV,4) show that $\mu_{N\theta}$ is positive; this is reasonable because when α is zero, fission is the only mechanism relating the reactor's neutron content and internal energy, and this mechanism requires a positive correlation between $N(t)$ and $\theta(t)$. When α is positive a large value of $\theta(t)$ implies a small neutron capture probability, and this mechanism increases the positive correlation between $N(t)$ and $\theta(t)$. Thus when α is positive $\Delta\mu_{N\theta}$ must be positive; conversely, when α is negative $\Delta\mu_{N\theta}$ must be negative. These conclusions are borne out by the linearized solutions, Eqs. (V,4). When α is positive, the first and third terms of Eq. (V,6) are positive, so the second term is negative. If α is less than α_C , $\Delta\bar{N}$ must be positive and, as α approaches α_C , $\Delta\bar{N}$ must become very large. But when $\alpha = \alpha_C$ no real solution exists, and when α is greater than α_C , Eq. (V,6) requires $\Delta\bar{N}$ to be negative which is unreasonable in view of the mechanisms involved. It is concluded that a stationary condition cannot exist when $\alpha \geq \alpha_C$.

If the stochastic aspects of the problem are ignored, i.e., if all moments except first moments are set equal to zero in Eqs. (V,1) and (V,6), then it is easily shown that: (1) $\Delta\bar{N} = \Delta\bar{\theta} = \Delta\bar{C}_1 = 0$; (2) This singular point is

unstable when $\alpha \geq \alpha_C$, as is shown above for the stochastic process; (3) The singular point is stable when $\alpha < \alpha_C$, as is not shown for the stochastic process. It is possible that the stochastic process is not stationary for some range of temperature coefficients less than α_C .

A large negative temperature coefficient tends to remove the positive correlation between $N(t)$ and $\theta(t)$ which was set up by the fission mechanism. Thus as α decreases $\mu_{N\theta} + \Delta\mu_{N\theta}$ decreases, and because $\mu_{N\theta} + \Delta\mu_{N\theta}$ is proportional to $\mu_{\theta\theta} + \Delta\mu_{\theta\theta}$ the latter moment decreases also. The linearized solutions, Eqs. (V,4), indicate that fluctuations are moderated rapidly by an increasingly negative temperature coefficient.

The critical temperature coefficient defined by Eq. (V,5) depends strongly on the fission rate, $\bar{N}\Lambda_f$, of the $\alpha = 0 = \gamma$ reactor. For the reactor of Table I operating at 32 mw with $\bar{N}\Lambda_f = 10^{18} \text{ sec}^{-1}$ and $S \leq 10^8 \text{ sec}^{-1}$,

$$\alpha_C \cong \frac{S}{10^{20}} \text{ } ^\circ\text{C}^{-1} \leq 10^{-12} \text{ } ^\circ\text{C}^{-1} \quad (\text{V},7)$$

and for all practical purposes no stationary condition is possible if the temperature coefficient is positive. However, if this reactor were operated as a critical experiment at about 300 w with $\bar{N}\Lambda_f = 10^{13} \text{ sec}^{-1}$,

$$\alpha_C \cong \frac{S}{10^{10}} \text{ } ^\circ\text{C}^{-1} \leq 10^{-2} \text{ } ^\circ\text{C}^{-1} \quad (\text{V},8)$$

and if the source strength is sufficiently large a wide range of positive temperature coefficients can yield a stationary condition. The temperature coefficient under discussion represents the change in reactivity when the temperature of the solid, fueled portion of the reactor is raised by $1 \text{ } ^\circ\text{C}$; its value is usually of order $-10^{-4} \text{ } ^\circ\text{C}^{-1}$, and for most practical combinations of K , S and

