DIFFRACTION SCATTERING AND FACTORIZATION
THIS SIDE OF ASYMPTOTIA

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ABSTRACT

Methods are described for calculating amplitudes involving the exchange of an "effective" Pomeron pole, which do not depend on assumptions about the actual asymptotic behavior with energy of these amplitudes. It is shown that an "effective" Pomeron pole should (at least approximately) factorize, and should not decouple from inelastic channels even in the forward direction.

I. Introduction

This paper is devoted to the development of techniques for calculating the amplitudes for high-energy hadronic diffraction processes in a way which is independent of specific models. In particular, we will not make any assumption about the asymptotic behavior of these amplitudes with energy, so that our results will not depend upon, for example, the question of whether or not total cross sections are asymptotically constant.

A popular technique for dealing with high-energy amplitudes is to describe them in terms of singularities in the complex angular momentum plane. Indeed, it is well known that powerful results such as factorization and the various decoupling theorems follow from the assumption that the Pomeron is a simple pole in the J plane.⁽¹⁾ However, since it is also well known that this assumption leads to various inconsistencies, and since in any case experiments are never performed by using infinite energy, it has been found useful⁽²⁾ to describe the Pomeron as an "effective" pole, that is, to observe that diffractive amplitudes at large but finite energy behave as if the leading singularity in the J plane were a simple pole, without making any commitment as to the actual asymptotic behavior with energy (equivalently, as to the actual nature of the J -plane singularity). One drawback of this approach is that one loses much of the calculational power of the J -plane method.⁽³⁾ For example, although it may be reasonable to suppose that, since the residue of a J -plane pole would factorize, the residue of an "approximate" pole should factorize approximately, this supposition is not supported by J -plane considerations; it is quite simple to construct examples of an effective pole whose residues

grossly violate factorization, which is composed of two closely-spaced J-plane poles each of which factorizes precisely.

In this paper we shall derive (after making several approximations) dynamical equations which high-energy diffraction amplitudes should satisfy, whose basic input is the assumption that elastic and quasi-elastic cross sections are reasonably constant over a large (but perhaps finite) range of energy; that is, we shall derive dynamical equations for effective poles. The derivation of these equations is given in some detail in the second section of this paper, and in outline just below. The fact that effective poles satisfy dynamical equations leads us to expect them to possess, at least approximately, features which actual J-plane poles should possess, and in the third section we present two results of this type: we argue that an effective Pomeron should (approximately) factorize, and that, if its intercept is very close to one, it should obey a type of decoupling theorem, namely that at vanishing momentum transfer it may not couple to elastic states unless it also couples to inelastic ones. In the fourth section we present a few concluding comments.

We will consider the amplitude $A(s, s_1, s_2, t_1, t_2)$ for the production process $a + b \rightarrow 1 + 2 + 3$, with all particles considered for simplicity to be spinless, as depicted in Figure 1. Our general intention is to take particles a and 1 to be the same type (so as to permit vacuum exchange in the t_1 -channel), but almost all of our discussion would be valid for quantum-number exchange also. We are interested in this amplitude for $s \equiv (p_a + p_b)^2$ large, $t_1 \equiv (p_a - p_1)^2$ small, and $s_2 \equiv (p_2 + p_3)^2$ near threshold. Define $a_{\ell m}(s_2; s, t_1)$ to be the projection of A onto states of the $(2, 3)$ system with angular momentum ℓ , and z-component of angular momentum m .

We will attempt to calculate a_{lm} as a function of s_2 in terms of its nearby singularities, and shall argue that the ones we need consider are a "right-hand" cut, arising from the s_2 threshold (Figure 2a), and "left-hand" cuts, arising from singularities of A in $t_2 \equiv (p_b - p_3)^2$ and in $u_2 \equiv (p_b - p_2)^2$ (Figure 2b). The two-body amplitudes for the processes $a + 2' \rightarrow 1 + 2$ and $a + 3' \rightarrow 1 + 3$ which appear inside the diagram in Figure 2b are to be evaluated with $s_1 \equiv (p_1 + p_2)^2$ and $s_1' \equiv (p_1 + p_3)^2$ of the order of s .

