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ON THE STRUCTURE OF THE
ISOTROPIC TRANSPORT OPERATORS IN
THREE INDEPENDENT SPACE VARIABLES

by

Erwin H. Bareiss and
Ibrahim K. Abu-Shumays

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Applied Mathematics Division

May 1967

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ON THE STRUCTURE OF THE ISOTROPIC TRANSPORT OPERATORS IN THREE INDEPENDENT SPACE VARIABLES

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ABSTRACT

Based on the idea of separation of variables, a spectral theory for the three-dimensional, stationary isotropic transport operator is presented in a vector space of complex-valued Borel functions resulting in continuous sets of regular and generalized eigenfunctions. Because of the non-self-adjoint nature of this operator, the results could not be anticipated intuitively from the known decomposition of this operator in the special case of plane geometry. The results obtained indicate a promising new approach to the analytic solution of the linear transport equation in higher space dimensions. This work summarizes and extends a previous Argonne National Laboratory Report (ANL-6914). Examples are given for slab geometry with and without axial symmetry, spherical symmetry, and cylindrical symmetries.

I. INTRODUCTION

Ever since Wiener and Hopf²² discovered a powerful method for the analytic solution of certain problems in radiative transfer, the search for more general, more powerful, and perhaps simplified methods has not ceased. In 1960, Case¹² adapted a method by Davison¹⁷ and Van Kampen,²⁴ based on dispersion theory, to the stationary isotropic transport operator in one space dimension with axial symmetry in the angular distribution of the solution space. The method has been extended to anisotropic scattering and multicell and multigroup problems.^{7,19,30,35-37,41-44,50} Mathematically, finding expansion modes of an operator is related to the spectral theory of the operator. Since the linear transport operator is the sum of several linear operators, different approaches to a spectral theory are possible and have been attempted.^{1,4,5,10,18,34,39,46-49} The introduction of the Dirac δ -function as in Ref. 12 to construct the singular modes has symbolic significance, since the nature of this δ -function in the solution of the transport equation is not obvious. However, it can be demonstrated⁴ that this δ -function has its origin in the application of Cauchy's second theorem when a variable moves from the inside of a closed contour across to the outside.

The use of elementary functions to construct solutions for more than one space dimension, if only for special geometries, was attempted in Refs. 25, 35, 36, and 38. Since often the continuous part of the spectrum was overlooked, at most asymptotically correct answers were found. On the

following pages, a decomposition of the transport operator in three space dimensions is carried out. The general idea is that of Refs. 12 and 17, for the senior author was unable to generalize his ideas in Ref. 4. The reason is that the transport operator is not self-adjoint, and therefore no general spectral theory has existed for it, as of today. The results of the expansion theorem proved rather surprising and could not be anticipated by intuition. On the other hand, the known expansion theorem for plane geometry with its much simpler structure is a special case of the general theory. The method used here is suggested by the principle of separation of variables, which is widely used for the decomposition of linear partial differential equations of mathematical physics. The solution space for the transport Eq. (1.4) below is mathematically defined as a set of complex-valued Borel functions. To attempt completeness, it was necessary to add to the set of regular eigenfunctions, $\phi_{\Lambda}^0(\Omega)$, and the corresponding regular expansion modes $\psi_{\Lambda}^0(r, \Omega)$ over a complex continuum, a set of generalized eigenfunctions, $\phi_{\Lambda, \Omega_0}(\Omega)$, and the corresponding generalized expansion modes $\psi_{\Lambda, \Omega_0}(r, \Omega)$ over a complex continuum of even higher dimension.

The regular eigenfunctions are characterized by a complex vector, which we may call a regular (or proper) eigenvector, Λ . These eigenvectors, however, have to satisfy the condition $\Lambda \cdot \Lambda = \lambda_0^2$, where λ_0^2 assumes exactly one value for any given equation; therefore $\pm\lambda_0$ can be called eigenvalues. To obtain the generalized eigenfunctions, it is necessary to evaluate the improper integral which occurs in Eq. (2.13) below. Its regular form as it appears in (2.9) can be reduced to integrals with known analytic expressions, but its improper form and its interpretation requires careful analysis in order to formulate the decomposition theorem. Essentially, the "difficult" integrals can be considered Cauchy principal-value integrals of higher dimension. Although principal-value integrals have been considered extensively for problems in potential theory,^{20,45} the integrals encountered here do not quite fit their pattern. What may make the following pages interesting is the fact that most results are not quite what intuition at first would suggest.

A basic result of these research efforts is the general solution to the transport Eq. (1.4) below in a sense analogous to that of the theory of differential equations. Special representations are obtained by imposing geometrical restrictions on the general solution. They enable the numerical analyst to construct exact test problems for checking the performance of general multidimensional transport codes, such as described in Refs. 3, 9, 11, 18, 23, 29, and 33, and therefore can be most helpful in error analysis. The calculation of singular integrals is not difficult if properly executed.⁶ There is little doubt that the results of the following pages can be generalized to anisotropic scattering, multi-region, and multigroup problems. A similar theory can also be conceived for the discrete ordinate approximations; their spectrum, modes, and general solutions can then be compared with the continuous case presented here to yield a measure of fitness for the approximate solutions.

Many papers on the spectral representation of the time-dependent equation give a general expansion in exponentials, although not much is said on the structure of the time-independent coefficient functions, which very often satisfy an equation of type (1.4). This gap is filled here.

We turn now to the description of the operator. Given is the transport equation for constant energy

$$\frac{\partial \psi}{\partial t} + v\Omega \cdot \nabla \psi + v\sigma\psi = cv\sigma \int_{\{\Omega'\}} f(\Omega \cdot \Omega') \psi(r, \Omega', t) d_2 \Omega' + Q(r, \Omega, v, t), \quad (1.1)$$

where

$\psi = \psi(r, \Omega, t)$ is the directional flux, a scalar function;

t is the time;

r is the position vector;

v is a constant velocity in the direction of the unit vector Ω ;

σ is the total macroscopic cross section at velocity v ;

c is the net number of particles produced per collision
 $\left(= \frac{\sigma_s + v\sigma_f}{\sigma} \right)$;

f is the scattering function;

Q is the production of particles by sources;

and

$\nabla \equiv \text{grad}$, operates with respect to r only.

The scattering function is so normalized that*

$$\int f(\Omega \cdot \Omega') d_2 \Omega' = 1. \quad (1.2)$$

We investigate the decomposition of Eq. (1.1) under the following simplifying assumptions:

(a) $f = 1/(4\pi)$; i.e., we have isotropic scattering;

(b) σ is constant;

(c) $Q(r, \Omega, v, t) = 0$;

(d) $\partial \psi / \partial t \equiv 0$ (stationary case).

(1.3)

*Subsequently, the operator $\int \cdot d_2 \Omega'$ will mean integration over the entire unit sphere in the sense defined in Sections III and IV.

Let the mean free path, $1/\sigma$, be introduced as unit length. This is equivalent to setting $\sigma = 1$ in Eq. (1.1). Since $v \neq 0$, Eq. (1.1) reduces to

$$\Omega \cdot \nabla \psi + \psi - \frac{c}{4\pi} \int_{\{\Omega'\}} \psi(r, \Omega') d_2 \Omega' = 0. \quad (1.4)$$

If assumption (d) does not hold, one often lets $\psi = e^{-\alpha t} \psi'$, which leads to a stationary equation with

$$\sigma' = \sigma \beta, \quad c' = c/\beta, \quad \beta = 1 - \alpha/\sigma v.$$

When α is taken complex, σ' and c' are complex, and the results of this report need appropriate modification.

II. SEPARATION OF VARIABLES AND THE CHARACTERISTIC EQUATION

We look for all solutions that have the form

$$\psi_{\Lambda}(r, \Omega) = R_{\Lambda}(r) \Phi_{\Lambda}(\Omega). \quad (2.1)$$

Substitution of Eq. (2.1) into Eq. (1.4) yields

$$-\frac{\Omega \cdot \nabla R_{\Lambda}}{R_{\Lambda}} = 1 - \frac{c}{4\pi \Phi_{\Lambda}} \int \Phi_{\Lambda}(\Omega') d_2 \Omega' \quad (2.2)$$

under the assumption $R_{\Lambda} \neq 0$, $\Phi_{\Lambda} \neq 0$. Since $\nabla R_{\Lambda}/R_{\Lambda}$ does not depend on Ω by definition (2.1) and does not depend on r by (2.2), it must be proportional to a parameter vector, $-\Lambda$, which will be in general a complex vector,

$$\nabla R_{\Lambda}/R_{\Lambda} = -\Lambda, \quad (2.3)$$

whence

$$R_{\Lambda} = \text{const} \cdot e^{-\Lambda \cdot r}. \quad (2.4)$$

Therefore, the right-hand side of Eq. (2.2) is equal to $\Omega \cdot \Lambda$, and Eq. (2.2) takes the form

$$(1 - \Omega \cdot \Lambda) \Phi_{\Lambda} - \frac{c}{4\pi} \int \Phi_{\Lambda}(\Omega') d_2 \Omega' = 0. \quad (2.5)$$

We normalize Φ_{Λ} such that

$$\int \Phi_{\Lambda}(\Omega') d_2 \Omega' = \kappa, \quad (2.6)$$

where κ is a conveniently chosen, but fixed normalization constant (e.g., $\kappa = 1$ or $4\pi/c$). Under the condition that

$$\Omega \cdot \Lambda \neq 1 \quad \text{for all } \Omega, \quad (2.7)$$

the solution of (2.5) satisfying Eq. (2.6) is denoted by Φ_{Λ}^0 and is

$$\Phi_{\Lambda}^0 = \frac{\kappa c}{4\pi} \frac{1}{1 - \Omega \cdot \Lambda}. \quad (2.8)$$

Substitution of Eq. (2.8) in Eq. (2.6) yields a condition for Λ ; namely, the *characteristic equation*,

$$\boxed{1 - \frac{c}{4\pi} \int \frac{d_2 \Omega}{1 - \Omega \cdot \Lambda} = 0.} \quad (2.9)$$

Definition 2.1.

The set of solutions of Eq. (2.9) satisfying Eq. (2.7) is denoted by $\{\Lambda^0\}$. An element $\Lambda \in \{\Lambda^0\}$ is called a *regular eigenvector*. The corresponding functions Φ_{Λ}^0 are called *regular eigenfunctions* of Eq. (2.5).

Hence, we can write for Eq. (2.1)

$$\psi_{\Lambda}^0 = e^{-\Lambda \cdot r} \Phi_{\Lambda}^0(\Omega), \quad \Lambda \in \{\Lambda^0\}, \quad (2.10)$$

and Φ_{Λ}^0 is given by Eq. (2.8). The function $\psi_{\Lambda}^0(r, \Omega)$ has been so normalized that $\psi_{\Lambda}^0(0, \Omega) = \Phi_{\Lambda}^0(\Omega)$.

Consider now the case where, for a given Λ , Eq. (2.7) is *not* satisfied. Then, at certain directions $\Omega = \Omega_0$, Φ_Λ in Eq. (2.8) is unbounded, and the meaning of the integral (2.6) must be redefined. Assuming this is done, we then obtain another characteristic equation with possible solutions for Λ , and a set of elementary solutions corresponding to Eq. (2.10), but unbounded in Ω . These functions are, however, a subset of an even wider class of elementary solutions of the general form (2.1), when we extend the class of admissible functions to be generalized functions. We extend our function space by the set of functions $\delta(\Omega - \Omega_0)$ having the following definition.

Definition 2.2.

Given any function $G(\Omega)$, which is continuous with respect to Ω , $|\Omega| = 1$, in a neighborhood $D(\Omega)$ of a given direction Ω_0 , then $\delta(\Omega - \Omega_0)$ is defined by

$$\int_{D(\Omega)} G(\Omega) \delta(\Omega - \Omega_0) d_2\Omega = \begin{cases} G(\Omega_0) & \text{for } \Omega_0 \in D(\Omega) \\ 0 & \text{for } \Omega_0 \notin D(\Omega). \end{cases}$$

Now, we introduce the *improper eigenfunctions* (generalized eigenfunctions) as

$$\Phi_{\Lambda, \Omega_0} = \frac{\kappa c}{4\pi} \frac{1}{1 - \Omega \cdot \Lambda} + \kappa G \delta(\Omega - \Omega_0), \quad (2.11)$$

where the *improper eigenvectors* Λ are such that

$$\Lambda \in \{\Lambda: 1 - \Omega_0 \cdot \Lambda = 0\}, \quad (2.12)$$

and G is a function possibly of Λ and Ω . Substitution of Eq. (2.11) into Eq. (2.5) under consideration of Eq. (2.6), and integration with respect to Ω over the unit sphere, show that the Φ 's of Eq. (2.11) are indeed weak solutions. The function G is defined by substituting Eq. (2.11) into Eq. (2.6). Hence

$$\boxed{G = 1 - \frac{c}{4\pi} \int \frac{d_2\Omega}{1 - \Omega \cdot \Lambda}} \quad (2.13)$$

and is a function of Λ only. An elementary solution corresponding to Eq. (2.11) is

$$\psi_{\Lambda, \Omega_0} = e^{-\Lambda \cdot r} \phi_{\Lambda, \Omega_0}, \quad (2.14)$$

where ϕ_{Λ, Ω_0} is given by Eq. (2.11), and Λ is an element from the set (2.12). The function ψ_{Λ, Ω_0} is so normalized that $\psi_{\Lambda, \Omega_0}(0, \Omega) = \phi_{\Lambda, \Omega_0}(\Omega)$.

It is seen from (2.5) and is readily verified from (2.6) to (2.9) and from (2.11) to (2.13) that if Λ is a (regular or improper) eigenvector, $-\Lambda$ is also an eigenvector with

$$\phi_{-\Lambda}^0(\Omega) = \phi_{\Lambda}^0(-\Omega), \quad \phi_{-\Lambda, -\Omega_0}(\Omega) = \phi_{\Lambda, \Omega_0}(-\Omega),$$

and if c is real, $\bar{\Lambda}$ is also an eigenvector with

$$\phi_{\bar{\Lambda}}^0(\Omega) = \overline{\phi_{\Lambda}^0(\Omega)}, \quad \phi_{\bar{\Lambda}, \Omega_0}(\Omega) = \overline{\phi_{\Lambda, \Omega_0}(\Omega)}$$

as corresponding eigenfunctions.

Any linear combinations of solutions of the forms of Eqs. (2.10) and (2.14) are again solutions of the linear Eq. (1.4). Therefore the general solution of Eq. (1.4), which can be represented by a linear combination of the regular expansion modes $\psi_{\Lambda}^0(r, \Omega)$ of Eq. (2.10) and the improper expansion modes $\psi_{\Lambda, \Omega_0}(r, \Omega)$ of Eq. (2.14), is given by

$$\psi(r, \Omega) = \int_{\{\Lambda^0\}} A^0(\Lambda) \psi_{\Lambda}^0 d_4 \Lambda + \int_{\{\Omega_0\}} d_2 \Omega_0 \int_{\{\Lambda: \Omega_0 \cdot \Lambda = 1\}} d_4 \Lambda A(\Lambda, \Omega_0) \psi_{\Lambda, \Omega_0}, \quad (2.15)$$

where $\{\Lambda^0\}$ is given by Eq. (2.9), and $\{\Omega_0\}$ is the set of all real directions. The first and second terms on the right-hand side of Eq. (2.15) are given explicitly below by Eqs. (5.12) and (6.9), respectively. Another representation is given in (7.19).

Conjecture.

Every solution of the homogeneous Eq. (1.4) in the class of functions that are integrable* with respect to Ω and continuous in $|r| < \infty$ have a representation of the form of Eq. (2.15).

The following sections will be specific as to the meaning of integration with respect to Ω and Λ .

*See footnote on p. 7.

III. INTEGRATION OF $(1 - \Omega \cdot \Lambda)^{-1}$

A. Geometrical Considerations

To evaluate Eqs. (2.9) and (2.13), a specific meaning is required for

$$I_{\Lambda} = \int \frac{d_2 \Omega}{1 - \Omega \cdot \Lambda}. \quad (3.1)$$

This integral is invariant under the rotation group R of Ω . The complex vector Λ can be represented as the sum of a real vector $\Lambda_1 \in E_3$ and an imaginary vector $i\Lambda_2$ with $\Lambda_2 \in E_3$; that is,

$$\Lambda = \Lambda_1 + i\Lambda_2. \quad (\text{Definition}) \quad (3.2)$$

The integral I_{Λ} is therefore only a function of

$$\left. \begin{aligned} \lambda_1 &= |\Lambda_1|; \\ \lambda_2 &= |\Lambda_2|; \\ \gamma &= \cos^{-1}[(\Lambda_1 \cdot \Lambda_2)/(\lambda_1 \lambda_2)]. \end{aligned} \right\} \quad (\text{Definition}) \quad (3.3)$$

It turns out to be convenient to choose Cartesian coordinates such that

$$\left. \begin{aligned} \Lambda_1 &= \lambda_1 \{0, 0, 1\}; \\ \Lambda_2 &= \lambda_2 \{\sin \gamma, 0, \cos \gamma\}; \\ \Omega &= \{\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta\}. \end{aligned} \right\} \quad (3.4)$$

This means that Λ_1 lies on the z -axis, Λ_2 lies in the x, z -plane, and Ω is a point on the unit sphere, with θ and ϕ representing spherical coordinates as illustrated in Fig. 3.1.