$\bar{N}\Lambda_f$ this value is much more negative than $-\alpha_C$. Thus in most practical situations fluctuations are moderated drastically by the negative temperature coefficient. There is, however, one interesting situation for which α_C is very large compared with $10^{-4} \text{ }^\circ\text{C}^{-1}$, and for which the fluctuations are not affected by a temperature coefficient of this order of magnitude. This is just the situation for which controlled measurements of fluctuations have been made (Ref 5). In this case $\bar{N}\Lambda_f$ is of order 10^6 sec^{-1} to 10^9 sec^{-1} , and S is of order 10^2 sec^{-1} to 10^4 sec^{-1} , whence if $K = 10^3 \text{ kw}/^\circ\text{C}$

$$1 \text{ }^\circ\text{C}^{-1} < \alpha_C < 10^8 \text{ }^\circ\text{C}^{-1} \quad (\text{V},9)$$

The linearized solutions of Eqs. (V,4) are adequate in this case, and they indicate a negligible effect of temperature coefficient on fluctuations. It may be noted that the assumption that coolant temperature does not fluctuate with the temperature of the solid parts of the reactor (Assumption A, Section II) is applicable to low power critical experiments.

VI. SUMMARY AND DISCUSSION

It has been shown for a particular reactor model that fluctuations are moderated by the presence of delayed neutrons, a large neutron source, and a large negative temperature coefficient. In agreement with experience, stationary fluctuations in a power reactor have been found to be negligible when the neutron source strength is about 10^8 sec^{-1} and the temperature coefficient is about $-10^{-4} \text{ }^\circ\text{C}^{-1}$.

Fluctuations are not negligible in other cases such as some of those illustrated in Fig. 1. Here $\sigma_N/\bar{N}$, a measure of the fluctuation, is plotted versus power level for three source strengths and three temperature coefficients. The single precursor type model and the parameters of Table I are used in these calculations. The exact Eq. (IV,4) is employed for the zero temperature coefficient cases, while the linearized solution, Eq. (V,4), is shown in Fig. 1 for its approximate range of validity when the temperature coefficient is finite. Each curve in Fig. 1 describes stationary values of $\sigma_N/\bar{N}$ which would be attained if the reactor's power level were increased very slowly while the source strength and temperature coefficient remain fixed. It is supposed that the reactivity is altered, e.g., a control rod is withdrawn, so as to achieve the indicated power level. At each successive control rod position sufficient time is allowed for the reactor populations to adopt the stationary condition.

When the power level is very small the multiplication also is small; then the neutron probability distribution is approximately random (Poisson) with

$$\sigma_N \cong \bar{N}^{1/2} \text{ and}$$

$$\log \frac{\sigma_N}{\bar{N}} = \log \Lambda_f^{1/2} - \frac{1}{2} \log \bar{N} \Lambda_f \quad . \quad (\text{VI},1)$$

As the multiplication (power level) increases, $\sigma_N/\bar{N}$ becomes larger than indicated

Fig. 1

by Eq. (VI,1) as a result of fluctuations contributed by the multiplication process. When the temperature coefficient is zero, the source strength is fixed, and the multiplication is very large, $\sigma_N/\bar{N}$ becomes insensitive to further increases in multiplication; this result is common to any subcritical multiplicative process where population loss is balanced by immigration (Ref 1). Had delayed neutrons been neglected, the linear dependence of $\log \sigma_N/\bar{N}$ on $\log \bar{N}\Lambda_f$ at low multiplication would join smoothly with the constant $\log \sigma_N/\bar{N}$ at high multiplication; in Fig. 1 a single, short plateau is found instead at the juncture, and this plateau results from the single precursor type employed in the calculations. At low power levels a finite temperature coefficient, α , has little effect on $\sigma_N/\bar{N}$, but when $\bar{N}\Lambda_f$ becomes of order

$$\bar{N}\Lambda_f = 0 \left(\sqrt{\frac{KS}{\bar{v}q\alpha}} \right) \quad (\text{VI},2)$$

the finite temperature coefficient strongly affects $\sigma_N/\bar{N}$. Over an interval of a few decades in power level, $\sigma_N/\bar{N}$ changes from a value nearly unaffected by the temperature coefficient to a value incompatible with a stationary state if $\alpha > 0$, or to a (presumably) small value if $\alpha < 0$.