Since the amplitude a_{lm} appears linearly in the expression for its own discontinuity across the right-hand cut (Figure 2a), a dispersion relation in s_2 yields a linear integral equation for a_{lm} . By solving this equation, we obtain an expression for a_{lm} in terms of two-body amplitudes at low energy ($2 + 3 \rightarrow 2 + 3$) and at high energy ($a + 2' \rightarrow 1 + 2$ and $a + 3' \rightarrow 1 + 3$). This expression constitutes the basic result of this paper.

We will choose the t_1 -channel to have vacuum quantum numbers. This will imply that the appropriate partial wave amplitude a_{lm} (namely a_{00} in our spinless example) will have a pole in s_2 at $s_2 = m_b^2$; the residue of this pole is given by the high-energy amplitude for the process $a + b \rightarrow 1 + b$, and can be calculated from our expression for a_{lm} . Thus there will be consistency conditions that two-body amplitudes must satisfy; these conditions will be shown to lead to factorization, both of two-body amplitudes (and hence of total cross sections) and of amplitudes for diffractive dissociation. Finally, an interesting approximation is to restrict the exchanged particles $2'$ and $3'$ in Figure 2b to be the same as the external particles 2 and 3 respectively. This approximation will be shown to lead to an inconsistency at $t_1 = 0$; this inconsistency will teach us that quasi-elastic diffractive dissociation must exist at high energy, even at $t_1 = 0$.

II. The Calculation of Diffraction Amplitudes

Let $A(s, s_1, s_2, t_1, t_2)$ be the amplitude for the process $a + b \rightarrow 1 + 2 + 3$, where all particles are for simplicity taken to be spinless, with the definitions $s \equiv (p_a + p_b)^2$, $s_1 \equiv (p_1 + p_2)^2$, $s_2 \equiv (p_2 + p_3)^2$, $t_1 \equiv (p_a - p_1)^2$, and $t_2 \equiv (p_b - p_3)^2$; this notation is displayed in Figure 1. We wish to project this amplitude onto states of definite angular momentum of the (2, 3) system. In the rest system of particles 2 and 3, let θ and ϕ be the spherical coordinates of the momentum vector of particle 3 with respect to a z axis taken along the direction of the momentum of particle b [i.e., the Jackson frame], and then define

$$a_{lm}(s_2; s, t_1) = \int d\Omega Y_{lm}(\theta, \phi) A(s, s_1, s_2, t_1, t_2). \quad (1)$$

Within the integral in Eq. (1) s_2 , s , and t_1 are fixed, while s_1 and t_2 depend on the integration variables, t_2 being linear in $\cos \theta$:

$$\begin{aligned} s_1 &= s_1(s_2, s, t_1, \theta, \phi) \\ t_2 &= t_2(s_2, t_1, \theta) \equiv G(s_2, t_1) + H(s_2, t_1) \cos \theta; \end{aligned} \quad (2)$$

a detailed discussion of the kinematics of five-point functions can be found in ref. (4).

Once we succeed in calculating the a_{lm} , we can of course sum them to recover the amplitude A . Let us restrict our attention to the region in which s is large, $|t_1|$ is small, and s_2 is near threshold. We wish to assume that $a_{lm}(s_2; s, t_1)$ can be calculated in this region from a knowledge of the nearby singularities in the s_2

variable. Unfortunately we do not know how to prove this to be true; however a highly analogous assumption has often been made in the treatment of partial wave amplitudes for two-body processes, with a similar lack of rigorous justification⁽⁵⁾, and has led to a considerable amount of success. If we would assume Reggeization in the limit $s \rightarrow \infty$, then, in this limit, $a_{\ell m}(s_2; s, t_1)$ would be the partial wave amplitude for the "two-body" process (Pomeron of mass-squared t_1) + $b \rightarrow 2 + 3$, evaluated at total center-of-mass energy squared equal to s_2 . We have taken s to be large, but will avoid examining the limit $s \rightarrow \infty$; nevertheless, we shall let ourselves be guided by experience gained from analysing two-body processes, and in fact our solution for $a_{\ell m}$ will be modeled on the N/D solution for two-body reactions.⁽⁶⁾