We define* the inner product $\Omega \cdot \Lambda = (\Omega, \Lambda)$ to be

$$\left. \begin{aligned} \Omega \cdot \Lambda &= \Omega \cdot \Lambda_1 + i\Omega \cdot \Lambda_2; \\ \Omega \cdot \Lambda_1 &= \lambda_1 \cos \theta; \\ \Omega \cdot \Lambda_2 &= \lambda_2 (\cos \phi \sin \theta \sin \gamma + \cos \theta \cos \gamma). \end{aligned} \right\} \quad (3.5)$$

*Note that this definition differs from the usual definition in complex inner-product spaces, where $\Omega \cdot \Lambda = \Omega \cdot \Lambda_1 - i\Omega \cdot \Lambda_2$.

Fig. 3.1. Reference System for
the Computation of
 $\int \frac{d\Omega}{1 - \Omega \cdot \Lambda}$

The integral I_{Λ} , expressed by Eq. (3.1), becomes improper when

$$1 - \Omega \cdot \Lambda = 0. \quad (3.6)$$

Separating real and imaginary parts, we obtain the following lemma:

Lemma 3.1.

Given: $\Lambda = \Lambda_1 + i\Lambda_2$. Then the integral I_{Λ} , expressed by Eq. (3.1), is improper if and only if

$$\Omega \cdot \Lambda_1 = 1, \quad \Omega \cdot \Lambda_2 = 0, \quad (3.7)$$

for some $\Omega \in \{\Omega: |\Omega| = 1\}$.

Equations (3.7) have simple geometrical interpretations. The condition $\Omega \cdot \Lambda_1 = 1$ is represented by the intersection of the unit sphere $|\Omega| = 1$ and a plane perpendicular to Λ_1 with distance $1/\lambda_1$ from the origin. The condition $\Omega \cdot \Lambda_2 = 0$ represents the intersection of the unit sphere and the plane through the origin, perpendicular to Λ_2 (see Fig. 3.2). The intersections, if any, of these two circles are the solutions of (3.7), which we denote by Ω_0^+ and Ω_0^- . The polar coordinates of Ω_0^+ are denoted by θ_0 and ϕ_0^+ . Obviously, $\cos \theta_0 = 1/\lambda_1$. We have

$$\left. \begin{array}{l} \text{no solutions when } \lambda_1 < 1 \\ \text{no solutions when } \sin \gamma < \cos \theta_0 \\ \text{one solution when } \sin \gamma = \cos \theta_0 \\ \text{two solutions when } \sin \gamma > \cos \theta_0 \\ \text{infinitely many solutions when } \lambda_2 = 0, \cos \theta_0 \end{array} \right\} = 1/\lambda_1, \lambda_1 \geq 1. \quad (3.8)$$

The last line of (3.8) is a consequence of $\Omega \cdot \Lambda_1 = 1$ of (3.7) only, since for $\lambda_2 = 0$, $\Omega \cdot \Lambda_2 = 0$ is always satisfied. We conclude

Corollary 3.1.*

The integral I_Λ of (3.1) is proper when $\lambda_1 < 1$ or $\lambda_1 \sin \gamma < 1$ for $\lambda_2 \neq 0$. It is improper, when $\lambda_1 \geq 1$ for $\lambda_2 = 0$ or $\lambda_1 \sin \gamma \geq 1$ for $\lambda_2 \neq 0$.

From Fig. 3.2, we can also deduce

$$\left. \begin{array}{l} \cos \theta_0 = 1/\lambda_1 \\ \cos (\pm \phi_0) = -\cot \theta_0 \cot \gamma, \end{array} \right\} \quad (3.9)$$

a useful relation to determine Ω_0 .

* Corollary 3.1 can also be deduced from Fig. 6.2 by noting, for example, that Λ_2 , for $\Lambda_2 \neq 0$, lies on a cone of aperture γ with Λ_1 as axis. The intersections of this cone with the plane $\Omega_0 \cdot \Lambda_2 = 0$, if any, are the desired solutions. An algebraic proof of Corollary 3.1 is given in Ref. 5.

Fig. 3.2

Geometric Solution of
 $1 - \Omega_0 \cdot \Lambda = 0;$
 $\Omega_0^\pm = (\theta_0, \pm\phi_0).$

We can now proceed with the actual evaluation of I_Λ , expressed by Eq. (3.1). From Eq. (3.5), we conclude that

$$I_\Lambda = \int \frac{d_2 \Omega}{1 - \Omega \cdot \Lambda} = \int_0^\pi \sin \theta \, d\theta \int_{-\pi}^\pi \frac{d\phi}{\alpha + \beta \cos \phi}, \quad (3.10)$$

where

$$\left. \begin{aligned} \alpha &= 1 - \lambda_1 \cos \theta - i\lambda_2 \cos \gamma \cos \theta; \\ \beta &= -i\lambda_2 \sin \gamma \sin \theta. \end{aligned} \right\} \quad (3.11)$$

First we evaluate

$$I_\theta = \int_{-\pi}^{+\pi} \frac{d\phi}{\alpha + \beta \cos \phi} \quad (3.12)$$

then we evaluate

$$I_\Lambda = \int_0^\pi I_\theta \sin \theta \, d\theta. \quad (3.13)$$

B. Evaluation of I_θ

If $\beta = 0$, then $I_\theta = 2\pi/\alpha$.

Assume $\beta \neq 0$, and introduce the transformation

$$z = e^{i\phi}, \quad dz = iz \, d\phi.$$

With $2 \cos \phi = z + z^{-1}$, the right side of (3.12) becomes

$$I_\theta = \frac{2}{i} \oint \frac{dz}{\beta z^2 + 2\alpha z + \beta} = -\frac{2i}{\beta} \oint \frac{dz}{(z - z_1)(z - z_2)}, \quad (3.15)$$

where the integral extends over the unit circle. A convenient form for the zeros z_1 and z_2 of the quadratic $z^2 + 2(\alpha/\beta)z + 1$ is

$$z_{1,2} = -\frac{\alpha}{\beta} \pm \frac{\alpha - \beta}{\beta} \sqrt{\frac{\alpha + \beta}{\alpha - \beta}}. \quad (3.16)$$

We define the "positive" square root z of a complex number w in the usual way,

$${}^+\sqrt{w} = z \Rightarrow \operatorname{Re} z \geq 0 \quad (3.17)$$

that is the w -plane is cut along the negative real axis so that

$$\operatorname{Re} z = 0 \Rightarrow \operatorname{Im} z > 0.$$

The "negative" square root of w is defined by

$${}^-\sqrt{w} = -{}^+\sqrt{w}.$$

It is obvious that $z_1 z_2 = 1$. Now,

$$1 \geq \left| \frac{1 - \sqrt{\frac{\alpha + \beta}{\alpha - \beta}}}{1 + \sqrt{\frac{\alpha + \beta}{\alpha - \beta}}} \right| = \left| -\frac{\alpha - (\alpha - \beta) \sqrt{\frac{\alpha + \beta}{\alpha - \beta}}}{\beta} \right| = |z_1|. \quad (3.18)$$

Thus, $|z_1| \leq 1$ and $|z_2| \geq 1$.

First, consider $|z_1| < 1$. Since z_1 is the only singularity in the unit circle, (3.15) yields, by *Cauchy's Residue Theorem*,*

$$I_\theta = \frac{2\pi}{(\alpha - \beta) \sqrt{\frac{\alpha + \beta}{\alpha - \beta}}} \quad (3.19)$$

Next consider $|z_1| = 1$. The equality sign in (3.18) holds only when

$\sqrt{(\alpha + \beta)/(\alpha - \beta)}$ is pure imaginary, implying $(\alpha/\beta) \in [-1, 1]$, and $|z_1| = |z_2| = 1$. In this case, the two singularities lie on the unit circle, and we define the integral of (3.15) by its *Cauchy's Principal Value*. If $-1 < \alpha/\beta < 1$, this is equivalent to deforming the contour of the integral in (3.15) slightly to bring the two singularities inside the contour, and taking half the sum of the residues. Since $\text{Res } z_1 = -\text{Res } z_2$, this sum is zero; thus,

$$I_\theta = 0. \quad (3.20)$$

If $\alpha/\beta = \pm 1$, we have $z_1 = z_2 = \pm 1$ in (3.15), and consequently I_θ does not exist as a Cauchy principal value. This result can also be obtained by direct inspection of the integrand in (3.12).

We express the condition $-1 < \alpha/\beta < 1$ in terms of the original parameters. From (3.11), it follows that α/β is real if and only if

$$1 - \lambda_1 \cos \theta = 0. \quad (3.21)$$

Furthermore, $-1 < \alpha/\beta < 1$ if and only if (3.21) holds and

$$-1 < \frac{\cos \gamma \cos \theta}{\sin \gamma \sin \theta} < 1. \quad (3.22)$$

Squaring, rearranging terms, and considering that $\cos \theta = 1/\lambda_1$ by (3.21), yields $\lambda_1^2 \sin^2 \gamma > 1$. Since $\sin \gamma \geq 0$, (3.22) reduces to

$$\lambda_1 \sin \gamma > 1. \quad (3.23)$$

*This part of our integral is known in the literature as

$I_\theta = 2\pi/\sqrt{\alpha^2 - \beta^2}$. See, e.g., Ref. 21.

For $\alpha/\beta = \pm 1$, the inequality sign in (3.23) is replaced by an equality sign.

In summary, combining (3.19) to (3.23) yields

$$I_{\theta} = \int_{-\pi}^{+\pi} \frac{d\phi}{\alpha + \beta \cos \phi} = \begin{cases} 0 & \text{for } \lambda_1 \cos \theta = 1 \text{ and} \\ & \lambda_1 \sin \gamma > 1 \\ \frac{2\pi}{(\alpha - \beta) \sqrt{\frac{\alpha + \beta}{\alpha - \beta}}} & \text{otherwise} \end{cases} \quad (3.24)$$

where α and β are defined by (3.11). Note that this representation includes the case $\beta = 0$, for which $I_{\theta} = 2\pi/\alpha$, and the case $\alpha/\beta = \pm 1$, or equivalently $\lambda_1 \cos \theta = 1$, $\lambda_1 \sin \gamma = 1$, for which $I_{\theta} \rightarrow \infty$ as $\alpha \rightarrow \pm\beta$. Equation (3.24) was derived in Ref. 5, using a different transformation.

IV. FURTHER INTEGRATION OF $(1 - \Omega \cdot \Lambda)^{-1}$

A. The Function $\sqrt{Z(\mu)}$

In Section III we gave a closed expression for I_{θ} . We must now consider I_{θ} as a function of θ , and determine I_{Λ} as indicated in Eq. (3.13). It will be convenient below to consider I_{θ} as a function of $\cos \theta$. Therefore we introduce the notation

$$\left. \begin{aligned} \mu &\equiv \cos \theta; \\ \mu_0 &\equiv \cos \theta_0 \equiv 1/\lambda_1, \quad \text{for } \lambda_1 \geq 1; \\ I_{\mu}(\mu) &\equiv I_{\theta}(\theta). \end{aligned} \right\} \quad (4.1)$$

To set the proper background for the integration of I_{μ} , we discuss first the function

$$\sqrt{Z(\mu)} \equiv (\alpha - \beta) \sqrt{\frac{\alpha + \beta}{\alpha - \beta}}. \quad (4.2)$$

From Eqs. (3.11) and (4.1), we have

$$\begin{aligned} \frac{\alpha + \beta}{\alpha - \beta} &= \frac{(\alpha + \beta)(\bar{\alpha} - \bar{\beta})}{|\alpha - \beta|^2} \\ &= \frac{(1 - \lambda_1 \mu)^2 + \lambda_2^2(\mu^2 - \sin^2 \gamma) - 2i\lambda_2(1 - \lambda_1 \mu)\sin \gamma \sin \theta}{(1 - \lambda_1 \mu)^2 + \lambda_2^2 \cos^2(\gamma + \theta)} \end{aligned} \quad (4.3)$$

as a function of μ . The definition of its square root was given in (3.17). The behavior of the positive square root of Eq. (4.3) as a function of μ is as follows:

a) If $\lambda_1 < 1$, it follows immediately from Eq. (4.3) that

$$\operatorname{Im} \frac{\alpha + \beta}{\alpha - \beta} < 0, \quad (4.4)$$

and hence $\sqrt{(\alpha + \beta)/(\alpha - \beta)}$ is continuous in the interval $-1 \leq \mu \leq +1$.

b) If $\lambda_1 \geq 1$, it follows from Eq. (4.3) that, respectively,

$$\operatorname{Im} \frac{\alpha + \beta}{\alpha - \beta} \begin{matrix} \leq \\ > \end{matrix} 0 \quad \text{for } \mu \begin{matrix} \leq \\ > \end{matrix} \mu_0. \quad (4.5)$$

At $\mu = \mu_0$,

$$\operatorname{Re} \frac{\alpha + \beta}{\alpha - \beta} \begin{matrix} \geq \\ < \end{matrix} 0 \quad \text{for } \lambda_1 \sin \gamma \begin{matrix} \leq \\ > \end{matrix} 1. \quad (4.6)$$

We set

$$\rho = \left| \sqrt{\frac{\mu_0^2 - \sin^2 \gamma}{\cos^2(\gamma + \theta)}} \right|. \quad (4.7)$$

As μ approaches μ_0 from both sides, we have in the limit

$$\sqrt{\frac{\alpha + \beta}{\alpha - \beta}} = \begin{cases} \rho & \text{for } \mu = \mu_0 \pm 0 \quad \text{and } \lambda_1 \sin \gamma < 1; \\ i\rho & \text{for } \mu = \mu_0 + 0 \\ -i\rho & \text{for } \mu = \mu_0 - 0 \end{cases} \quad \text{and } \lambda_1 \sin \gamma > 1. \quad (4.8)$$

The square root is $\rho = 0$ for $\mu = \mu_0 \pm 0$ and $\lambda_1 \sin \gamma = 1$. Hence, (4.8) is continuous in the interval $-1 \leq \mu \leq 1$ for $\lambda_1 \sin \gamma \leq 1$, but has a discontinuity at $\mu = \mu_0$ in the case $\lambda_1 \sin \gamma > 1$.

Since $(\alpha - \beta)$ is a continuous function in μ , we can conclude that $\sqrt{Z(\mu)}$ is continuous in $-1 \leq \mu \leq 1$ for $\{\lambda_1 < 1\}$, and $\{\lambda_1 \geq 1, \lambda_1 \sin \gamma < 1\}$, but has a discontinuity at $\mu = \mu_0$ for $\{\lambda_1 > 1, \lambda_1 \sin \gamma > 1\}$. In the case of $\lambda_1 \sin \gamma = 1$, $\sqrt{Z(\mu_0)} = 0$, and $\sqrt{Z(\mu)}$ is therefore also continuous.

We are now in a position to attribute signs to $\sqrt{Z(\mu)}$.

The sign of the square root of $Z(\mu)$ is determined by substituting $\mu = \pm 1$ in Eq. (4.2). We obtain immediately from Eq. (4.3), observing that $\mu = \pm 1$ implies $\sin \theta = 0$,

$$(\alpha + \beta)/(\alpha - \beta) = 1, \quad \text{for } \mu = \pm 1; \quad (4.9)$$

and from (3.11)

$$\alpha - \beta = (1 \mp \lambda_1) \mp i\lambda_2 \cos \gamma, \quad \text{for } \mu = \pm 1. \quad (4.10)$$

Hence, by the discussion above, we have for

$$\underline{\lambda_1 < 1}, \quad \sqrt{Z} = \sqrt[+]{Z} \quad \text{for } -1 \leq \mu \leq 1; \quad (4.11)$$

and for $\underline{\lambda_1 \geq 1}$,

$$\sqrt{Z} = \begin{cases} \sqrt[+]{Z} & \text{for } -1 \leq \mu < \mu_0 \\ \sqrt[-]{Z} & \text{for } \mu_0 < \mu \leq 1, \end{cases} \quad (4.12)$$

$$\sqrt[+]{Z(\mu_0 - 0)} = \sqrt[-]{Z(\mu_0 + 0)} \quad \text{for } \lambda_1 \sin \gamma < 1, \quad (4.12a)$$

$$\sqrt[+]{Z(\mu_0 - 0)} = -\sqrt[-]{Z(\mu_0 + 0)} \quad \text{for } \lambda_1 \sin \gamma > 1, \quad (4.12b)$$

$$\sqrt{Z(\mu_0 \pm 0)} = 0 \quad \text{for } \lambda_1 \sin \gamma = 1; \quad (4.12c)$$

where the last three equations are a consequence of Eq. (4.8) and the subsequent remarks.

We are now ready to integrate I_μ with respect to μ .

B. Integration of I_μ

Comparison of Eqs. (3.24), (4.1), and (4.2) shows that

$$I_\mu = 2\pi/\sqrt{Z(\mu)} \quad (4.13)$$

for $\lambda_1\mu \neq 1$, $\lambda_1\mu = 1$ and $\lambda_1 \sin \gamma \leq 1$.

For $\lambda_1 \sin \gamma = 1$ and $\lambda_1\mu \rightarrow 1$, I_μ is unbounded. For $\lambda_1 \sin \gamma > 1$ and $\lambda_1\mu = 1$, I_μ is zero. If we use Riemannian integration, we have, under consideration of Eqs. (4.11), (4.12), and (3.24), two cases, which can be integrated, a) where $\sqrt{Z}$ is continuous, and b) where $\sqrt{Z}$ has a discontinuity:

a) $\lambda_1 < 1$, or $\lambda_1 \geq 1$ and $\lambda_1 \sin \gamma < 1$:

$$I_\Lambda = 2\pi \int_{-1}^{+1} \frac{d\mu}{\sqrt{Z(\mu)}} \quad (4.14)$$

b) $\lambda_1 \geq 1$, $\lambda_1 \sin \gamma > 1$:

$$I_\Lambda = 2\pi \left\{ \int_{-1}^{\mu_0-0} \frac{d\mu}{\sqrt{+Z(\mu)}} + \int_{\mu_0+0}^1 \frac{d\mu}{\sqrt{-Z(\mu)}} \right\} \quad (4.15)$$

The situation $\lambda_1 \geq 1$, $\lambda_1 \sin \gamma = 1$, leads to an improper integral, which must be obtained by a limiting process. This process is carried through in Section C(c) below.