The results which are summarized in Fig. 1 have been obtained for a particular reactor model which is restricted by Assumptions A, B, and C of Sections II and III. Assumptions A and B require that reactor parameters such as Λ_f , Λ_{CO} , θ_C , q , etc. are independent of the random variables $N(t)$, $\theta_R(t)$, etc. In a real situation the coolant temperature, θ_C , must vary to some extent with $\theta_R(t)$, and the coolant temperature coefficient must affect the fluctuations. The remainder of Assumptions A and B seem to be adequate in certain situations. If the fluctuations are not large we are justified in regarding the temperature coefficient as independent of temperature. It is probable that fluctuations in K , C_R , θ_C , q

in which are independent of fluctuations in $\theta_R(t)$ do not affect the results to any great extent. Similarly, the assumption that Λ_f is fixed and all fluctuations in multiplication arise from fluctuations in Λ_c is adequate as a first approximation.

We have considered fluctuations in reactivity, $\rho(t)$, arising from fluctuations in solid temperature, $\theta_R(t)$; in view of the very small average reactivities, -10^{-10} to -10^{-16} , required for large, stationary power levels, are there other causes of fluctuations in $\rho(t)$? A vibrating control rod would have such an effect. Similarly, ordinary thermodynamic fluctuations in composition, density, etc., will affect $\rho(t)$; one such effect will be examined in detail.

It has been assumed that the coolant acts as an infinite heat reservoir, and we now assume that the coolant also acts as a constant pressure environment. In this case the solid portions of the reactor, e.g., the fuel plates and support structure, can be described by an isothermal, isobaric ensemble (Ref 6) when one ignores temperature fluctuations produced by fissions. From this ensemble the relative standard deviation of the fluctuating volume, $v(t)$, of the solid part of the reactor can be calculated (Ref 6).

$$\frac{\sigma_v}{\bar{v}} \cong \sqrt{\frac{k \theta_R K}{\bar{v}}} \quad (\text{VI}, 3)$$

Here k is Boltzmann's constant and K is the isothermal compressibility, about $10^{-12} \text{ cm}^2 \text{ dyne}^{-1}$ for most metals. For a typical reactor, θ_R might be 300°K to 400°K , while $\bar{v}$ might be about 10^5 cm^3 ; thus $\sigma_v/\bar{v}$ is of order 10^{-15} or 10^{-16} .

Isothermal fluctuations in volume of the solid portions of the reactor affect neutron leakage. For a simple dependence of reactivity on leakage,

$$\rho(t) = 1 - \frac{\Lambda_a}{\bar{v} \Lambda_f} \frac{1 + M^2(t) B^2(t)}{\quad} \quad (\text{VI}, 4)$$

where $\Lambda_a \delta t$ is the mean probability of neutron absorption in the absence of leakage, $M(t)$ is the fluctuating migration length, and $B^2(t)$ is the fluctuating buckling. To a first approximation $B^{-3}(t)$ is proportional to $v(t)$, while $M^3(t)$ varies less rapidly with $v(t)$ when, as is usually the case, the migration length is dependent primarily on the spatial average of the coolant density. Then, from Eq. (VI,4)

$$\alpha_p \approx \left[1 - \frac{\Lambda_a}{\bar{v} \Lambda_f} \right] \frac{2}{3} \frac{\sigma_v}{\bar{v}} \cong 10^{-16} \text{ or } 10^{-17} . \quad (\text{VI},5)$$

In this particular model, thermodynamic fluctuations affect the results of Sections IV and V when the average reactivity, $\bar{\rho}$, is of order -10^{-16} , i.e., when the source strength is very small and the power level is very high.

Assumption C, Section III, requires that the stochastic process can be analyzed in terms of $P(N, C_1, \dots, \theta, Z; t)$. In the language of statistical mechanics, such a probability distribution represents a grand ensemble where, for some reason, the distribution of representative points in the phase space of each petit ensemble ($N, C, \dots, Z$ chosen to be particular integers) is non-fluctuating and the same for each petit ensemble. Such an approximation is applicable if the neutron, precursor, etc., content of the reactor fluctuates as a whole, i.e., each fluctuation represents a magnification or demagnification of the spatial distributions. There is some evidence that this is approximately the case when the multiplication is large; however, the assumption remains unverified.

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