We have thus to discuss the singularities of $a_{\ell m}(s_2, s, t_1)$ in s_2 , and pay particular attention to those which lie close to the values of s_2 in which we are interested, that is, values close to the threshold value $s_2 = (m_2 + m_3)^2$. The singularities in s_2 can be arranged into three classes: (i) singularities arising from singularities of $A(s, s_1, s_2, t_1, t_2)$ in s_2 , which we call "right-hand" singularities; (ii) singularities arising from the intersection of singularities of A in t_2 or in $u_2 \equiv (p_b - p_2)^2$ with the end-points of the integral in Eq. (1), which we call "left-hand" singularities, and (iii) other singularities with no analogues in the two-body case, which, we will argue, are far from the threshold in s_2 . We proceed to discuss each of these in turn.

(i) The amplitude A will have branch points in s_2 at the thresholds for states in the s_2 channel, and these branch points will appear in $a_{\ell m}$ also. The discontinuities

across these branch cuts are given by the standard multiparticle discontinuity formulae.⁽⁷⁾

We will consider explicitly only the cut associated with the state consisting of the particle 2 and 3⁽⁸⁾ (the extension to a finite number of two-body intermediate states is straightforward); this cut begins at $s_2 = (m_2 + m_3)^2$, and its discontinuity is given by

$$\Delta a_{\ell m}(s_2; s, t_1) = i p(s_2) a_{\ell}^*(s_2) a_{\ell m}(s_2; s, t_1), \quad (3)$$

where p is the two-body phase space and a_{ℓ} is the partial-wave amplitude for the process $2 + 3 \rightarrow 2 + 3$. The discontinuity across this "right-hand cut" is pictured in Figure 2a. Equation (3) follows directly from the multiparticle discontinuity formulae, and does not require the assumption of any "Reggeon unitarity" condition. In addition to this branch point, $a_{\ell m}$ may have poles in s_2 corresponding to single-particle states in the s_2 channel.

(ii) The "left-hand" singularities closest to the s_2 threshold will arise from the least massive states in the t_2 (or u_2) channel; we will consider explicitly only the single-particle states. Denote by $2'$ a particle in the t_2 channel; then A has a pole in t_2 at $t_2 = m_{2'}^2$, and this pole term is given (in the case particle $2'$ has zero spin) by

$$A(s, s_1, s_2, t_1, t_2) \rightarrow T(s_1, t_1) / (t_2 - m_{2'}^2), \quad (4)$$

where $T(s_1, t_1)$ is the amplitude for the two-body process $a + 2' \rightarrow 1 + 2$. If we substitute this contribution to A into Eq. (1), and use Eq. (2), we find a contribution to $a_{\ell m}$ given by

$$a_{\ell m}(s_2; s, t_1) = \int_{-1}^1 d \cos \theta \int_0^{2\pi} d\phi Y_{\ell m}(\theta, \phi) \frac{T(s_1(s_2, s, t_1, \theta, \phi), t_1)}{G(s_2, t_1) + H(s_2, t_1) \cos \theta - m_2^2} .$$

(5)

Thus $a_{\ell m}$ has branch points at values of s_2 which make the denominator in Eq. (5) vanish for $\cos \theta = \pm 1$. The positions of these branch points are exactly the same as for the kinematically-equivalent 2-body process $t_1 + m_b^2 \rightarrow m_2^2 + m_3^2$, and so can be found, for example, from the results of ref. 9. For given t_1 , the positions of these branch points are independent of m_a and of m_1 . The discontinuity across the branch cuts is given by

$$\Delta a_{\ell m}(s_2; s, t_1) = \frac{2i}{H(s_2, t_1)} \int_0^{2\pi} d\phi Y_{\ell m}(\theta, \phi) T(s_1(s_2, s, t_1, \theta, \phi), t_1) ,$$