To obtain an analytic expression for the desired integral, we observe from Eq. (4.2) that

$$Z = \alpha^2 - \beta^2. \quad (4.16)$$

From (3.11) and (4.1), it follows then that

$$Z(\mu) = a\mu^2 - 2b\mu + c, \quad (4.17)$$

where

$$\left. \begin{aligned} a &= \lambda_1^2 - \lambda_2^2 + 2i\lambda_1\lambda_2 \cos \gamma, \\ b &= \lambda_1 + i\lambda_2 \cos \gamma, \\ c &= 1 + \lambda_2^2 \sin \gamma. \end{aligned} \right\} \quad (4.18)$$

The indefinite integral of $Z(\mu)^{-1/2}$ is (as can be verified)

$$\int \frac{d\mu}{\sqrt{Z}} = \pm \frac{1}{\sqrt{a}} \log(a\mu - b \pm \sqrt{a} \sqrt{Z}) + \text{const}, \quad \text{for } b^2 \neq ac. \quad (4.19a)$$

$$= \frac{1}{\sqrt{a}} \log(a\mu - b) + \text{const}, \quad \text{for } b^2 = ac. \quad (4.19b)$$

Equation (4.19b) is contained in (4.19a), provided the sign of $\sqrt{a}$ is chosen so that

$$\sqrt{Z} = \sqrt{a} \left(\mu - \frac{b}{a} \right). \quad (4.19c)$$

Since I_A of (3.10) can be directly evaluated for $b^2 = ac$ without reference to (4.19b), we treat this case first to have it available for comparison with the limit cases of the general integrals (4.14) and (4.15).

$b^2 = ac$:

Using (4.18), the condition $b^2 = ac$ becomes

$$\lambda_2^2 \sin^2 \gamma (\lambda_1^2 - \lambda_2^2 - 1) = 0, \quad \text{for } \text{Re } b^2 = \text{Re } ac \quad (4.20a)$$

$$\lambda_1 \lambda_2^3 \cos \gamma \sin^2 \gamma = 0, \quad \text{for } \text{Im } b^2 = \text{Im } ac. \quad (4.20b)$$

Conditions (4.20a) and (4.20b) are simultaneously satisfied if at least one of the following equations is true:

$$(i) \quad \lambda_2 = 0; \quad (4.21a)$$

$$(ii) \quad \sin \gamma = 0; \quad (4.21b)$$

$$(iii) \quad \cos \gamma = 0 \text{ and } \lambda_1^2 - \lambda_2^2 = 1. \quad (4.21c)$$

(i) If we set $\lambda_2 = 0$, then from (3.10) and (3.11),

$$I_{\Lambda} = 2\pi \int_{-1}^{+1} \frac{d\mu}{1 - \lambda_1 \mu} = \frac{2\pi}{\lambda_1} \log \left| \frac{1 + \lambda_1}{1 - \lambda_1} \right|, \quad \text{for } \lambda_2 = 0 \quad (4.22)$$

according to results obtained in the case of plane geometry. The Cauchy principal value was used to obtain the result for $\lambda_1 > 1$.

(ii) If we set $\sin \gamma = 0$, it follows from (3.11) that

$$I_{\Lambda} = 2\pi \int_{-1}^{+1} \frac{d\mu}{1 - (\lambda_1 \pm i\lambda_2)\mu} = \frac{2\pi}{\lambda_1 \pm i\lambda_2} \log \frac{1 + \lambda_1 \pm i\lambda_2}{1 - \lambda_1 \mp i\lambda_2}, \quad (4.23)$$

for $\sin \gamma = 0$,

where the upper sign corresponds to the value $\cos \gamma = +1$ and the lower sign corresponds to $\cos \gamma = -1$. It will be shown that for the proper integral, $\sin \gamma = 0$ implies $\lambda_1 \lambda_2 = 0$ [see Eq. (5.5)], and that for the improper integral, (4.23) has meaning only as a limit case [$\lambda_1 \rightarrow \infty$, implying $I_{\Lambda} = 0$; see Eq. (4.41)].

(iii) If we set $\cos \gamma = 0$ ($\sin \gamma = 1$) and $\lambda_1^2 - \lambda_2^2 = 1$, then

$$\lambda_1 \sin \gamma \geq 1.$$

From (4.16) to (4.18), we obtain $Z = \alpha^2 - \beta^2 = (\mu - \lambda_1)^2$, and consequently from (4.12), (4.12b), and (4.15)

$$\sqrt{Z} = \begin{cases} +\sqrt{Z} = \lambda_1 - \mu & \text{for } -1 \leq \mu < \mu_0; \\ -\sqrt{Z} = \mu - \lambda_1 & \text{for } \mu_0 < \mu \leq 1. \end{cases}$$

Thus

$$\begin{aligned}
I_{\Lambda} &= 2\pi \int_{-1}^{\mu_0-0} \frac{d\mu}{\sqrt{+Z}} + 2\pi \int_{\mu_0+0}^1 \frac{d\mu}{\sqrt{-Z}} = 2\pi \int_{-1}^{1/\lambda_1-0} \frac{d\mu}{\lambda_1 - \mu} + 2\pi \int_{1/\lambda_1+0}^1 \frac{d\mu}{\mu - \lambda_1} \\
&= 2\pi \log \frac{\lambda_1 + 1}{\lambda_1 - \frac{1}{\lambda_1}} + 2\pi \log \frac{\lambda_1 - 1}{\lambda_1 - \frac{1}{\lambda_1}} = 2\pi \log \frac{\lambda_1^2}{\lambda_1^2 - 1} = 2\pi \log \frac{\lambda_1^2}{\lambda_2^2}.
\end{aligned}$$

Thus,

$$I_{\Lambda} = 4\pi \log \frac{\lambda_1}{\lambda_2}, \quad \text{for } \cos \gamma = 0 \text{ and } \lambda_1^2 - \lambda_2^2 = 1. \quad (4.24)$$

We return now to the general case

$b^2 \neq ac$:

As explained at the beginning of this section, we distinguish only two cases, a) and b) corresponding to (4.14) and (4.15), respectively:

a) To obtain the integral (4.14), we must evaluate $\sqrt{Z}$ at $\mu = \pm 1$ under the conditions (4.11) and (4.12). This can be done using Eqs. (4.17) and (4.18). Then

$$Z(\pm 1) = a \mp 2b + c = (1 \mp b)^2.$$

Hence, by Eqs. (4.11) and (4.12),

$$\sqrt{Z}(\pm 1) = 1 \mp b, \quad (4.25)$$

and by (4.19a), we obtain for (4.14)

$$\begin{aligned}
\int_{-1}^{+1} \frac{d\mu}{\sqrt{Z}} &= \frac{1}{\sqrt{a}} \log \frac{a - b + \sqrt{a}(1 - b)}{-a - b + \sqrt{a}(1 + b)} \\
&= \frac{1}{\sqrt{a}} \log \frac{(1 + \sqrt{a})(\sqrt{a} - b)}{(1 - \sqrt{a})(\sqrt{a} - b)}.
\end{aligned} \quad (4.26)$$

We assign the negative sign to $\sqrt{a}$. In this case, even for $\lambda_2 \rightarrow 0$, $(\sqrt{a} - b) \neq 0$ and can be canceled.

Now we define the inner product $\Lambda^2 \equiv \Lambda \cdot \Lambda$ to be, for $\Lambda = \Lambda_1 + i\Lambda_2$,

$$\left. \begin{aligned} \Lambda^2 &\equiv (\Lambda_1 + i\Lambda_2) \cdot (\Lambda_1 + i\Lambda_2) \\ &\equiv \Lambda_1^2 - \Lambda_2^2 + 2i\Lambda_1 \cdot \Lambda_2 \\ &= \lambda_1^2 - \lambda_2^2 + 2i\lambda_1\lambda_2 \cos \gamma. \end{aligned} \right\} \quad (4.27)$$

Comparing this expression with (4.18), we see that

$$\Lambda^2 = a. \quad (4.28)$$

We substitute*

$$\sqrt{a} = -\sqrt{a} = -\sqrt{\Lambda^2} = \sqrt{\Lambda^2} \quad (4.29)$$

in Eq. (4.26), simplify, and multiply by 2π to conform with (4.14). This yields the desired explicit expression for the proper integral in the characteristic Eq. (2.9),

$$\boxed{I_\Lambda = \int \frac{d\Omega}{1 - \Omega \cdot \Lambda} = \int_{-1}^{+1} I_\mu d\mu = \frac{2\pi}{\sqrt{\Lambda^2}} \log \frac{1 + \sqrt{\Lambda^2}}{1 - \sqrt{\Lambda^2}} = \frac{4\pi}{\sqrt{\Lambda^2}} \tanh^{-1} \sqrt{\Lambda^2}} \quad (4.30)$$

for $\{\lambda_1 < 1\}$ and $\{\lambda_1 \geq 1, \lambda_1 \sin \gamma < 1\}$,

where Λ^2 is defined by (4.27), and the Λ^2 -plane is cut along the negative axis. Note that $\sqrt{\Lambda^2}$ in the final form (4.30) can be either $-\sqrt{\Lambda^2}$ or $+\sqrt{\Lambda^2}$.

b) To obtain the integral (4.15), we must evaluate $\sqrt{Z}$ at $\mu = \pm 1$ and $\mu = \mu_0 \pm 0$. We obtain, from (4.1), (4.17), (4.18), and (4.12b)

$$\sqrt{Z(\mu_0 \pm 0)} = \mp \lambda_2 \sqrt{\sin^2 \gamma - \mu_0^2}, \quad (4.31)$$

* Note that $\sqrt{\Lambda^2}$ is a complex scalar and not the vector Λ .

while

$$\sqrt{Z(\pm 1)} = 1 \mp b \quad (4.25)$$

as before. Therefore, by Eqs. (4.18) and (4.28),

$$\begin{aligned} & \int_{-1}^{\mu_0-0} \frac{d\mu}{\sqrt{Z}} + \int_{\mu_0+0}^{+1} \frac{d\mu}{\sqrt{Z}} = \\ &= \frac{1}{\sqrt{a}} \log \frac{a\mu_0 - b + \sqrt{a}\lambda_2 \sqrt{\sin^2 \gamma - \mu_0^2}}{-a - b + \sqrt{a}(1+b)} + \frac{1}{\sqrt{a}} \log \frac{a - b + \sqrt{a}(1-b)}{a\mu_0 - b - \sqrt{a}\lambda_2 \sqrt{\sin^2 \gamma - \mu_0^2}} \\ &= \frac{1}{\sqrt{a}} \log \frac{(1+\sqrt{a})(\sqrt{a}-b)}{(1-\sqrt{a})(\sqrt{a}-b)} \cdot \frac{\lambda_2[-\lambda_2/\lambda_1 + i \cos \gamma + (1/\lambda_1)\sqrt{a} \sqrt{\lambda_1^2 \sin^2 \gamma - 1}]}{\lambda_2[-\lambda_2/\lambda_1 + i \cos \gamma - (1/\lambda_1)\sqrt{a} \sqrt{\lambda_1^2 \sin^2 \gamma - 1}]} \\ &= \frac{1}{\sqrt{\Lambda^2}} \log \frac{1+\sqrt{\Lambda^2}}{1-\sqrt{\Lambda^2}} \cdot \frac{-\lambda_2 + i\lambda_1 \cos \gamma + \sqrt{\Lambda^2} \sqrt{\lambda_1^2 \sin^2 \gamma - 1}}{-\lambda_2 + i\lambda_1 \cos \gamma - \sqrt{\Lambda^2} \sqrt{\lambda_1^2 \sin^2 \gamma - 1}} \quad (4.32) \end{aligned}$$

Substituting the last line of (4.32) into (4.15) yields the desired explicit expression for the improper integral in (2.13),

$$\begin{aligned} I_{\Lambda} &= \int \frac{d_2 \Omega}{1 - \Omega \cdot \Lambda} = \int_{-1}^{+1} I_{\mu} d\mu \\ &= \frac{2\pi}{\sqrt{\Lambda^2}} \log \left[\frac{\sqrt{\Lambda^2} + 1}{\sqrt{\Lambda^2} - 1} \cdot \frac{\sqrt{\Lambda^2} \sqrt{\lambda_1^2 \sin^2 \gamma - 1} - \lambda_2 + i\lambda_1 \cos \gamma}{\sqrt{\Lambda^2} \sqrt{\lambda_1^2 \sin^2 \gamma - 1} + \lambda_2 - i\lambda_1 \cos \gamma} \right] \\ &\quad \text{for } \{\lambda_1 \geq 1, \text{ and } \lambda_1 \sin \gamma > 1\}, \end{aligned} \quad (4.33)$$

where Λ^2 is defined by (4.28), and the Λ^2 -plane is cut along the negative axis.

As in (4.30), we see from the final Eq. (4.33) that the sign of $\sqrt{a}$ is of no practical interest; i.e., replacing $\sqrt{\Lambda^2}$ by $-\sqrt{\Lambda^2}$ leaves this equation unchanged.

We note that

$$-\lambda_2 + i\lambda_1 \cos \gamma = \sqrt{\lambda_1^2 \sin^2 \gamma - a}. \quad (4.34)$$

Use of Eq. (4.34) would give Eq. (4.33) a more symmetric appearance, but no additional insight.

C. Limit Cases

In this section we consider all limit cases of interest. First we see (a) how the special cases, when $b^2 = ac$, are related. Then, (b) we consider limits of the proper integral (4.30) and its relation to the special cases, and finally, (c) we study limits of the improper integral (4.33) and its relation to (4.30), the special cases and their limits.

(a) If we let λ_2 go to zero in Eq. (4.23), we have

$$\lim_{\lambda_2 \rightarrow 0} I_\Lambda = \frac{2\pi}{\lambda_1} \log \left| \frac{1 + \lambda_1}{1 - \lambda_1} \right| + \begin{cases} 0 & \text{for } \lambda_1 < 1 \\ i\pi & \text{for } \cos \gamma = 1 \\ -i\pi & \text{for } \cos \gamma = -1 \end{cases} \text{ and } \lambda_1 > 1. \quad (4.35)$$

Hence, I_Λ in Eq. (4.22) is the arithmetic mean of I_Λ in Eq. (4.35).

(b) Now we investigate Eq. (4.30) and obtain, by (4.27),

$$\lim_{\lambda_2 \rightarrow 0} \frac{2\pi}{\sqrt{\Lambda^2}} \log \frac{1 + \sqrt{\Lambda^2}}{1 - \sqrt{\Lambda^2}} = \frac{2\pi}{\lambda_1} \log \left| \frac{1 + \lambda_1}{1 - \lambda_1} \right| + \begin{cases} 0 & \text{for } \lambda_1 < 1 \\ i\pi & \text{for } \cos \gamma > 0 \\ -i\pi & \text{for } \cos \gamma < 0 \end{cases} \text{ and } \lambda_1 > 1. \quad (4.36)$$

Therefore, Eq. (4.35) is a special case of (4.36) since the case $\sin \gamma = 0$ corresponds exactly to Eq. (4.35).

Next we consider the case $\cos \gamma = 0$ in (4.30). This case occurs in Section V in connection with the regular eigenfunctions. Note that $\cos \gamma = 0$ for the proper eigenvectors Λ_0 implies

$$\lambda_1 \sin \gamma = \lambda_1 < 1;$$

therefore (4.30) reduces to

$$I_{\Lambda} = \frac{2\pi}{\sqrt{\lambda_1^2 - \lambda_2^2}} \log \frac{1 + \sqrt{\lambda_1^2 - \lambda_2^2}}{1 - \sqrt{\lambda_1^2 - \lambda_2^2}} \quad \text{for } \cos \gamma = 0, \lambda_1 < 1. \quad (4.37)$$

As $\lambda_2 \rightarrow 0$, Eq. (4.37) reduces to Eqs. (4.36), (4.35), and (4.22) for $\lambda_1 < 1$.

For the sake of completeness, we mention that the limit $\lambda_1 \rightarrow 0$ does not cause any difficulties, since in

$$\lim_{\lambda_1 \rightarrow 0} \sqrt{\Lambda^2} = \pm i\lambda_2$$

either sign may be used in Eq. (4.30).

(c) In a similar manner, we investigate Eq. (4.33), where $\lambda_1 \geq 1$ always. We have

$$\lim_{\lambda_2 \rightarrow 0} I_{\Lambda} = \frac{2\pi}{\lambda_1} \log \frac{\lambda_1 + 1}{\lambda_1 - 1} \frac{\sqrt{\lambda_1^2 \sin^2 \gamma - 1 + i \cos \gamma}}{\sqrt{\lambda_1^2 \sin^2 \gamma - 1 - i \cos \gamma}} \quad (4.38)$$

The limit depends on the angle γ . For $\cos \gamma = 0$, Eq. (4.38) reduces to Eq. (4.22) for $\lambda_1 \geq 1$. One can also consider I_{Λ} in Eq. (4.22) as the mean value of I_{Λ} in Eq. (4.33) with respect to γ .

As $(\lambda_1 \sin \gamma - 1) \rightarrow 0+$,

$$I_{\Lambda} = \frac{2\pi}{\sqrt{\Lambda^2}} \log \frac{1 + \sqrt{\Lambda^2}}{1 - \sqrt{\Lambda^2}}. \quad (4.39)$$

Hence Eq. (4.33) changes continuously to Eq. (4.30).

From Eqs. (4.38) and (4.39), we conclude

$$\lim_{\lambda_2 \rightarrow 0} \lim_{\lambda_1 \sin \gamma \rightarrow 1+} I_{\Lambda} = \lim_{\lambda_1 \sin \gamma \rightarrow 1+} \lim_{\lambda_2 \rightarrow 0} I_{\Lambda}, \quad (4.40)$$

and the result coincides with Eq. (4.36) for $\lambda_1 > 1$.

If $\gamma \rightarrow 0$, it follows from $\lambda_1 \sin \gamma > 1$ that $\lambda_1 \rightarrow \infty$. In this case, from Eq. (4.33),

$$\lim_{\lambda_1 \rightarrow \infty} I_{\Lambda} \rightarrow 0, \quad (\gamma \rightarrow 0) \quad (4.41)$$

Similarly, $\lim_{\lambda_2 \rightarrow \infty} I_\Lambda \rightarrow 0$.

Another type of limit case for (4.33) is $\sqrt{\Lambda^2} \rightarrow 1$ with $\lambda_2 \neq 0$. Let $\lambda_1^2 - \lambda_2^2 = 1 + \varepsilon$ and $\cos \gamma = \eta$, where ε and $\eta \rightarrow \pm 0$. Then

$$\Lambda^2 = 1 + \varepsilon + 2i\lambda_1\lambda_2\eta,$$

$$\sqrt{\Lambda^2} = 1 + \frac{\varepsilon}{2} + i\lambda_1\lambda_2\eta + O(\varepsilon^2, \varepsilon\eta, \eta^2),$$

and

$$\sqrt{\lambda_1^2 \sin^2 \gamma - 1} = \lambda_2 + \frac{\varepsilon}{2\lambda_2} + O(\varepsilon^2, \eta^2).$$

Inserting these expressions in (4.33), rearranging terms, and simplifying yields

$$\lim_{\varepsilon \rightarrow 0} \lim_{\eta \rightarrow 0} I_\Lambda = \lim_{\eta \rightarrow 0} \lim_{\varepsilon \rightarrow 0} I_\Lambda = 4\pi \log \frac{\lambda_1}{\lambda_2} \quad (4.42)$$

in agreement with the special case (4.24). For $\lambda_2 \rightarrow 0$, (4.42) is unbounded.

We note that in (4.30) the limit case $\sqrt{\Lambda^2} \rightarrow 1$ corresponds to $\lambda_1^2 - \lambda_2^2 = 1 - \varepsilon(\varepsilon \rightarrow 0)$ and $\cos \gamma \rightarrow 0$. This requires $\lambda_2 \rightarrow 0$, $\lambda_1 \rightarrow 1$. We conclude then from (4.37) that I_Λ of (4.30) is always unbounded for $\Lambda^2 = 1$.

V. SOLUTION OF THE CHARACTERISTIC EQUATION AND THE REGULAR EIGENFUNCTIONS

In Eqs. (2.7) to (2.9), we defined the regular eigenfunctions and the characteristic equation. In Corollary 3.1, we asserted that the integral I_Λ is proper if and only if

$$\lambda_1 < 1; \text{ or } \lambda_1 \sin \gamma < 1 \text{ and } \lambda_2 \neq 0. \quad (5.1)$$

We shall prove

Lemma 5.1. Regular eigenfunctions of the form (2.8) exist only if $\lambda_1 < 1$.

In Section IV we showed that the integral I_Λ , appearing in the characteristic Eq. (2.9) and satisfying conditions (5.1), is given by Eq. (4.30). Therefore, we can write the following expression for the characteristic equation:

$$1 - (c/z) \tanh^{-1} z = 0; \quad \sqrt{\Lambda^2} = z, \Lambda \in \{\Lambda^0\}. \quad (5.2)$$

This is the same equation in z as was found for the characteristic equation in plane geometry, and has two solutions, λ_0 and $-\lambda_0$, for given c as follows:

$$\left. \begin{array}{l} \text{a) } 0 < c < 1 \quad \lambda_0 \text{ is real, and } \lambda_0^2 < 1; \\ \text{b) } c = 1 \quad \lambda_0 = 0, \text{ and } \lambda_0^2 = 0; \\ \text{c) } 1 < c < \infty \quad \lambda_0 \text{ is imaginary, and } \lambda_0^2 < 0. \end{array} \right\} \quad (5.3)$$

Hence, λ_0 can be considered known for given c , and thus

$$\{\Lambda^0\} = \{\Lambda : \Lambda^2 = \lambda_0^2, \forall \Omega \rightarrow \Omega \cdot \Lambda \neq 1\}. \quad (\text{Definition}) \quad (5.4)$$

This means that every vector Λ that satisfies $\Lambda^2 = \lambda_0^2$ and $\Omega \cdot \Lambda \neq 1$ for each Ω is an element of $\{\Lambda^0\}$. Since λ_0^2 is real, we have, by (4.27), the condition

$$\lambda_1^2 - \lambda_2^2 = \lambda_0^2, \quad \lambda_1 \lambda_2 \cos \gamma = 0, \quad (5.5)$$

for $\Lambda^2 = \lambda_0^2$. The condition $\Omega \cdot \Lambda \neq 1$ for each Ω , yielded the conditions (5.1). These conditions impose only restrictions on the magnitudes λ_1 and λ_2 of the real and imaginary components Λ_1 and Λ_2 , respectively, of $\Lambda = \Lambda_1 + i\Lambda_2$ and the mutual angle γ in E_3 . From Eq. (5.5), if $\lambda_1 \lambda_2 \neq 0$, then $\cos \gamma = 0$. Therefore, in (5.1), $\sin \gamma = 1$ and for $\lambda_2 \neq 0$ necessarily

$$\lambda_1 < 1. \quad (5.6)$$

This proves Lemma 5.1.

We treat the three cases of (5.3) separately in the following paragraphs. They contain the cases $\lambda_1 \lambda_2 = 0$ and $\lambda_1 \lambda_2 \neq 0$ but $\cos \gamma = 0$.

$$\text{a) } \underline{0 < c < 1}$$

From Eq. (5.5), we obtain

$$\lambda_1 = \sqrt{\lambda_2^2 + \lambda_0^2} \quad (5.7a)$$

and from (5.3) and (5.6), we obtain

$$\lambda_0 \leq \lambda_1 < 1; \quad 0 \leq \lambda_2 < \sqrt{1 - \lambda_0^2}. \quad (5.7b)$$

b) $c = 1$

By the same equations as used in a), we obtain

$$\lambda_1 = \lambda_2; \quad (5.8a)$$

$$0 \leq \lambda_1 < 1; \quad (5.8b)$$

$$0 \leq \lambda_2 < 1.$$

c) $1 < c < \infty$

Again, by the same equations as used in a), we obtain

$$\lambda_1 = \sqrt{\lambda_2^2 - |\lambda_0|^2}; \quad (5.9a)$$

$$0 \leq \lambda_1 < 1; \quad |\lambda_0| \leq \lambda_2 < \sqrt{1 + |\lambda_0|^2}. \quad (5.9b)$$

Because $\lambda_0^2 < 0$, we can write Eq. (5.9a) as

$$\lambda_1 = \sqrt{\lambda_2^2 + \lambda_0^2} \quad (5.9c)$$

to conform with Eq. (5.7a).

The relationship between λ_1 and λ_2 is illustrated in Fig. 5.1.

We mentioned at the beginning of Section III that I_Λ is invariant under the rotation group R on Ω . This is also true of the characteristic equation. Hence, if $\Lambda \in \{\Lambda^0\}$, then $R\Lambda \in \{\Lambda^0\}$ also. Thus, if $\phi_\Lambda^0(\Omega)$ is an eigenfunction, so is any element of $\{\phi_{R\Lambda}^0(\Omega)\}$. The range of $\{\Lambda^0\}$ is indicated in Figs. 5.2 and 5.3 for the cases $c < 1$ and $c > 1$, respectively, for the projections of $\{\Lambda^0\}$ into E_3 .

Fig. 5.1
Relationship between λ_1 and λ_2
for $c < 1$, $c = 1$, and $c > 1$

The functions $\psi(r, \Omega)$ that can be represented by the regular eigenfunctions (2.8) are given by the first integral in Eq. (2.15). We select a convenient reference direction from $\{\Lambda^0\}$, say $w^0 + iu^0$, such that, by Eqs. (5.7a) and (5.9c),

$$\begin{aligned} \Lambda^0 &= \Lambda_1^0 + i\Lambda_2^0 \\ &= \sqrt{\lambda_2^2 + \lambda_0^2} w^0 + i\lambda_2 u^0, \end{aligned} \quad (5.10)$$

where

$$\begin{aligned} w^0 &= \Lambda_1^0 / \lambda_1, \\ \text{and} \\ u^0 &= \Lambda_2^0 / \lambda_2. \end{aligned} \quad (5.11)$$

Note that $w^0 \cdot u^0 = 0$ since $\cos \gamma = 0$. Let an element of R be denoted by R . Then

$$\int_{\{\Lambda^0\}} A^0(\Lambda) \psi_{\Lambda}^0 d_4 \Lambda = \int_{\lambda_2 = \begin{cases} \sqrt{1-\lambda_0^2} \\ 0 \text{ for } c \leq 1 \\ |\lambda_0| \text{ for } c \geq 1 \end{cases}}^{\lambda_2 = 1} d\lambda_2 \int_R d_3 R A^0(R \Lambda^0) \psi_{R \Lambda^0}(\Omega). \quad (5.12)$$

Fig. 5.2. Range of $\Lambda_1 + i\Lambda_2 = \Lambda \in \{\Lambda^0\}$ for $c < 1$

Fig. 5.3. Range of $\Lambda_1 + i\Lambda_2 = \Lambda \in \{\Lambda^0\}$ for $c > 1$

The order of integration on the right-hand side of this equation may be interchanged.

For applications, it may be convenient to choose for the reference direction an orthonormal system; e.g.,

$$\left. \begin{aligned} u^0 &= \{1, 0, 0\}; \\ v^0 &= \{0, 1, 0\}; \\ w^0 &= \{0, 0, 1\}. \end{aligned} \right\} \quad (5.13)$$

Then Λ_1^0 is in the "z-direction," and Λ_2^0 in the "x-direction," and an element R is given by the transformation matrix that transforms the orthonormal system (5.13) into another orthonormal system (u, v, w) ; i.e., R is an orthogonal transformation. The reduction of $\int A(\Lambda^0) \psi_{R\Lambda^0}^0(\Omega) d_3 R$ to a triple integral of the Riemannian type will depend on the geometric properties of the problem at hand.

VI. GENERALIZED EIGENFUNCTIONS AND THE GENERAL SOLUTION OF THE HOMOGENEOUS EQUATION

In Eqs. (2.11) and (2.14), we defined the generalized eigenfunction and the generalized expansion modes, respectively. In Section IV, parts B and C, we determined the integral term in Eq. (2.13) to be I_Λ , as given by Eq. (4.33) or its limit cases. As has been anticipated, G is a function of only λ_1 , λ_2 , and γ , and does not depend on the particular direction of Ω_0 or the complex vector, Λ . It depends only on the relative position of the projections of the real and imaginary parts of Λ into E_3 . Hence,

$$G = G(\Lambda) = G(\lambda_1, \lambda_2, \gamma) = G(|\Lambda_1|, |\Lambda_2|, \Lambda_1 \cdot \Lambda_2), \quad (6.1)$$

or explicitly

$$G = 1 - (c/4\pi) I_\Lambda, \quad (6.2)$$

where I_Λ is given by Eq. (4.33) for $\lambda_1 \geq 1$ and $\lambda_1 \sin \gamma \geq 1$ [and by Eq. (4.22) for $\lambda_2 \equiv 0$ and $\lambda_1 \geq 1$].

To obtain all generalized eigenfunctions that are unbounded at Ω_0 , we observe that, by (2.12),

$$1 - \Omega_0 \cdot \Lambda = 0. \quad (6.3)$$

This means that the end points of the real part of the complex vector, Λ , lie in the plane

$$\Omega_0 \cdot \Lambda_1 = 1, \quad (6.4)$$

which is perpendicular to, and passes through the end point of Ω_0 , as illustrated in Fig. 6.1. The end points of the imaginary part of Λ lie in the plane

$$\Omega_0 \cdot \Lambda_2 = 0, \quad (6.5)$$

which is perpendicular to Ω_0 and passes through the origin, as illustrated also in Fig. 6.1. We see therefore that these vectors, Λ , satisfy the conditions given in Lemma 3.1 and Corollary 3.1.

Fig. 6.1. Representation of the Complex Vectors, Λ , that Satisfy $\Omega_0 \cdot \Lambda = 1$

Definition 6.1.

All vectors lying in the real plane $\Omega_0 \cdot \Lambda_2 = 0$ shall be denoted by $\{\Lambda^{\Omega_0}\}$.

With this definition, every vector Λ_1 satisfying Eq. (6.5) is defined by

$$\Lambda_2 = \Lambda_2^{\Omega_0}, \quad (6.6)$$

and every vector Λ_1 satisfying (6.4) is defined by

$$\Lambda_1 = \Omega_0 + \Lambda_1^{\Omega_0}, \quad (6.7)$$

since the two planes (6.4) and (6.5) are parallel (see Fig. 6.2).

Therefore, the second term on the right-hand side of Eq. (2.15) has the representation

$$\begin{aligned} & \int_{\{\Omega_0\}} d_2 \Omega_0 \int_{\{\Lambda: \Omega_0 \cdot \Lambda = 1\}} d_4 \Lambda A(\Lambda, \Omega_0) \psi_{\Lambda, \Omega_0}(r, \Omega) = \\ & \int_{\{\Omega_0\}} d_2 \Omega_0 \int_{\{\Lambda^{\Omega_0}\}} d_2 \Lambda_1^{\Omega_0} \int_{\{\Lambda^{\Omega_0}\}} d_2 \Lambda_2^{\Omega_0} A(\Omega_0 + \Lambda_1^{\Omega_0} + i\Lambda_2^{\Omega_0}, \Omega_0) \psi_{\Lambda, \Omega_0}(r, \Omega) = \quad (6.8) \\ & \kappa \int_{\{\Lambda^{\Omega}\}} d_2 \Lambda_1^{\Omega} \int_{\{\Lambda^{\Omega}\}} d_2 \Lambda_2^{\Omega} A(\Lambda, \Omega) G(\Lambda) \exp[-(r, \Omega + \Lambda_1^{\Omega} + i\Lambda_2^{\Omega})] + \\ & + \kappa \int_{\{\Omega_0\}} d_2 \Omega_0 \int_{\{\Lambda^{\Omega_0}\}} d_2 \Lambda_1^{\Omega_0} \int_{\{\Lambda^{\Omega_0}\}} d_2 \Lambda_2^{\Omega_0} \frac{\exp[-(r, \Omega_0 + \Lambda_1^{\Omega_0} + i\Lambda_2^{\Omega_0})]}{1 - (\Omega, \Omega_0 + \Lambda_1^{\Omega_0} + i\Lambda_2^{\Omega_0})} A(\Lambda, \Omega_0), \end{aligned} \quad (6.9)$$

where $\Lambda = \Omega + \Lambda_1^{\Omega} + i\Lambda_2^{\Omega}$ in $A(\Lambda, \Omega)$ and $G(\Lambda)$ of the first part of (6.9). G is reduced further by Eq. (6.1) [or Eq. (6.12) below].

The first term in (6.9) was obtained from (6.8) by interchanging Ω and Ω_0 in Definition 2.2 and corresponds to the second term of Φ_{Λ, Ω_0} in Eq. (2.11).

If the integration over Ω_0 is carried out in polar coordinates, we can replace Definition 2.2 by

$$\delta(\Omega - \Omega_0) = \delta(\mu - \mu_0) \delta(\phi - \phi_0) \quad (6.10)$$

and thus reduce the two-dimensional δ -function to the product of two one-dimensional δ -functions in a natural way.

In both terms of (6.9), the order of integration with respect to $\Lambda_1^{\Omega_0}$ and $\Lambda_2^{\Omega_0}$ can be interchanged. The actual integration in the $\{\Lambda^{\Omega_0}\}$ -plane can be carried out in any way convenient for the specific problem at hand, e.g., in two-dimensional polar coordinates or two-dimensional Cartesian coordinates.