(6)

where in Eq. (6) θ is the solution of $G(s_2, t_1) + H(s_2, t_1) \cos \theta - m_2^2 = 0$. Now it turns out that, for the parts of the branch cuts near the s_2 threshold, the values of s_1 in Eq. (6) (although they may become complex), are always on the order of s : $s_1 \cong sf(s_2, t_1, \theta, \phi)$, where f is a finite function. If we assume that, for s_1 on the order of s , the amplitude $T(s_1, t_1)$ can be well approximated by an effective Regge pole (we do not have to assume this for $s_1 \rightarrow \infty$), then within the integral in Eq. (6) we can replace $T(s_1, t_1)$ by $\gamma(t_1) s_1^{\alpha(t_1)}$, and then re-write Eq. (6) in the form

$$\Delta a_{\ell m}(s_2; s, t_1) = K(s_2, t_1) \gamma(t_1) s^{\alpha(t_1)} \quad (7)$$

where K depends only on the kinematics of the problem, and γ is determined by

the high-energy behavior of the process $a + 2' \rightarrow 1 + 2$.

There will also be cuts arising from poles of A in u_2 ; the discontinuity across these cuts can be obtained in a manner analogous to that described above, by interchanging the roles of the particles labeled 2 and 3.

(iii) States in the channels labeled by the variables $v \equiv (p_0 - p_3)^2$ and $w \equiv (p_0 - p_2)^2$ can, in the same way, produce singularities in a_{lm} . However, for s large, these singularities cannot be close to the s_2 threshold and so will not be considered further. On the other hand, singularities arising from states in the channels of $s_1 \equiv (p_1 + p_2)^2$ or of $s_1' \equiv (p_1 + p_3)^2$ may be close to the s_2 threshold; indeed, a (multiparticle) state in the s_1 channel with threshold $\bar{s}_1$ can give rise to a singularity in a_{lm} which coincides with the s_2 threshold when the condition (for large s) $\bar{s}_1 = m_2^2/(m_2 + m_3)$ is satisfied. However, to the extent that these high-mass thresholds were important, we would not have approximately-constant or even smooth cross sections at large but finite energies. In fact, if any sort of Regge representation in s_1 is valid, containing poles and cuts in any combination, then singularities at large but finite values of s_1 can be ignored, since the Regge representation admits only singularities at small and at infinite values of s_1 . That is, we expect that all of the high-mass thresholds together can be replaced, to a high degree of accuracy, within a single branch cut with no branch points at large but finite values of s_1 . This means that we can ignore the branch points of a_{lm} near the s_2 threshold arising from states in the s_1 or s_1' channel. ⁽¹⁰⁾

We have thus argued, not rigorously but hopefully persuasively, that we need only consider the singularities of a_{lm} in classes (i) and (ii). The discontinuity across the left-hand cut is given by a sum of terms, each of which has the form of Eq. (7), and

is known if two-body amplitudes are known; call this known discontinuity $B_{\ell m}(s_2; s, t_1)$. Equation (3), together with the elastic unitarity condition satisfied by a_ℓ [namely $\Delta a_\ell(s_2) = i\rho(s_2) a_\ell^*(s_2) a_\ell(s_2)$] implies that $(\Delta a_\ell)/a_\ell = (\Delta a_{\ell m})/a_{\ell m}$ for s_2 on the right-hand cut. Let $D_\ell(s_2)$ be the Omnes function for the elastic process $2+3 \rightarrow 2+3$ (so that D_ℓ has only a right-hand cut on which $[D_\ell]^{-1}$ has the same phase as does a_ℓ) and define

$$N_{\ell m}(s_2; s, t_1) = \frac{1}{2\pi i} \int \frac{ds'_2}{s'_2 - s_2} D_\ell(s'_2) B_{\ell m}(s'_2; s, t_1), \quad (8)$$

where the integral is taken over the left-hand cut. Then the solution⁽¹¹⁾ to Eqs. (3) and (6) is clearly given by