Further simplifications are possible, e.g., as follows: Let us denote the angle between $\Lambda_1^{\Omega_0}$ and $\Lambda_2^{\Omega_0}$ by γ^{Ω_0} , as indicated in Fig. 6.2. Now

$$\left. \begin{aligned} \lambda_1 &= \sqrt{1 + \left(\lambda_1^{\Omega_0}\right)^2}, \\ \Lambda_1 \cdot \Lambda_2 &= \left(\Lambda_1^{\Omega_0} + \Omega_0, \Lambda_2^{\Omega_0}\right) \\ &= \lambda_1 \lambda_2 \cos \gamma = \lambda_1^{\Omega_0} \lambda_2^{\Omega_0} \cos \gamma^{\Omega_0}. \end{aligned} \right\} \quad (6.11)$$

Hence, λ_1 , λ_2 , and γ are determined by $\lambda_1^{\Omega_0}$, $\lambda_2^{\Omega_0}$, and γ^{Ω_0} , respectively, and we can write

$$G = G(\Lambda) = G\left(\lambda_1^{\Omega_0}, \lambda_2^{\Omega_0}, \gamma^{\Omega_0}\right). \quad (6.12)$$

Thus the first integral in (6.9) can be represented by

$$\begin{aligned} & \int_{\{\Lambda^{\Omega_0}\}} d_2 \Lambda_1^{\Omega_0} \int_{\{\Lambda^{\Omega_0}\}} d_2 \Lambda_2^{\Omega_0} A(\Lambda, \Omega) G(\Lambda) e^{-(r, \Lambda)} = \\ &= \int_0^\infty d\lambda_1^{\Omega_0} \int_0^\infty d\lambda_2^{\Omega_0} \int_0^{2\pi} d\gamma^{\Omega_0} G \int_{R^{\Omega_0}} dR^{\Omega_0} A(R^{\Omega_0} \Lambda^I, \Omega) e^{-(r, R^{\Omega_0} \Lambda^I)}, \end{aligned} \quad (6.13)$$

Interchanging Λ and Λ' in this equation, and taking the difference between these equations, yields a *general orthogonality relation*,

$$(\Lambda - \Lambda') \cdot \int \Omega \Phi_{\Lambda} \Phi_{\Lambda'} d_2 \Omega = 0. \quad (7.3)$$

Now, let $\Lambda' = \zeta \Lambda$, where $\zeta = \xi + i\eta$ is a nonvanishing complex scalar. Then (7.3) and (7.2), respectively, lead to the "*class orthogonality relations*,"

$$\left. \begin{aligned} \int \Lambda \cdot \Omega \Phi_{\Lambda} \Phi_{\zeta \Lambda} d_2 \Omega &= N_{\Lambda} \delta(\zeta - 1) \\ \text{and} \quad \int \Phi_{\Lambda} \Phi_{\zeta \Lambda} d_2 \Omega &= \frac{c\kappa^2}{4\pi} + N_{\Lambda} \delta(\zeta - 1). \end{aligned} \right\} \quad (7.4)$$

Equations (7.4) as written above are only symbolic. Their exact meaning is explained below in Eqs. (7.23), (7.24), (7.26), and (7.31) and the remarks following. The class orthogonality relations (7.4) permit an elegant determination of the expansion coefficients in some important cases, as illustrated below and in Section VIII. The representation of the general solution of Section VI is further simplified in these cases if the integrations are performed according to "classes of eigenvectors."

Definition 7.1.

The eigenvectors Λ and Λ' belong to the same class if they are linearly dependent over the field of complex numbers.

Thus, if Λ and Λ' belong to the same class, there exists a complex number $\zeta = \xi + i\eta \neq 0$, such that $\Lambda' = \zeta \Lambda$. We select as base elements for the classes the quasi-proper vectors Λ^q , which are defined as follows.

Definition 7.2.

Quasi-proper vectors, Λ^q , are eigenvectors that satisfy the relations

$$(\Lambda^q)^2 = \lambda_0^2, \quad (7.5)$$

where λ_0 is a solution of (5.2).

Thus, a quasi-proper vector, Λ^q , whose real part has magnitude less than unity ($\lambda_1^q < 1$) is a proper eigenvector, Λ^0 , and vice versa. We concentrate on the case $c \neq 1$. Then, the following lemma is true.

Lemma 7.1. Let $c \neq 1$ in Eq. (2.5). Then (1) each quasi-proper vector and its negative define one and only one class; (2) the set of all quasi-proper vectors defines all possible classes uniquely; (3) if one element of a class is real (or pure imaginary), then all other elements are either real or pure imaginary.

This lemma, therefore, asserts that each improper eigenvector has a unique representation $\Lambda = \zeta \Lambda^q$.

To prove the lemma, we project the real and complex parts of the vectors into E_3 as was previously done in Section III. Then from

$$\Lambda' = \Lambda_1' + i\Lambda_2'; \quad \Lambda = \Lambda_1 + i\Lambda_2; \quad \zeta = \xi + i\eta \quad (7.6)$$

follows the transformation of $\Lambda' = \zeta \Lambda$ in the projection space E_3 :

$$\left. \begin{aligned} \Lambda_1' &= \xi \Lambda_1 - \eta \Lambda_2; \\ \Lambda_2' &= \eta \Lambda_1 + \xi \Lambda_2. \end{aligned} \right\} \quad (7.7)$$

Thus, if $\Lambda_1 \neq 0$ and $\Lambda_2 \neq 0$, both components Λ_1' and Λ_2' lie in the plane spanned by Λ_1 and Λ_2 in E_3 . If either $\Lambda_1 = 0$ or $\Lambda_2 = 0$, then Λ_1' , Λ_2' , and Λ are parallel to each other in E_3 .

After these general remarks, we turn to the proof of part (3) of the lemma. Assume to the contrary that $\Lambda_1' \neq 0$, $\Lambda_2' \neq 0$. Since they are parallel to each other, $1 - \Lambda' \cdot \Omega \neq 0$ for all Ω . This implies that Λ' is a proper eigenvector, and from (5.5) follows $\Lambda_1' \cdot \Lambda_2' = 0$, which contradicts the fact that Λ_1' and Λ_2' are parallel. Thus either $\Lambda_1' = 0$ or $\Lambda_2' = 0$, q.e.d.

To prove part (2) of the lemma, we demonstrate that for any vector Λ there exist unique ζ_q , Λ^q such that, except for sign, $\Lambda = \zeta_q \Lambda^q$. Then $\Lambda^q = \zeta \Lambda$, where $\zeta = \zeta_q^{-1}$. Let Λ' in (7.6) be a quasi-proper vector, $\Lambda' = \Lambda^q$. Then by (7.5), $(\Lambda_1^q)^2 - (\Lambda_2^q)^2 = \lambda_0^2$ and $\Lambda_1^q \cdot \Lambda_2^q = 0$. Substituting in these equations the right sides of (7.7) yield

$$(\xi^2 - \eta^2)(\Lambda_1^2 - \Lambda_2^2) - 4\xi\eta\Lambda_1 \cdot \Lambda_2 = \lambda_0^2, \quad (7.8)$$

and

$$(\xi^2 - \eta^2)\Lambda_1 \cdot \Lambda_2 + \xi\eta(\Lambda_1^2 - \Lambda_2^2) = 0, \quad (7.9)$$

respectively. From these two equations follows

$$(\xi^2 - \eta^2) = \frac{\lambda_0^2(\Lambda_1^2 - \Lambda_2^2)}{(\Lambda_1^2 - \Lambda_2^2)^2 + 4(\Lambda_1 \cdot \Lambda_2)^2} \equiv 2a \quad (7.10)$$

$$\xi\eta = -\frac{\lambda_0^2(\Lambda_1 \cdot \Lambda_2)}{(\Lambda_1^2 - \Lambda_2^2)^2 + 4(\Lambda_1 \cdot \Lambda_2)^2} \equiv b, \quad (7.11)$$

or, observing that $\xi^2 > 0$, $\eta^2 > 0$, and letting $\epsilon = \pm 1$,

$$\begin{aligned} \xi &= \epsilon \left[a + (a^2 + b^2)^{1/2} \right]^{1/2}; \\ \eta &= \epsilon \left[-a + (a^2 + b^2)^{1/2} \right]^{1/2}. \end{aligned} \quad (7.12)$$

Thus $\Lambda^q = \zeta \Lambda$ with $\zeta = \xi + i\eta$ as given by (7.12) is uniquely determined, except for the sign, q.e.d.

To prove part (1) of the lemma, we assume that in $\Lambda' = \xi \Lambda$ of (7.7), both Λ' and Λ are quasi-proper vectors. We must show that necessarily $\zeta = \pm 1$. We proceed as in the proof for part (2), but since Λ is also a proper eigenvector, we have the additional conditions $\Lambda_1^2 - \Lambda_2^2 = \lambda_0^2$ and $\Lambda_1 \cdot \Lambda_2 = 0$ to impose on (7.10) and (7.11). It follows that $2a = 1$ and $b = 0$; hence $\xi = \pm 1$, $\eta = 0$. This concludes the proof of lemma 7.1.

We have just proved that each improper eigenvector has a unique representation, $\Lambda = \zeta \Lambda^q$.* We ask now for the range of ζ such that for a given Λ^q , $\zeta \Lambda^q = \Lambda$ is an improper eigenvector. The transformation (7.7) takes the form

$$\left. \begin{aligned} \Lambda_1 &= \xi \Lambda_1^q - \eta \Lambda_2^q \\ \Lambda_2 &= \eta \Lambda_1^q + \xi \Lambda_2^q \end{aligned} \right\} \quad (7.13)$$

subject to the conditions of Eqs. (7.5) for Λ^q ,

$$\lambda_1^2 - \lambda_2^2 = \lambda_0^2, \quad (\lambda_1 = |\lambda_1^q|, \lambda_2 = |\lambda_2^q|); \quad \Lambda_1^q \cdot \Lambda_2^q = 0, \quad (7.14)$$

and of Corollary 3.1 for the improper eigenvector, Λ ,

*The sign of ζ is fixed as soon as the base of the class, say Λ^q , is chosen.

$$\Lambda_1^2 \geq 1 \quad \text{for } \Lambda_2 = 0 \quad (7.15a)$$

or

$$\Lambda_1^2 \sin^2 \gamma \geq 1 \quad \text{for } \Lambda_2 \neq 0. \quad (7.16a)$$

First assume (7.15a) to hold. Then, by lemma 7.1, part (3), either

$$\Lambda_2^q = 0 \text{ and } \Lambda = \Lambda_1 = \xi \Lambda_1^q,$$

or

$$\Lambda_1^q = 0 \text{ and } \Lambda = \Lambda_1 = -\eta \Lambda_2^q.$$

Thus $\zeta = \xi$ or $\zeta = i\eta$, where

$$\xi^2 \lambda_1^2 \geq 1 \text{ or } \eta^2 \lambda_2^2 \geq 1 \quad (7.15b)$$

in order that $|\Lambda_1| > 1$. Now, assume (7.16a) to hold. Multiplying (7.16a) by Λ_2^2 yields

$$\Lambda_1^2 \Lambda_2^2 - (\Lambda_1 \cdot \Lambda_2)^2 \geq \Lambda_2^2.$$

Substituting (7.13) in this equation, simplifying, and rearranging terms, leads to the condition

$$(\xi^2 + \eta^2)^2 \geq \xi^2 / \lambda_1^2 + \eta^2 / \lambda_2^2. \quad (7.16b)$$

This is a quadratic form in ξ^2 and η^2 and contains the condition of Eq. (7.15b) as a special case when either $\xi = 0$ or $\eta = 0$. By the usual methods of analytic geometry it can be shown that (7.16b) represent the exterior of a parabola, which under the additional conditions $\xi^2 > 0$ and $\eta^2 > 0$ defines the range for $\zeta = \xi + i\eta$. For computational purposes in applications, this range is explicitly given by

$$\xi^2 \geq \begin{cases} -\eta^2 + \left(1 + \sqrt{1 + 4\eta^2 \lambda_0^2 \lambda_1^2} \right) / 2\lambda_1^2 & \text{for } \eta^2 \leq 1/\lambda_2^2 \\ 0 & \text{for } \eta^2 \geq 1/\lambda_2^2, \end{cases} \quad (7.16c)$$

or equivalently

$$\eta^2 \geq \begin{cases} -\xi^2 + \left(1 + \sqrt{1 - 4\xi^2 \lambda_0^2 \lambda_2^2 / \lambda_1^2} \right) / 2\lambda_2^2 & \text{for } \xi^2 \leq 1/\lambda_1^2 \\ 0 & \text{for } \xi^2 \geq 1/\lambda_1^2, \end{cases} \quad (7.16d)$$

where λ_0 , λ_1 , and λ_2 are defined in (7.14). These equations contain (7.15b) as a special case. Therefore, the integration in (2.15) can be performed over the set of all quasi-proper vectors defined by (7.5) and the range of the complex plane as given by any one of Eqs. (7.16b-d). Since to each complex improper Λ belong two singular directions, Ω_0^+ and Ω_0^- as shown in Fig. 3.2 [and calculated by (3.9)], it is necessary to specify for each ζ which Ω_0 is used in $A(\Lambda, \Omega_0) = A(\zeta \Lambda^q, \Omega_0^{\pm})$. Furthermore, when $\Lambda = \zeta \Lambda^q$ is real, there are infinitely many singular directions Ω_0 associated with Λ . The entire problem is simplified if, instead of calculating Ω_0 from ζ , we calculate ζ from Ω_0 . Since $\Lambda = \zeta \Lambda^q$ is an improper eigenvector, Eq. (2.12) requires

$$\Lambda \cdot \Omega_0 = \zeta (\Lambda^q \cdot \Omega_0) = 1, \quad (7.17)$$

and thus

$$\zeta = \frac{1}{\Lambda \cdot \Omega_0} = \frac{(\Lambda_1^0 - i\Lambda_2^0) \cdot \Omega_0}{(\Lambda_1 \cdot \Omega_0)^2 + (\Lambda_2 \cdot \Omega_0)^2}. \quad (7.18)$$

Hence, Eq. (2.15) takes the form

$$\psi(r, \Omega) = \int_{\{(\Lambda^0)^2 = \lambda_0^2\}} A^0(\Lambda^0) \psi_{\Lambda^0} d_4 \Lambda^0 + \int_{\{(\Lambda^q)^2 = \lambda_0^2\}} d_4 \Lambda^q \int_{\{\Omega_0\}} d_2 \Omega_0 A\left(\frac{\Lambda^q}{\Lambda^q \cdot \Omega_0}, \Omega_0\right) \psi_{\Lambda^q / \Lambda^q \cdot \Omega_0} \quad (7.19)$$

where $\psi_{\Lambda^q / \Lambda^q \cdot \Omega_0}$ is short for $\psi_{\Lambda^q / \Lambda^q \cdot \Omega_0, \Omega_0}$ (r, Ω). This second term corresponds to (6.8) and (6.9) when the order of integration over Λ and Ω_0 is interchanged.

Consider now the case when $\psi(r, \Omega)$ can be represented by eigenfunctions of only one class, such that

$$\psi(r, \Omega) = \psi(r, \Omega; \Lambda^0) = a_{\Lambda^0} \psi_{\Lambda^0} + a_{-\Lambda^0} \psi_{-\Lambda^0} + \int_{\{\Omega_0\}} d_2 \Omega_0 A_{\Lambda^0}(\Omega_0) \psi_{\Lambda^0 / \Lambda^0 \cdot \Omega_0} \quad (7.20)$$

if the base vector is the proper eigenvector Λ^0 , or such that

$$\psi(r, \Omega) = \psi(r, \Omega; \Lambda^q) = \int_{\{\Omega_0\}} d_2 \Omega_0 \Lambda_{\Lambda^q}^{\Lambda^q}(\Omega_0) \psi_{\Lambda^q/\Lambda^q \cdot \Omega_0}(r, \Omega) \quad (7.21)$$

if the base vector is a quasi-proper vector Λ^q with $\lambda_1^q \geq 1$. [By Lemma 7.1, no proper eigenfunctions exist for (7.21)]. Now, let $\psi(0, \Omega)$ in (7.20) be given. Then because of the assumption that (7.20) is valid,

$$\psi(0, \Omega) = a_{\Lambda^0 \Lambda^0} \Phi_{\Lambda^0}(\Omega) + a_{-\Lambda^0 -\Lambda^0} \Phi_{-\Lambda^0}(\Omega) + \int_{\{\Omega^0\}} d_2 \Omega_0 \Lambda_{\Lambda^0}^{\Lambda^0}(\Omega_0) \Phi_{\Lambda^0/\Lambda^0 \cdot \Omega_0}(\Omega). \quad (7.22)$$

It follows that the coefficients in Eq. (7.20) can be calculated, making use of the class orthogonality relations (7.4). Substituting (7.18) in (7.4), yields explicitly

$$\int_{\{\Omega\}} \Lambda^0 \cdot \Omega \left[\Phi_{\pm \Lambda^0}(\Omega) \right]^2 d_2 \Omega = N_{\lambda_0}, \quad (7.23a)$$

$$\left. \begin{aligned} \int_{\{\Omega\}} \Lambda^0 \cdot \Omega \Phi_{\Lambda^0}(\Omega) \Phi_{-\Lambda^0}(\Omega) d_2 \Omega &= 0, \\ \int_{\{\Omega\}} \Lambda^0 \cdot \Omega \Phi_{\pm \Lambda^0}(\Omega) \Phi_{\Lambda^0/\Lambda^0 \cdot \Omega_0}(\Omega) d_2 \Omega &= 0, \end{aligned} \right\} \quad (7.23b)$$

and

$$\int \Lambda^q \cdot \Omega \Phi_{\Lambda^q/\Lambda^q \cdot \Omega_0}(\Omega) \Phi_{\Lambda^q/\Lambda^q \cdot \Omega_1}(\Omega) d_2 \Omega = N_{\lambda_1^q} (\Lambda^q \cdot \Omega_1) \delta(\Omega_0 - \Omega_1). \quad (7.24)$$

Therefore, by (7.22) and (7.23),

$$a_{\pm \Lambda^0} = \frac{1}{N_{\lambda_0}} \int_{\Omega} \Lambda^0 \cdot \Omega \Phi_{\pm \Lambda^0}(\Omega) \psi(0, \Omega). \quad (7.25)$$

Carrying out (7.23a) with methods analogous to those used in Sections III and IV yields

$$N_{\lambda_0} = \frac{c^2 \kappa^2}{4\pi} \frac{\lambda_0^2}{1 - \lambda_0^2}. \quad (7.26)$$

To obtain $A_{\Lambda^0}(\Omega)$, we consider first the evaluation of the integral

$$U = \int d_2 \Omega \int d_2 \Omega_0 \frac{\Lambda^q \cdot \Omega}{\Lambda^q \cdot \Omega_0} \frac{\Phi_{\Lambda^q/\Lambda^q \cdot \Omega_0}(\Omega) \Phi_{\Lambda^q/\Lambda^q \cdot \Omega_0}(\Omega)}{\Lambda^q \cdot \Omega_0} A_{\Lambda^q}(\Omega_0). \quad (7.27a)$$

We make use of the explicit forms (2.11) for Φ_{Λ, Ω_0} , (2.13) for $G(\Lambda)$, and (3.1) for I_{Λ} , respectively,

$$\frac{\Phi_{\Lambda^q/\Lambda^q \cdot \Omega_0}}{\Lambda^q \cdot \Omega_0} = \frac{\kappa c}{4\pi} \frac{1}{1 - \frac{\Lambda^q \cdot \Omega}{\Lambda^q \cdot \Omega_0}} + \kappa G\left(\frac{\Lambda^q}{\Lambda^q \cdot \Omega}\right) \delta(\Omega - \Omega_0), \quad (7.28a)$$

$$G\left(\frac{\Lambda^q}{\Lambda^q \cdot \Omega}\right) = 1 - \frac{c}{4\pi} \int \frac{d_2 \Omega_1}{1 - \frac{\Lambda^q \cdot \Omega_1}{\Lambda^q \cdot \Omega}} = G_{\lambda_1^q}(\Lambda^q \cdot \Omega), \quad (7.28b)$$

and

$$I_{\Lambda^q/\Lambda^q \cdot \Omega} = \int \frac{d_2 \Omega_1}{1 - \frac{\Lambda^q \cdot \Omega_1}{\Lambda^q \cdot \Omega}} = I_{\lambda_1^q}(\Lambda^q \cdot \Omega). \quad (7.28c)$$

The last expression, (7.28c), could be evaluated using (4.33). There I_{Λ} depends only on λ_1 , λ_2 , and γ . From (7.18) and (7.7) it follows that λ_1 , λ_2 , and γ of the vector $\Lambda = \Lambda^q/\Lambda^q \cdot \Omega_1$ depend on λ_1^q (or λ_2^q), $\Lambda_1^q \cdot \Omega_1$ and $\Lambda_2^q \cdot \Omega_1$ [λ_2^q follows from the relation $(\lambda_1^q)^2 - (\lambda_2^q)^2 = \lambda_0^2$]. Thus, for any fixed class characterized by Λ^q , $I_{\Lambda^q/\Lambda^q \cdot \Omega_1}$ and $G_{\lambda_1^q}(\Lambda^q \cdot \Omega_1)$ are functions of $\Lambda^q \cdot \Omega_1$ only. Appropriate substitutions of (7.28) in (7.27a) and some algebraic manipulations yield

$$U = N_{\lambda_1^q}(\Lambda^q \cdot \Omega_1) \cdot A_{\Lambda^q}(\Omega_1), \quad (7.27b)$$

where

$$N_{\lambda_1^q}(\Lambda^q \cdot \Omega_1) = \Lambda^q \cdot \Omega_1 \kappa^2 \left\{ \left[G_{\lambda_1^q}(\Lambda^q \cdot \Omega_1) \right]^2 + H_{\lambda_1^q}(\Lambda^q \cdot \Omega_1) \right\}, \quad (7.29a)$$

$$H_{\lambda_1^q}(\Lambda^q \cdot \Omega_1) = \left(\frac{c}{4\pi} \right)^2 \int d_2 \Omega \int d_2 \Omega_0 \left\{ \frac{1}{\left(1 - \frac{\Lambda^q \cdot \Omega}{\Lambda^q \cdot \Omega_1} \right) \left(1 - \frac{\Lambda^q \cdot \Omega_0}{\Lambda^q \cdot \Omega} \right)} - \frac{1}{\left(1 - \frac{\Lambda^q \cdot \Omega_0}{\Lambda^q \cdot \Omega_1} \right) \left(1 - \frac{\Lambda^q \cdot \Omega}{\Lambda^q \cdot \Omega_0} \right)} \right\}. \quad (7.29b)$$

An explicit form for $N_{\lambda_1^q}(\Lambda^q \cdot \Omega_1)$ is

$$2N_{\lambda_1^q}(\Lambda^q \cdot \Omega_1) = \Lambda^q \cdot \Omega_1 \kappa^2 \left\{ 2 - \frac{c}{\pi} I_{\Lambda^q/\Lambda^q \cdot \Omega_1} + \left(\frac{c}{2\pi} \right)^2 \int \frac{I_{\Lambda^q/\Lambda^q \cdot \Omega_1} - I_{\Lambda^q/\Lambda^q \cdot \Omega}}{1 - \frac{\Lambda^q \cdot \Omega}{\Lambda^q \cdot \Omega_1}} d_2 \Omega \right\}. \quad (7.29c)$$

From this expression it follows that $N_{\lambda_1^q}(\Lambda^q \cdot \Omega_1)$ is bounded and analytic in the complex variable $v_1 = \Lambda^q \cdot \Omega_1$. To derive a shorter expression for $N_{\lambda_1^q}$ and for later reference, let us consider briefly the two limit cases for λ_1^q .

$\Lambda^q = \Lambda^0$: $\Lambda_1^0 = 0$ or $\Lambda_2^0 = 0$; i.e., Λ^0 is either real or pure imaginary. Then $\Lambda^0 \cdot \Omega = v$, where $|v| \leq |1/\lambda_0|$. Equations (7.29a,b) take the form

$$N_{\lambda_1^0}(v_1) = v_1 \kappa^2 \left\{ \left[G_{\lambda_1^0}(v_1) \right]^2 + \frac{c^2}{4} \left(\int dv_0 \int dv \frac{v_1 v_0}{(v_1 - v_0)(v_0 - v)} - \int dv \int dv_0 \frac{v_1 v_0}{(v_1 - v_0)(v_0 - v)} \right) \right\}.$$

Because of the Poincaré-Bertrand transformation formula,⁴⁰ this is

$$N_{\lambda_1^0}(v_1) = v_1 \kappa^2 \left\{ \left[G_{\lambda_1^0}(v_1) \right]^2 + \left[\frac{c\pi}{2} v_1 \right]^2 \right\}, v_1 = \Lambda^0 \cdot \Omega_1. \quad (7.30)$$

Since $N_{\lambda_1^q}(\Lambda^q \cdot \Omega_1)$ depends only on λ_1^q and $\Lambda^q \cdot \Omega_1$, and since it is analytic in $\Lambda^q \cdot \Omega_1$, we conclude by reason of analytic continuation from (7.30) and (7.29) that

$$N_{\lambda_1^q}(\Lambda^q \cdot \Omega_1) = \Lambda^q \cdot \Omega_1 \kappa^2 \left\{ \left[G_{\lambda_1^q}(\Lambda^q \cdot \Omega_1) \right]^2 + \left[\frac{c\pi}{2} \Lambda^q \cdot \Omega_1 \right]^2 \right\}. \quad (7.31)$$

We return now to the determination of $A_{\Lambda^0}(\Omega)$, and treat $A_{\Lambda^q}(\Omega)$ of (7.21) concurrently. Because of (7.20), (7.21), and the orthogonality relations (7.23),

$$\int \Lambda^q \cdot \Omega \Phi_{\Lambda^q/\Lambda^q \cdot \Omega_1}(\Omega) \psi'(0, \Omega) d_2 \Omega = -(0 + 0) + U,$$

where $\psi'(0, \Omega)$ stands for $\psi(0, \Omega; \Lambda^0) - (a_{\Lambda^0} \Phi_{\Lambda^0} + a_{-\Lambda^0} \Phi_{-\Lambda^0})$ when $\Lambda^q = \Lambda^0$, and for $\psi(0, \Omega; \Lambda^q)$ when $\Lambda^q \notin \{\Lambda^0\}$. Thus by (7.27b),

$$A_{\Lambda^q}(\Omega_1) = \frac{1}{N_{\lambda_1^q}(\Lambda^q \cdot \Omega_1)} \int \Lambda^q \cdot \Omega \Phi_{\Lambda^q/\Lambda^q \cdot \Omega_1}(\Omega) \psi'(0, \Omega) d_2 \Omega. \quad (7.32)$$

If we interchange the order of integration in (7.27a) and consider (7.24), then

$$U = \int d_2 \Omega_0 N_{\lambda_1^q}(\Lambda^0 \cdot \Omega_0) \delta(\Omega_0 - \Omega_1) A_{\Lambda^q}(\Omega_0) = N_{\lambda_1^q}(\Lambda^q \cdot \Omega_1) A_{\Lambda^q}(\Omega_1).$$

Therefore, the "orthogonality" condition (7.24) "legalizes" the interchange of variables in (7.27a). Thus, when we operate on (7.22) (or its equivalent for $\Lambda^q \notin \{\Lambda^0\}$) by

$$\int d_2 \Omega \Lambda^q \cdot \Omega \Phi_{\Lambda^q/\Lambda^q \cdot \Omega_1}(\Omega) \cdot ,$$

and treat the integrals as if they were proper integrals, and apply then the orthogonality relations (7.23b) and (7.24), we obtain the correct answer for the coefficient function $A_{\Lambda^q}(\Omega)$.

Inserting (7.28a) into (7.20) and (7.21) for $r = 0$ yields

$$G_{\lambda_1^q}(\Lambda^q \cdot \Omega) A_{\Lambda^q}(\Omega) + \frac{c}{4\pi} \int \frac{d_2 \Omega_0 A_{\Lambda^q}(\Omega_0)}{1 - \frac{\Lambda^q \cdot \Omega}{\Lambda^q \cdot \Omega_0}} = \psi'(0, \Omega). \quad (7.33)$$

This is a singular integral equation of the second kind for $A_{\Lambda^q}(\Omega)$.

Therefore the function space whose elements $\psi(r, \Omega)$ have a representation (7.20) or (7.21) and are simultaneously elements of the solution space of the transport equation (1.4), consists of all $\psi(r, \Omega)$ such that a solution $A_{\Lambda^q}(\Omega)$ exists for (7.33). However, a solution exists if the right side of (7.32) is meaningful, or explicitly, if the expression

$$A_{\Lambda^q}(\Omega_1) = \kappa \frac{\Lambda^q \cdot \Omega_1 \left[G_{\lambda_1^q}(\Lambda^q \cdot \Omega_1) \psi'(0, \Omega_1) + \frac{c}{4\pi} \int \frac{\Lambda^q \cdot \Omega \psi'(0, \Omega)}{\Lambda^q \cdot \Omega_1 - \Lambda^q \cdot \Omega} d_2 \Omega \right]}{N_{\lambda_1^q}(\Lambda^q \cdot \Omega_1)} \quad (7.34)$$

has a meaningful interpretation. In the one-dimensional case, it is sufficient for $\psi(0, \Omega)$ to satisfy the H condition.

Let arbitrary functions $\psi(0, \Omega; \Lambda^q)$ be given such that (7.34) exists for all quasi-proper vectors $\{\Lambda^q\}$. Then for each Λ^0 and $\Lambda^q \notin \{\Lambda^0\}$ exist, respectively, representations (7.20) and (7.21), namely $\psi(r, \Omega; \Lambda^q)$. A superposition of these solutions $\psi(r, \Omega; \Lambda^q)$ for all Λ^q yields the general solution of the transport Eq. (1.4) given in the form of Eq. (7.19). Now, take the special case when $\psi(0, \Omega)$ is a given function and $\psi(0, \Omega; \Lambda^q) = \psi(0, \Omega)$ is true for given sets $\{\Lambda^0\}_g$ and $\{\Lambda^q \neq \Lambda^0\}_g$. An arbitrary weighted superposition of these $\psi(r, \Omega; \Lambda^q)$ belonging to $\psi(0, \Omega)$ yields again a solution of the transport equation with the given $\psi(0, \Omega)$ as the angular distribution at the origin. Such a solution can be expressed as

$$\psi(r, \Omega) = \frac{\int_{\{\Lambda^0\}} w(\Lambda) \psi(r, \Omega; \Lambda) d_4 \Lambda}{\int_{\{\Lambda^0\}} w(\Lambda) d_4 \Lambda} + \frac{\int_{\{\Lambda^q \neq \Lambda^0\}} w(\Lambda) \psi(r, \Omega; \Lambda) d_4 \Lambda}{\int_{\{\Lambda^q \neq \Lambda^0\}} w(\Lambda) d_4 \Lambda}, \quad (7.35)$$

Here $w(\Lambda^q)$ is an arbitrary weight function depending on Λ^q (including Λ^0) only.

We turn now to the case when $c = 1$ in (2.5). By (5.3), the corresponding eigenvalue is $\lambda_0 = 0$. Thus by (7.11) and (7.12), the components ξ and η of the factor $\zeta = \xi + i\eta$, which transforms an arbitrary improper eigenvector Λ into the quasi-proper vector Λ^q of its class, are both zero, unless $\Lambda_1^2 - \Lambda_2^2 = 0$ and $\Lambda_1 \cdot \Lambda_2 = 0$. In other words, if $|\Lambda_1| \neq |\Lambda_2|$ and/or $\Lambda_1 \cdot \Lambda_2 \neq 0$, $\zeta \Lambda = \Lambda^0 \Rightarrow \xi = \eta = 0$, $\Lambda^0 = 0$. But if $|\Lambda_1| = |\Lambda_2|$ and $\Lambda_1 \cdot \Lambda_2 = 0$, Λ is itself a quasi-proper vector. It will be shown that all quasi-proper vectors whose components Λ_1^q and Λ_2^q lie in a given plane of the projection space E_3 belong to one of two classes, which are complex conjugates of each other. Let

$$\Lambda^q = \Lambda_1^q + i\Lambda_2^q, \quad |\Lambda_1^q| = |\Lambda_2^q|, \quad \Lambda_1^q \cdot \Lambda_2^q = 0 \quad (7.36)$$

be the base vector. Then $\zeta \Lambda^q$, where ζ is any complex number, is also quasi-proper. Using (7.36) in (7.7) yields $(\Lambda_1^q)^2 = (\Lambda_2^q)^2 = (\xi^2 + \eta^2)(\Lambda_1^q)^2$ and $\Lambda_1^q \cdot \Lambda_2^q = 0$. The same result is obtained when we replace (7.36) by its complex conjugate, $\bar{\Lambda}^q = \Lambda_1^q - i\Lambda_2^q$. However, $\bar{\Lambda}^q = \zeta \Lambda^q$ has no solution as is easily verified. Thus we have

Lemma 7.2. Let $c = 1$ in Eq. (2.5) and consider any plane through the origin in E_3 . Then

(1) All quasi-proper vectors of the given plane belong to exactly one of two mutually conjugate complex classes, and

(2) The improper eigenvectors in the given plane that are not quasi-proper cannot be generated from the quasi-proper vectors by multiplication with a complex number.

Since the classification of the eigenfunctions according to quasi-proper vectors breaks down in this case, we have several choices for a substitute.

One such choice is to replace the set of base vectors $\{\Lambda^q\}$ defined by Eq. (7.5) by a new set of base vectors $\{\Lambda^c\}$ such that

$\Lambda^c = \Lambda_1^c + i\Lambda_2^c$, $\Lambda_1^c \cdot \Lambda_2^c = 0$ and either $(|\Lambda_1^c| = 1, |\Lambda_2^c| = 0)$ or $(|\Lambda_1^c| \geq 1, |\Lambda_2^c| = 1)$; i.e.,

$$\{\Lambda^c\} \equiv \{\Lambda: \Lambda_1 \cdot \Lambda_2 = 0 \text{ and } \Lambda_1^2 = 1, \Lambda_2 = 0 \text{ or } \Lambda_1^2 \geq 1, \Lambda_2^2 = 1\}.$$

(7.37)

The class of improper eigenvectors is represented by $|\Lambda_1^c| = |\Lambda_2^c| = 1$. Finally, consider the case $c = 0$ in (2.5). Here, no proper eigenfunction exists, and the improper eigenvectors are given by

$$\Phi_{\Lambda, \Omega_0}^{c=0} = \kappa \delta(\Omega - \Omega_0) \quad (7.38a)$$

with

$$\Psi_{\Lambda, \Omega}^{c=0} = \kappa e^{-\Lambda \cdot r} \delta(\Omega - \Omega_0). \quad (7.38b)$$

In Section VIII we shall see that the sets $\{\Lambda^0\}$ and $\{\Lambda: \Lambda \cdot \Omega_0 = 1\}$ and consequently the general solution space $\{\psi\}$ of (1.4) are often considerably simplified when the solutions ψ are subject to geometrical invariance conditions.