$$a_{\ell m}(s_2; s, t_1) = N_{\ell m}(s_2; s, t_1) / D_\ell(s_2). \quad (9)$$

We have thus succeeded in calculating $a_{\ell m}$ in terms of high-energy two-body amplitudes (the functions γ in Eq. (7)) and low-energy two-body elastic amplitudes. In this calculation, we have not explicitly included any contribution corresponding to poles in $a_{\ell m}$ arising from single-particle states in the s_2 channel. When the t_1 channel has vacuum quantum numbers, there will be at least one such pole, since then particle b has the quantum numbers of the s channel. If necessary, this contribution could be explicitly included in $B_{\ell m}$. However, we prefer to assume that this pole will arise in $a_{\ell m}$ from a zero in D_ℓ ; that is, we will assume that particle b is dynamically a bound state of the $(2, 3)$ system, since D_ℓ is also the D function for the elastic $(2, 3)$ amplitude. If this were not the case, we would want to take more

intermediate states in the s_2 channel, so that particle b would be a bound state (in the sense of appearing as a zero in the D function) of the states we do consider.

III. Some Applications

We have seen that, if the particle b is a bound state of the $(2, 3)$ system (and this is an assumption about the two-body system only), then the pole in the appropriate $a_{\ell m}$ at $s_2 = m_b^2$ will arise from a zero of D_ℓ , and so its residue can be calculated by the methods developed in the preceding section. This residue is just the amplitude for the two-body process $a + b \rightarrow 1 + b$, and so we can obtain linear relations between the input and output two-body amplitudes. With appropriate choices of particle types, these relations can lead to factorization, as we now show.

Let particles a and 1 be of the same type, which we shall call x ; let particles b and 3 be nucleons, and particle 2 a pion, so that we study $x + N \rightarrow x + \pi + N$, i.e. low-mass diffraction dissociation of the nucleon by some projectile. We will include only the (πN) intermediate state in the right-hand discontinuity relation for the $a_{\ell m}$, so that we are assuming that the N will appear as a bound state of πN in a single-channel calculation.¹²

Let us include as exchanged particles the π in the t_2 channel, and the N and some of its isobars in the u_2 channel; we denote the exchanged baryons by N_i , with the nucleon itself being denoted N_1 . In this approximation, we can calculate the diffraction dissociation of the nucleon, in terms of the high-energy amplitudes for the two-body processes $\pi + x \rightarrow \pi + x$ and $N_i + x \rightarrow N_i + x$ ($i = 1, 2, \dots$) and of the low-energy D functions for elastic πN scattering. In particular, we can calculate the residue of the nucleon pole in the appropriate $a_{\ell m}$, and thus can obtain an expression for the high-energy amplitude for the process $N_1 + x \rightarrow N_1 + x$.

Let us see how this expression will depend on the particle type x . The only place that particle x enters our calculation is in the high-energy amplitude that determines the left-hand cut. Let us represent, at large but not infinite energy, the amplitude for $\pi + x \rightarrow \pi + x$ as $\gamma_{x\pi}(t_1) s_1^{\alpha(t_1)}$ and that for $N_i + x \rightarrow N_i + x$ as $\gamma_{xi}(t_1) s_1^{\alpha(t_1)}$ (so that γ_{xi} represents elastic Nx scattering). Since the left-hand discontinuity depends linearly on the γ 's (Eq. (7)) and since $a_{\ell m}$ depends linearly on its left-hand discontinuity (Eqs. (8) and (9)), then anything that we calculate will depend linearly on $\gamma_{x\pi}$ and the γ_{xi} . In particular, the output Nx elastic amplitude, which we have parametrized as $\gamma_{xi}(t_1) s_1^{\alpha(t_1)}$, depends linearly on $\gamma_{x\pi}$ and the γ_{xi} , so that we can write

$$\gamma_{xi}(t_1) = \alpha \gamma_{x\pi}(t_1) + \sum_{i=1} \beta_i \gamma_{xi}(t_1), \quad (10)$$

where α and the β_i are complicated objects (which depend on the (πN) D functions among other things), but do not depend on the particle type x .