VIII. REDUCTION OF THE GENERAL SOLUTION TO SPECIAL CASES

A. Slab Geometry with and without Axial Symmetry

The requirement that $\psi(r, \Omega)$ be a plane solution of Eq. (1.4) means, by definition, that $\psi(r, \Omega)$ be constant for all r whose end points lie on parallel planes. We rotate our coordinate system so that these planes are orthogonal to the z -axis. It follows, then, from Eqs. (2.10) and (2.14), that

$$\frac{\partial \psi_{\Lambda}^0}{\partial x} = \frac{\partial \psi_{\Lambda}^0}{\partial y} = 0$$

and

$$\frac{\partial \psi_{\Lambda, \Omega_0}}{\partial x} = \frac{\partial \psi_{\Lambda, \Omega_0}}{\partial y} = 0,$$

if and only if

$$\Lambda_x = \Lambda_y = 0. \quad (8.1)$$

Hence the elementary solutions have the form

$$\left. \begin{aligned} \psi_{\Lambda}^0 &= e^{-\Lambda_z z} \Phi_{\Lambda}^0(\Omega) && \text{(regular expansion modes);} \\ \psi_{\Lambda, \Omega_0} &= e^{-\Lambda_z z} \Phi_{\Lambda, \Omega_0}(\Omega) && \text{(generalized expansion modes).} \end{aligned} \right\} \quad (8.2)$$

We must now submit the regular eigenfunctions to the conditions (5.5) that resulted from the characteristic equation. Because of Eqs. (3.2), (3.3), and (8.1),

$$|\operatorname{Re} \Lambda_z| = \lambda_1;$$

and

$$|\operatorname{Im} \Lambda_z| = \lambda_2.$$

We show that either λ_1 or λ_2 must be zero. From Eq. (8.1) it follows that $\sin \gamma = 0$, since Λ_{1z} is necessarily parallel to Λ_{2z} when all other components of Λ are zero. This means $\cos \gamma = \pm 1$. But then the second equation of (5.5) requires

$$\lambda_1 \lambda_2 = 0, \quad \text{q.e.d.}$$

From Eqs. (5.3) and the first equation of (5.5), we deduce that

$$\begin{aligned} \text{a) } \lambda_2 &= 0 && \text{for } 0 < c < 1 \quad (\lambda_0 \text{ real}); \\ \text{b) } \lambda_1 &= \lambda_2 = 0 && \text{for } c = 1 \quad (\lambda_0 = 0); \\ \text{c) } \lambda_1 &= 0 && \text{for } 1 < c < \infty \quad (\lambda_0 \text{ imaginary}). \end{aligned}$$

Hence the regular expansion modes are

$$\psi_{\Lambda}^0 = e^{\pm \lambda_0 z} \Phi_{\Lambda}^0(\Omega),$$

where

$$\phi_{\Lambda}^0 = \frac{\kappa c}{4\pi} \frac{1}{1 \pm \lambda_0 \Omega_z} = \frac{\kappa c}{4\pi} \frac{1}{1 \pm \lambda_0 \mu} \quad (8.3)$$

if Ω is defined as in (3.4) and $\mu = \cos \theta$.

Next we must reduce the generalized eigenfunctions to the case of plane geometry.

From Eq. (8.1) it follows, as before, that $\sin \gamma = 0$ since Λ_{1z} and Λ_{2z} are necessarily parallel. But then Eqs. (6.4) and (6.5) cannot be satisfied simultaneously in E_3 , unless $\lambda_2 = 0$. Thus, by Lemma 7.1, Eqs. (7.18), (7.28a), (3.4), (4.1), (7.28b), and (6.10),

$$\phi_{\Lambda^0/\Lambda^0 \cdot \Omega_0} = \frac{\kappa c}{4\pi} \frac{1}{1 - \mu/\mu_0} + \kappa G(\mu_0) \delta(\mu - \mu_0) \delta(\phi - \phi_0). \quad (8.4)$$

Again by Lemma (7.1) and the remarks above, no other class of quasi-proper vectors exists. Therefore the general solution of (1.4) in slab geometry *without axial symmetry* has the form (7.20) and is in the assumed coordinate system,

$$\begin{aligned} \psi(z, \mu, \phi) = & a_+ e^{-\lambda_0 z} \phi_{\Lambda^0}(\mu) + a_- e^{+\lambda_0 z} \phi_{-\Lambda^0}(\mu) \\ & + \int_0^{2\pi} d\phi_0 \int_{-1}^1 d\mu_0 A(\mu_0, \phi_0) e^{-z/\mu_0} \phi_{\Lambda^0/\Lambda^0 \cdot \Omega_0}(\mu, \phi). \end{aligned} \quad (8.5)$$

The corresponding orthogonality relations are given by (7.23), (7.26), (7.24), and (7.30), respectively. In the case of *axial symmetry* $\psi(z, \mu, \phi) = \psi(z, \mu)$ and $A(\mu, \phi) = A(\mu)$ are independent of ϕ . If we further define

$$\phi_{\pm \lambda_0} = 2\pi \phi_{\pm \Lambda^0}, \quad \phi_{1/\mu_0}(\mu) = \int_0^{2\pi} \phi_{\Lambda^0/\Lambda^0 \cdot \Omega_0}(\mu, \phi) d\phi_0, \quad (8.6a)$$

then (8.5) reduces to

$$\psi(z, \mu) = a_{+\lambda_0} \phi_{\lambda_0}(\mu) e^{-\lambda_0 z} + a_{-\lambda_0} \phi_{-\lambda_0}(\mu) e^{\lambda_0 z} + \int_{-1}^1 d\mu_0 A(\mu_0) \phi_{1/\mu_0}(\mu) e^{-z/\mu_0} \quad (8.6b)$$

which is identical to the results obtained in Refs. 4 and 12.

B. Spherical Symmetry

For spherical symmetry, the angular distribution ψ is invariant under rotation of the (r, Ω) -system and therefore is only a function of the distance from the origin, $\rho = (r \cdot r)^{1/2}$, and the cosine of the angle between r and Ω , $\mu = \Omega \cdot r / \rho$. The transport Eq. (1.4) reduces under these symmetry conditions to

$$\mu \frac{\partial}{\partial \rho} \psi(\rho, \mu) + \frac{1 - \mu^2}{\rho} \frac{\partial}{\partial \mu} \psi(\rho, \mu) + \psi(\rho, \mu) = \frac{c}{2} \int_{-1}^{+1} \psi(\rho, \mu') d\mu'. \quad (8.7)$$

Necessary and sufficient for (2.15) to be the general solution of (8.7) is the condition that the expansion coefficients $A^0(\Lambda)$ and $A(\Lambda, \Omega)$ are independent of the (complex) directions of Λ , which implies

$$A^0(\Lambda) = A^0(\lambda_1), \quad (8.8a)$$

where $\Lambda \in \{\Lambda^0\}$; i.e., $\lambda_1 = |\Lambda_1| < 1$, $|\lambda_2| = \lambda_2$, $\lambda_1^2 = \lambda_2^2 + \lambda_0^2$, $\cos \gamma = 0$ (from 5.5), and

$$A(\Lambda, \Omega_0) = A(\lambda_1, \lambda_2, \gamma; \Omega_0^+), \quad (8.8b)$$

where Λ is an improper eigenvector, satisfying either $\lambda_1 \sin \gamma \geq 1$ or $\lambda_1 > 1$, $\lambda_2 = 0$ (2.12 and Corollary 3.1). Preceding Corollary 3.1 and in Fig. 3.2 we have seen that for each given Λ , ($\lambda_2 \neq 0$) two singular directions, Ω_0^+ and Ω_0^- exist, such that $\Lambda_1, \Lambda_2, \Omega_0^+$ and $\Lambda_1, \Lambda_2, \Omega_0^-$ form a right-handed and left-handed coordinate system, respectively. Conversely, given any $\Omega_0^+, \lambda_1, \lambda_2, \gamma$ or $\Omega_0^-, \lambda_1, \lambda_2, \gamma$ specifies the orientation of all associated $\Lambda_1, \Lambda_2, \Omega_0$ systems uniquely as right- or left-handed. Thus, Eq. (2.15) takes the form

$$\begin{aligned}
\psi(\rho, \mu) = & \int d\lambda_1 A^0(\lambda_1) \psi_{\lambda_1}(\rho, \mu) \\
& \lambda_1 = \begin{cases} \lambda_0 & \text{for } c \leq 1 \\ 0 & \text{for } c \geq 1 \end{cases} \\
& + \int_1^\infty d\lambda_1 \int_0^\infty d\lambda_2 \int_{\sin \gamma \geq 1/\lambda_1} d\gamma \left\{ A(\lambda_1, \lambda_2, \gamma, \Omega_0^+) \psi_{\lambda_1, \lambda_2, \gamma; \Omega_0^+}(\rho, \mu) \right. \\
& \left. + A(\lambda_1, \lambda_2, \gamma, \Omega_0^-) \psi_{\lambda_1, \lambda_2, \gamma; \Omega_0^-}(\rho, \mu) \right\}
\end{aligned} \tag{8.9a}$$

where

$$\psi_{\lambda_1}(\rho, \mu) = \int_R d_3 R \psi_{R\Lambda^0}(r, \Omega), \tag{8.9b}$$

and

$$\psi_{\lambda_1, \lambda_2, \gamma, \Omega_0^\pm}(\rho, \mu) = \int_{\{\Omega_0\}} d_2 \Omega_0^\pm \int_R dR \psi_{R\Lambda^I, \Omega_0^\pm}(r, \Omega). \tag{8.9c}$$

The first term on the right side of (8.9a), together with (8.9b), corresponds to (5.12) and the remarks preceding (5.12) apply. The integration over λ_1 , instead of λ_2 as in (5.12), is just a matter of changing variables of integration, and is an example of an alternate approach to (5.12). That $\psi_{\lambda_1}(\rho, \mu)$ is a function of λ_1, ρ , and μ only shall be explained briefly for the sake of completeness, although it may appear obvious. Let us select our Cartesian reference system such that

$$\mathbf{r} = \rho(0, 0, 1) \tag{8.10a}$$

and

$$\Omega = (\sin \theta, 0, \cos \theta), \tag{8.10b}$$

where

$$\cos \theta = \mu. \tag{8.10c}$$

Furthermore, let a rotated eigenvector RA^0 in this system be given by $RA^0 = \Lambda'$,

$$\left. \begin{aligned} \Lambda'_1 &= \lambda_1 (\cos \phi_1 \sin \theta_1, \sin \phi_1 \sin \theta_1, \cos \theta_1) \\ \Lambda'_2 &= \lambda_2 (\cos \phi_2 \sin \theta_2, \sin \phi_2 \sin \theta_2, \cos \theta_2) \end{aligned} \right\} \quad (8.10d)$$

under the orthogonality constraint $\Lambda'_1 \cdot \Lambda'_2 = 0$,

$$\cos(\phi_1 - \phi_2) = \cot \theta_1 \cot \theta_2. \quad (8.10e)$$

Then

$$\psi_{RA^0}(r, \Omega) = \frac{c\kappa}{4\pi} \frac{e^{-r \cdot \Lambda'}}{1 - \Lambda' \cdot \Omega}, \quad (8.11a)$$

where

$$\left. \begin{aligned} r \cdot \Lambda' &= \rho (\lambda_1 \cos \theta_1 + i \lambda_2 \cos \theta_2), \\ \text{and} \\ \Lambda' \cdot \Omega &= \lambda_1 (\sin \theta \cos \phi_1 \sin \theta_1 + \cos \theta \cos \theta_1) \\ &\quad + i \lambda_2 (\sin \theta \cos \phi_2 \sin \theta_2 + \cos \theta \cos \theta_2). \end{aligned} \right\} \quad (8.11b)$$

The integration of (8.9b) in this coordinate system is now explicitly

$$\psi_{\lambda_1}(\rho, \mu) = \frac{c\kappa}{4\pi} \int_0^{2\pi} d\phi_1 \int_0^{\pi} d\theta_1 \int_0^{2\pi} d\phi_2 \frac{e^{-r \cdot \Lambda'}}{1 - \Lambda' \cdot \Omega}. \quad (8.12a)$$

Therefore, $\psi_{\lambda_1}(\rho, \mu)$ is by (8.12a) and (8.11b) a function of ρ , λ_1 , λ_2 , $\sin \theta$, and $\cos \theta$, but because of (8.10c) and (5.5), the independent variables reduce to λ_1 , ρ , and μ only. If we let $\lambda_1 \rightarrow \lambda_0$ for $c \leq 1$ and $\lambda_1 \rightarrow 0$ for $c \geq 1$, then (8.9c) becomes³⁸

$$\psi_{\lambda_1}(\rho, \mu) = \text{const.} \int_{-1}^{+1} \frac{e^{\pm \lambda_0 \rho v \mu}}{1 \pm \lambda_0 v} I_0 \left(\rho \lambda_0 \sqrt{(1 - \mu^2)(1 - v^2)} \right) dv, \quad (8.12b)$$

where $\eta = \cos \theta$ ($\theta = \theta_1$ for $c \leq 1$, $\theta = \theta_2$ for $c \geq 1$), and $I_0(z)$ is the modified Bessel function of the first kind.

The second term on the right of (8.9a), together with (8.9c), corresponds to (6.8) and the remarks following it. The integration over λ_1 instead of λ_1^Ω , as in (6.8), is just a matter of changing variables of integration [cf. (6.11)] and is an example of an alternate approach to (6.8). To show that $\psi_{\lambda_1, \lambda_2, \gamma, \Omega_0^\pm}(\rho, \mu)$ is a function of the indicated parameters and variables only, we use the same reference system as before [Eqs. (8.10)]. The integrand is given by

$$\begin{aligned} \psi_{R\Lambda^I, \Omega_0^\pm}(r, \Omega) = e^{-r \cdot \Lambda} & \left[\frac{c\kappa}{4\pi} \frac{1}{1 - \Lambda' \cdot \Omega} \right. \\ & \left. + \kappa G(\lambda_1, \lambda_2, \gamma) \delta(\mu - \mu_0^\pm) \delta(\phi - \phi_0^\pm) \right]. \end{aligned} \quad (8.13)$$

Here Λ^I is a convenient reference vector as in Section VI (e.g., Λ_1^I is taken co-planar to r and Ω). The δ -function corresponds to (6.10), and the integral [referring to the second term in (8.13)]

$$\int d_2 \Omega_0^\pm \int dR^{\Omega_0^\pm} e^{-r \cdot \Lambda} \kappa G(\mu - \mu_0^\pm) \delta(\phi - \phi_0^\pm)$$

correspond to the first integral in (6.9) and to (6.13). We note that R^{Ω_0} represents the plane rotation group with $\Omega_0^\pm$ as axis of rotation.

Referring to the first term in (8.13), the integral

$$\int d_2 \Omega_0 \int dR^{\Omega_0} \frac{c\kappa}{4\pi} \frac{e^{-r \cdot \Lambda'}}{1 - \Lambda' \cdot \Omega}$$

correspond to the second part of (6.9). They have the same structure as (8.9b), since in our notation

$$\int d_2 \Omega_0 \int dR^{\Omega_0} = \int d_3 R.$$

The same reasoning as was applied to (8.9b) can be applied to (8.9c) to prove that $\psi_{\lambda_1, \lambda_2, \gamma, \Omega_0^\pm}(\rho, \mu)$ is indeed a function of $\lambda_1, \lambda_2, \gamma, \rho$, and μ

only. This proves that the equations (8.8) are sufficient for spherical geometry. The proof that they are also necessary to yield the general solution of (8.7) is omitted.

C. Cylindrical Symmetries

The space vector r in cylindrical geometry will be specified by the coordinates (ρ, ϕ, z) , and the direction of motion Ω by the two angles (θ, ϕ_Ω) . If z is taken as the axis of the cylinder, then cylindrical symmetry requires that ψ remains invariant under rotation about this axis. Thus ψ can depend only on the difference $\phi = \phi_\Omega - \phi_0$ of the azimuthal angles. Under these conditions, the transport Eq. (1.4) becomes

$$\sin \theta \left[\cos \phi \frac{\partial \psi(\rho, \theta, \phi, z)}{\partial \rho} - \frac{\sin \phi}{\rho} \frac{\partial \psi(\rho, \theta, \phi, z)}{\partial \phi} \right] + \cos \theta \frac{\partial \psi(\rho, \theta, \phi, z)}{\partial z} + \psi(r, \theta, \phi, z) = \frac{c}{4\pi} \int_0^\pi \sin \theta' d\theta' \int_0^{2\pi} \psi(r, \theta', \phi', z) d\phi'. \quad (8.14)$$

Necessary and sufficient for (2.15) to be the general solution of (8.14) is to have the expansion coefficients, $A^0(\Lambda)$ and $A(\Lambda, \Omega)$, satisfy

$$A^0(\Lambda) = A^0(\lambda_1, \Lambda_z^+), \quad (8.15a)$$

where $\Lambda \in \{\Lambda^0\}$ as in (8.8a) and

$$A(\Lambda, \Omega) = A(\lambda_1, \lambda_2, \gamma, \Lambda_z^\pm, \Omega_0^\pm), \quad (8.15b)$$

where Λ is an improper eigenvector as in (8.8b). The Λ_z^+ and Λ_z^- identify the two possible cases where the vector triples, consisting of the z -axis, Λ_1 , and Λ_2 , form right-handed or left-handed systems, respectively. With these coefficients, (2.15) takes the form

$$\psi(\pi, \theta, \phi, z) =$$

$$\begin{aligned} & \int_{-1}^1 d\lambda_1 \int_{-\lambda_1}^{+\lambda_1} d\Lambda_{1z} \int_{-\sqrt{(\lambda_1^2 - \lambda_0^2)(\lambda_1^2 - \Lambda_{1z}^2)}/\lambda_1}^{+\sqrt{(\lambda_1^2 - \lambda_0^2)(\lambda_1^2 - \Lambda_{1z}^2)}/\lambda_1} d\Lambda_{2z} \left[A^0(\lambda_1, \Lambda_z^+) \psi_{\lambda_1, \Lambda_z^+} + A^0(\lambda_1, \Lambda_z^-) \psi_{\lambda_1, \Lambda_z^-} \right] \\ & \lambda_1 = \begin{cases} \lambda_0 & \text{for } c \leq 1 \\ 0 & \text{for } c > 1 \end{cases} \\ & + \int_1^\infty d\lambda_1 \int_{-\lambda_1}^{+\lambda_1} d\Lambda_{1z} \int_0^\infty d\lambda_2 \int_{\begin{cases} \lambda_1 \sin \gamma \geq 1 \\ 0 \leq \gamma \leq 2\pi \end{cases}} d\gamma \int_{\begin{cases} \left(\Lambda_{1z} \cos \gamma + \sqrt{\lambda_1^2 - \Lambda_{1z}^2} \sin \gamma \right) \lambda_2 / \lambda_1 \\ \left(\Lambda_{1z} \cos \gamma - \sqrt{\lambda_1^2 - \Lambda_{1z}^2} \sin \gamma \right) \lambda_2 / \lambda_1 \end{cases}} d\Lambda_{2z} \left[A(\lambda_1, \lambda_2, \gamma, \Lambda_z^+, \Omega_0^+) \psi_{\lambda_1, \lambda_2, \gamma, \Lambda_z^+, \Omega_0^+} \right. \\ & + A(\lambda_1, \lambda_2, \gamma, \Lambda_z^-, \Omega_0^+) \psi_{\lambda_1, \lambda_2, \gamma, \Lambda_z^-, \Omega_0^+} + A(\lambda_1, \lambda_2, \gamma, \Lambda_z^+, \Omega_0^-) \psi_{\lambda_1, \lambda_2, \gamma, \Lambda_z^+, \Omega_0^-} \\ & \left. + A(\lambda_1, \lambda_2, \gamma, \Lambda_z^-, \Omega_0^-) \psi_{\lambda_1, \lambda_2, \gamma, \Lambda_z^-, \Omega_0^-} \right], \end{aligned}$$

(8.16a)

where

$$\psi_{\lambda_1, \Lambda_z^\pm} \equiv \psi_{\lambda_1, \Lambda_z^\pm}(\rho, \theta, \phi, z) = \int_0^{2\pi} d\phi_1 \psi_{\Lambda^\pm}; \quad \Lambda^\pm \in \{\Lambda^0\}; \quad (8.16b)$$

$$\psi_{\lambda_1, \lambda_2, \gamma, \Lambda_z^\pm, \Omega_0^\pm} \equiv \psi_{\lambda_1, \lambda_2, \gamma, \Lambda_z^\pm, \Omega_0^\pm}(\rho, \theta, \phi, z) = \int_0^{2\pi} d\phi_1 \psi_{\Lambda^\pm, \Omega_0^\pm}; \quad (8.16c)$$

and $\Lambda^\pm$ refers to those vectors for which the triple $(\Lambda_1, \Lambda_2, z\text{-axis})$ form right- and left-handed systems, respectively. In these formulas, we have the Cartesian coordinate system chosen so that r lies in the (z, x) plane; thus,

$$r = (\rho, 0, z), \quad \Omega = (\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta) \quad (8.17a)$$

and

$$\Lambda_j = \left(\cos \phi_j \sqrt{\lambda_j^2 - \Lambda_{jz}^2}, \sin \phi_j \sqrt{\lambda_j^2 - \Lambda_{jz}^2}, \Lambda_{jz} \right), \quad j = 1, 2 \quad (8.17b)$$

under the condition $\Lambda_1 \cdot \Lambda_2 = \lambda_1 \lambda_2 \cos \gamma$,

$$\cos(\phi_1 - \phi_2) = \frac{\lambda_1 \lambda_2 \cos \gamma - \Lambda_{1z} \Lambda_{2z}}{\sqrt{(\lambda_1^2 - \Lambda_{1z}^2)(\lambda_2^2 - \Lambda_{2z}^2)}} \quad (8.17c)$$

Thus, in the course of integration for each choice $\lambda_1, \Lambda_{1z}, \lambda_2, \gamma, \Lambda_{2z}$, and ϕ_1 , two possibilities exist for Λ_2 , namely Λ_2^+ on the positive side of the (z, Λ_1) -plane and Λ_2^- on the negative side of the (z, Λ_1) -plane. Once the relative position of Λ_2 is fixed, we know two singular directions Ω_0^+ and Ω_0^- exist. All possible combinations account for the different terms under the integral signs in (8.16a).

To show that the elementary solutions for cylindrical symmetry, given by Eqs. (8.16b,c), are functions of the indicated arguments only, we write ψ_{Λ_0} and ψ_{Λ, Ω_0} explicitly and insert the explicit expressions for $r \cdot \Lambda$ and $\Lambda \cdot \Omega$. It is then seen in a similar manner as for spherical symmetry that here ψ depends on $\rho, \sin \theta, \cos \theta, \sin \phi, \cos \phi$, and z only, which proves the assertion.

Often, additional invariance properties are imposed on the solutions of (8.14) such as (a) invariance under reflection on any plane through the z -axis, (b) invariance under translation along the z -axis, and (c) symmetry with respect to a plane orthogonal to the z -axis.

(a) Let us impose on the solution ψ of (8.16a) the additional requirement of symmetry with respect to any plane through the axis of symmetry. This means ψ is required to be an even function in ϕ in our adopted coordinate system. It is easily verified that when $\psi(\rho, \theta, \phi, z)$ is a solution, $\psi(\rho, \theta, -\phi, z)$ is also a solution, and the sum of these two solutions is an even function in ϕ . The implication of this remark on the coefficient functions, A^0 and A , is seen by considering the explicit expressions of $r \cdot \Lambda$ and $\Omega \cdot \Lambda$ as above. It is observed that all terms in $r \cdot \Lambda$ and $\Omega \cdot \Lambda$ are even in ϕ , ϕ_1 , and ϕ_2 , except

$$\sin \theta \sin \phi \left[\sin \phi_1 \sqrt{\lambda_1^2 - \Lambda_{1z}^2} + i \sin \phi_2 \sqrt{\lambda_2^2 - \Lambda_{2z}^2} \right]. \quad (8.18)$$

Replacing ϕ by $-\phi$ in (8.18) is equivalent to replacing ϕ_1 by $-\phi_1$ and ϕ_2 by $-\phi_2$ while all other terms in $r \cdot \Lambda$ and $\Omega \cdot \Lambda$ remain unchanged. This transformation is achieved by a reflection with respect to the (z, r) -plane of the vectors Λ . If the original vector was Λ^+ , the reflected one is Λ^- . If the vector is an improper eigenvector, then Ω_0^+ reflected becomes Ω_0^- . Similar remarks hold for Λ^- and Ω_0^- . It follows now that in (8.16) the elementary solutions are

$$\psi_{\lambda_1, \Lambda_z^+}(\rho, \theta, \phi, z) = \psi_{\lambda_1, \Lambda_z^-}(\rho, \theta, -\phi, z), \quad (8.19a)$$

and

$$\psi_{\lambda_1, \lambda_2, \gamma, \Lambda_z^+, \Omega_0^+}(\rho, \theta, \phi, z) = \psi_{\lambda_1, \lambda_2, \gamma, \Lambda_z^-, \Omega_0^-}(\rho, \theta, -\phi, z). \quad (8.19b)$$

Thus the conditions for (8.16a) to be an even function in ϕ are

$$A^0(\lambda_1, \Lambda_z^+) = A^0(\lambda_1, \Lambda_z^-), \quad (8.20a)$$

and

$$A(\lambda_1, \lambda_2, \gamma, \Lambda_z^+, \Omega_0^+) = A(\lambda_1, \lambda_2, \gamma, \Lambda_z^-, \Omega_0^-), \quad (8.20b)$$

where the upper and lower symbols $+$ and $-$ correspond, respectively.

It is of interest to know the behavior of ψ in (8.16a) when $\Lambda_{1z} = \lambda_1$. The elementary solutions (8.16b,c) will satisfy (8.19) and (8.20). In this case, the solution is simply obtained by

$$\psi(\rho, \theta, \phi, z) = \int_{\lambda_1 = \begin{cases} \lambda_0 & \text{for } c \leq 1 \\ 0 & \text{for } c > 1 \end{cases}} d\lambda_1 A^0(\lambda_1) \psi_{\lambda_1} + \int_1^\infty d\lambda_1 \int_0^\infty d\lambda_2 \int_{\{\lambda_1 \sin \gamma \geq 1\}} d\gamma$$

$$\cdot \left[A(\lambda_1, \lambda_2, \gamma, \Omega_0^+) \psi_{\lambda_1, \lambda_2, \gamma, \Omega_0^+} + A(\lambda_1, \lambda_2, \gamma, \Omega_0^-) \psi_{\lambda_1, \lambda_2, \gamma, \Omega_0^-} \right] \quad (8.21a)$$

$$(\Lambda_{1z} = \lambda_1, \Lambda_{1x} = \Lambda_{1y} = 0),$$

where

$$\psi_{\lambda_1} = \int_0^{2\pi} d\phi_2 \psi_\Lambda, \quad \psi_{\lambda_1, \lambda_2, \gamma, \Omega_0^\pm} = \int_0^{2\pi} d\phi_2 \psi_{\Lambda, \Omega_0^\pm} \quad (8.21b, c)$$

(b) In the case of cylindrical symmetry with translational invariance with respect to the z-axis, Λ_z must be set equal to zero. Equation (8.16a) simplifies to

$$\psi(\rho, \theta, \phi) = \int_{\lambda_1 = \begin{cases} \lambda_0 & \text{for } c \leq 1 \\ 0 & \text{for } c > 1 \end{cases}} d\lambda_1 \{ A_+^0(\lambda_1) \psi_{\lambda_1^+} + A_-^0(\lambda_1) \psi_{\lambda_1^-} \}$$

$$+ \int_1^\infty d\lambda_1 \int_0^\infty d\lambda_2 \int_{\left\{ \begin{array}{l} \lambda_1 \sin \gamma \geq 1 \\ 0 \leq \gamma \leq \pi \end{array} \right\}} d\gamma \left[A_+(\lambda_1, \lambda_2, \gamma, \Omega_0^+) \psi_{\lambda_1, \lambda_2, \gamma, \Omega_0^+}^+ + A_+(\lambda_1, \lambda_2, \gamma, \Omega_0^-) \psi_{\lambda_1, \lambda_2, \gamma, \Omega_0^-}^+ \right.$$

$$\left. + A_-(\lambda_1, \lambda_2, \gamma, \Omega_0^+) \psi_{\lambda_1, \lambda_2, \gamma, \Omega_0^+}^- + A_-(\lambda_1, \lambda_2, \gamma, \Omega_0^-) \psi_{\lambda_1, \lambda_2, \gamma, \Omega_0^-}^- \right], \quad (8.22)$$

where the $A_\pm^0$, $A_\pm$, and $\psi_{\lambda_1, \lambda_2, \gamma, \Omega_0^\pm}^\pm$ have the meaning of (8.15a, b) and (8.16b, c), respectively, with $\lim \Lambda_z = 0$.

However, if $\Lambda_z = 0$, all Λ_1 and Λ_2 are co-planar, and the solutions of (8.14) are suitable to be represented in Λ^q -classes as given by (7.19). To give an explicit representation, we choose the coordinate system as in (8.17). As base vectors for the Λ^q -classes, we take the reference vectors $\Lambda^{q,0}$ such that their real components $\Lambda_1^{q,0}$ are along the y-axis

and their imaginary components along the x-axis. The other base vectors are then obtained from these by rotation around the z-axis. If we denote an element of the set $\{\Lambda^q, 0\}$ by $\Lambda^{\lambda, 0}$, then

$$\Lambda^{\lambda, 0} = \lambda(0, 1, 0) + i \sqrt{\lambda^2 - \lambda_0^2} (1, 0, 0), \quad (8.23)$$

where the parameter λ is a point in the interval defined by

$$|\lambda| \geq \begin{cases} \lambda_0 & \text{for } c \leq 1 \\ 0 & \text{for } c \geq 1. \end{cases} \quad (8.24a)$$

It follows from this that $|\lambda| = |\Lambda_1^{\lambda, 0}| = \lambda_1$ and $\sqrt{\lambda^2 - \lambda_0^2} = |\Lambda_2^{\lambda, 0}| = \lambda_2$ in the usual notation. Letting λ take positive and negative values enables us to obtain a left-handed and right-handed reference system in one representation only.* All other base vectors can be obtained from (8.23) by rotation of angle ϕ_1 such that $R_{\phi_1} \Lambda^{\lambda, 0} = \Lambda^{\lambda, \phi_1}$; i.e.,

$$\Lambda^{\lambda, \phi_1} = \lambda(-\sin \phi_1, \cos \phi_1, 0) + i \lambda_2 (\cos \phi_1, \sin \phi_1, 0), \quad (8.24b)$$

where

$$\lambda_2 = +\sqrt{\lambda^2 - \lambda_0^2}, \quad 0 \leq \phi_1 \leq 2\pi, \quad (8.24c)$$

and the range of λ is given by (8.24a). From (8.17a) and (8.24b) follows

$$\mathbf{r} \cdot \Lambda^{\lambda, \phi_1} = \rho(-\lambda \sin \phi_1 + i \lambda_2 \cos \phi_1), \quad (8.25a)$$

$$\Omega \cdot \Lambda^{\lambda, \phi_1} = \sin \theta [\lambda \sin(\phi - \phi_1) + i \lambda_2 \cos(\phi - \phi_1)], \quad (8.25b)$$

$$\zeta_{\Omega_0, \phi_1} = 1/\Omega_0 \cdot \Lambda^{\lambda, \phi_1}, \quad (8.25c)$$

$$\psi_{\Lambda^{\lambda, \phi_1}} = \frac{c\kappa}{4\pi} \frac{\exp(-\mathbf{r} \cdot \Lambda^{\lambda, \phi_1})}{1 - \Omega \cdot \Lambda^{\lambda, \phi_1}}, \quad |\lambda| < 1, \quad (8.26a)$$

and

*In a similar way, we could have introduced "negative lengths" in (8.16) and (8.9) to avoid the distinction between Ω_0^+ and Ω_0^- .

$$\psi_{\Lambda^{\lambda, \phi_1}, \Omega_0} = \frac{c\kappa}{4\pi} \frac{\exp\left(-\frac{r \cdot \Lambda^{\lambda, \phi_1}}{\Omega_0 \cdot \Lambda^{\lambda, \phi_1}}\right)}{1 - \frac{\Omega \cdot \Lambda^{\lambda, \phi_1}}{\Omega_0 \cdot \Lambda^{\lambda, \phi_1}}} + \kappa G(\lambda, \mu_0, \phi_0 - \phi_1) \delta(\mu - \mu_0) \delta(\phi - \phi_0), \quad (8.26b)$$

where

$$\mu = \cos \theta, \text{ and } \mu_0 = \cos \theta_0 \quad (8.26c)$$

as before. Considerations similar to those in the previous paragraphs show that the coefficient functions in (8.15) now become

$$A^0(\Lambda) = A^0(\lambda) \quad (8.27a)$$

and

$$A(\Lambda, \Omega_0) = A\left(\frac{\Lambda^{\lambda, \phi_1}}{\Omega_0 \cdot \Lambda^{\lambda, \phi_1}}, \Omega_0\right) = A(\lambda, \mu_0, \phi_0), \quad (8.27b)$$

and (8.22) is replaced by

$$\psi(\rho, \theta, \phi) = \int_{1 > |\lambda| \geq \begin{cases} \lambda_0 & \text{for } c \leq 1 \\ 0 & \text{for } c \geq 1 \end{cases}} d\lambda A^0(\lambda) \psi_\lambda + \int_{|\lambda| \geq 1} d\lambda \int_0^{2\pi} d\phi_0 \int_{-1}^{+1} d\mu_0 A(\lambda, \mu_0, \phi_0) \psi_{\lambda, \phi_0, \mu_0}, \quad (8.28a)$$

where

$$\psi_\lambda = \psi_\lambda(\rho, \theta, \phi) = \int_0^{2\pi} d\phi_1 \psi_{\Lambda^{\lambda, \phi_1}}, \quad (8.28b)$$

and

$$\psi_{\lambda, \phi_0, \mu_0} = \psi_{\lambda, \phi_0, \mu_0}(\rho, \theta, \phi) = \int_0^{2\pi} d\phi_1 \psi_{\Lambda^{\lambda, \phi_1}, \Omega_0}. \quad (8.28c)$$

If we impose the additional condition on ψ to be even in ϕ ,

$$\psi(\rho, \theta, \phi) = \psi(\rho, \theta, -\phi), \quad (8.29a)$$

then

$$A^0(\lambda) = A^0(-\lambda); \quad A(\lambda, \mu_0, \phi_0) = A(-\lambda, \mu_0, -\phi_0). \quad (8.29b, c)$$

That this is true follows from the fact that Eqs. (8.25) and (8.26) remain invariant under the simultaneous substitution of λ , ϕ , and ϕ_1 by $-\lambda$, $-\phi$, and $-\phi_1$, respectively.

(c) The case where we require symmetry with respect to a plane orthogonal to the z -axis, in addition to cylindrical symmetry, is obtained from (8.15) by letting

$$\left. \begin{aligned} A^0(\lambda_1, \Lambda_z^\pm) &= A^0(\lambda_1, -\Lambda_z^\pm) \\ \text{and} \\ A(\lambda_1, \lambda_2, \gamma, \Lambda_z^\pm, \Omega_0^\pm) &= A(\lambda_1, \lambda_2, \gamma, -\Lambda_z^\pm, \Omega_0^\mp). \end{aligned} \right\} \quad (8.30)$$

These equations can then be applied to (8.16), (8.19), and (8.20) to obtain symmetry with respect to planes through z -axis in addition to cylindrical symmetry and symmetry with respect to a plane orthogonal to the z -axis.

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