We can also examine the appropriate $a_{\ell m}$ for values of s_2 near to the squared mass of the nucleon isobars, and so determine in our output the amplitude for $N + x \rightarrow N_i + x$. This will also be linear in the input γ 's, so we can write, as a generalization of Eq. (10),

$$\gamma_{xi}(t_1) = \alpha_i \gamma_{x\pi}(t_1) + \sum_{j=1} \beta_{ij} \gamma_{xj}(t_1), \quad (11)$$

where the α_i and β_{ij} are independent of the particle type x . If we let β denote a matrix with matrix elements β_{ij} , then the solution of Eq. (11) is

$$\gamma_{xi}(t_1) = \gamma_{x\pi}(t_1) [1 - \beta]_{ij}^{-1} \alpha_j \quad (12)$$

We have thus written $\gamma_{xi}(t_1)$ as the product of two factors, one depending on x and the other on i ; this is factorization. For example, Eq. (12) implies $\gamma_{xi}(t_1)/\gamma_{x\pi}(t_1)$ is independent of x ; taking $i = 1$, $x = \pi$ and then $i = 1$, $x = N$, and using the fact that γ_{x1} and $\gamma_{x\pi}$ at $t_1 = 0$ are total cross sections, we get

$$\frac{\sigma_{\text{tot}}(\pi N)}{\sigma_{\text{tot}}(\pi\pi)} = \frac{\sigma_{\text{tot}}(NN)}{\sigma_{\text{tot}}(\pi N)} \quad (13)$$

Also, we have seen that whatever we calculate is linear in $\gamma_{x\pi}$ and the γ_{xi} ; from Eq. (12), this means linear in $\gamma_{x\pi}$. Thus the entire diffraction dissociation amplitude, not just its resonant part, factorizes, since it can be written as $\gamma_{x\pi}(t_1)$ times something which is independent of x . (13)

Finally, we discuss the approximation of taking as the exchanged particles $2'$ and $3'$ only the same particles as the external particles 2 and 3 ; in the example discussed above, this would mean allowing only π and N exchange. This would seem like a sensible approximation, not only because these are the least-massive states that can be exchanged, but also because the contribution of an exchanged particle ($2'$, say) to the left-hand cut is proportional to the high-energy amplitude for the process $a + 2' \rightarrow 1 + 2$, and we know (at least in the case in which a and 1 are the same type) that this amplitude is biggest when 2 and $2'$ are the same type. In this approximation, there is an extremely simple pictorial representation of our calculation [see Figure 2b]: particle b virtually dissociates into its two constituents 2 and 3 , either of which

then scatters on particle a .⁽¹⁴⁾ However, it should be pointed out that the amplitude T in Eq. (4) is the mass-shell amplitude, so that our calculation does not really involve virtual particles.

Unfortunately, this approximation must break down as $t_1 \rightarrow 0$. In this limit, the left-hand cuts arising from exchange of particles of the same masses as the external particles 2 and 3 collapse to become poles at $s_2 = m_b^2$; in the example discussed above, the functions α and β_1 in Eq. (10) simplify, and in fact if $\alpha(t_1 = 0) = 1$ Eq. (10) reduces to

$$\sigma_{\text{tot}}(xN) = \sigma_{\text{tot}}(x\pi) + \sigma_{\text{tot}}(xN) . \quad (14)$$

This result is what should be expected from the simple picture of virtual dissociation discussed above, and is essentially the same problem with Pomeron decoupling as was discussed in ref. (15) in the context of the $s \rightarrow \infty$ limit. It shows that there must be contributions to the left-hand cut of states $2'$ (and/or $3'$) different from the external particles. Since these contributions are proportional to high-energy amplitudes for inelastic diffractive processes, we learn that not all of these processes can vanish; even at $t_1 \approx 0$ the effective Pomeron cannot decouple from all inelastic channels.

IV. Remarks

(1) The most important aspect of the viewpoint that has been advanced here is that low-mass diffractive amplitudes may be analysed by using techniques which have been developed for analysing low-energy two-body amplitudes, without first making the asymptotic factorization assumption that allows one to define amplitudes with external Reggeons. To obtain manageable equations it is convenient, just as it is in the two-body case, to make drastic truncations on the sums over intermediate states in the right-hand discontinuity formula, and also over the exchanged states.

(2) Other possible applications of our equations, besides a direct calculation of diffraction cross sections, might include studies of the dependence of the amplitude upon the spin of the produced system, or on the momentum transfer. In regard to this last point, once inelastic exchange terms are included in the input, there does not seem to be any reason to expect dips in forward diffractive processes.

(3) This same approach might be useful in analysing amplitudes with even greater numbers of particles. For example, an amplitude with three incoming and three outgoing particles could be analysed near the threshold of two of the particles; in this case, the left-hand cut discontinuities would involve integrals of four- and five-point amplitudes. Since the six-point amplitude would be linear in its left-hand cut, presumably it would exhibit factorization once the four- and five-point amplitudes do. In this way, inclusive distributions might be shown to exhibit factorization.

Footnotes

1. See F. E. Low in Proceedings of the XVI International Conference on High Energy Physics (NAL, 1972), Vol. 3, p. 459.
2. For example, J. W. Dash, Phys. Rev. 9, 200 (1974).
3. This point has been stressed by D. Amati, Phys. Lett. B48, 253 (1974).
4. E. Byckling and K. Kajantie, Particle Kinematics (Wiley, 1973).
5. For example, it is not possible to prove that two-body partial wave amplitudes have sufficiently good asymptotic behavior to justify a nearest-singularity approximation; we will proceed as if the asymptotic behavior of a_{lm} is as good as we need it to be.
6. N/D equations for amplitudes with an external Reggeon have been previously studied by R. Dashen and S. Frautschi, Phys. Rev. 152, 1450 (1966). Also, the analyticity of these amplitudes has been investigated by P. Hoyer and J. Kwieciński, Nucl. Phys. B60, 26 (1973).
7. See R. J. Eden, P. V. Landshoff, D. I. Olive and J. C. Polkinghorne, The Analytic S Matrix (Cambridge University Press, 1966).
8. This is the analogue of the elastic unitarity (or single-channel) approximation often made in calculations of two-body reactions.
9. J. Kennedy and T. D. Spearman, Phys. Rev. 126, 1596 (1962).
10. The branch cut arising from states in the s_1 or s_1' channel can then be distorted so that it does not come close to the s_2 threshold. It can also be shown that there are no pinch singularities near the s_2 threshold.

11. If, as is usually assumed, $D_{\ell} \rightarrow 1$ and $a_{\ell m} \rightarrow 0$ as $|s_2| \rightarrow \infty$, the solution can be shown to be unique: Let $a_{\ell m}^{(1)}$ and $a_{\ell m}^{(2)}$ be two solutions, then $(a_{\ell m}^{(1)} - a_{\ell m}^{(2)})D$ is an entire function which vanishes as $|s_2| \rightarrow \infty$.
12. G. F. Chew, Phys. Rev. Letters 9, 233(1962).
13. This derivation of factorization clearly depends on our having allowed only one exchange, namely π exchange, in the t_2 -channel. More exchanges would require more equations; if we were to allow two exchanges, it would suffice to use the extra equations obtained by letting particle b be any of the N_i .
14. Compare the discussion by V. N. Gribov in Proceedings of the XVI International Conference on High Energy Physics (NAL, 1972), Vol. 3, p. 491.
15. R. E. Brower and J. H. Weis, Phys. Lett. 41B, 631 (1972).

Figure Captions

Figure 1. Kinematics for the reaction $a + b \rightarrow 1 + 2 + 3$.

Figure 2. (a) The right-hand discontinuity of σ_{lm} .

(b) Two contributions to the left-hand discontinuity of σ_{lm} .

(a)

(b)