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NUCLEAR EVENTS AT MODERATELY HIGH
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November 17, 1953

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ABSTRACT

The statistical model, introduced by Fermi, is used to calculate the probabilities for various nuclear events at moderately high energies. Tables are presented giving the probabilities for various nuclear processes.

NUCLEAR EVENTS AT MODERATELY HIGH ENERGIES

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I. Introduction

The statistical model has been used by Fermi¹ to study multiple processes

¹ Enrico Fermi, Progress of Theoretical Physics 5, 4, 570 (1950).

occurring in high energy nucleon encounters. The method consists in assuming that in such an encounter there is a localization of energy in a small spatial volume which endures for a long enough time so that when the system decays it does so in accordance with the statistical laws governing the partition of energy among the various degrees of freedom in an equilibrium situation. It is hoped that such a situation may be at least partially reached because of the intensity of nuclear interactions.

In view of the mathematical and physical uncertainties involved in more detailed field theoretic approaches to such problems the method offers a simple means of providing a much needed background for discussion of nuclear events at the moderately high energies which are now accessible to laboratory study by means of the various high energy particle accelerators.

Fermi has explored the consequences of the model most fully for cases of extremely high energy where the statistical model may be replaced by a thermodynamic one. When using the more detailed statistical approach he treats nucleons as being non-relativistic, pions as being extremely relativistic; furthermore the momentum of the pions is neglected. In this paper we treat to some extent the consequences of including pion momentum conservation when pions alone are produced. Production of mesons other than pions is also considered.

II. Method

Consider a collision between two nucleons in the laboratory system. We may imagine that one nucleon is at rest surrounded by a spherical pion cloud of radius R ($R \sim \hbar/\mu c = 1.4 \times 10^{-13}$ cm.), the other is approaching with a severely contracted cloud due to the Lorentz effect. The total collision cross section will be of the order of magnitude of the geometrical cross section of the pion cloud.

$$\sigma_{\text{Total}} = \pi R^2 \quad (1)$$

Although R may be expected to be about $\hbar/\mu c$ it may be regarded as an adjustable parameter in the theory.

In the center of momentum system the two nucleons approach each other with equal and opposite momenta so that if E is the total energy involved in the collision in this system the pion clouds will occupy a volume

$$\left. \begin{aligned} \Omega &= \Omega_0 \frac{2M}{E} \\ \Omega_0 &= \frac{4}{3} \pi R^3 \end{aligned} \right\} \quad (2)$$

It will be assumed that these volumes coalesce at the moment of collision.

It is assumed that, after the collision, the system decays in such a way that the probability for emission of n particles is proportional to the number of states per unit energy of this type contained in the accessible region of phase space. According to the general principles of quantum theory a volume $(2\pi\hbar)^3$ of phase space corresponds to a single state; therefore in a region, $\int d^3x d^3p$, there are

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$$\frac{d}{dE} \int \frac{d^3\vec{x} d^3\vec{p}}{(2\pi\hbar)^3} \quad (4)$$

states of a single particle per unit energy. Thus the number of states per unit energy for n particles of mass M_i and multiplicity m_i ($n = \sum_i m_i$) is

$$\frac{1}{(2\pi)^{3n}} \frac{d}{dE} \prod_{i,m_i} \int d^3\vec{x}_{m_i}^i d^3\vec{p}_{m_i}^i \quad (5)$$

In this formula $\hbar$ has been set equal to 1; units will be chosen in this paper such that $\hbar = c = 1$, where c is the velocity of light. This expression is however the number of states per unit energy if all particles are considered as independent. The integral over phase space must be limited to include only that volume compatible with the constraints on the system. We may, for example, choose the energy E and the total angular momentum as fixed; on the hand the integrals are considerably simplified if we choose the energy E and restrict the total linear momentum to be zero and ignore angular momentum conservation. If this is done each spatial integral merely contributes a factor Ω . The momentum space integrals may be conveniently expressed by introducing in the integrand a factor

$$\delta\left(\sum_{i,m_i} \vec{p}_{m_i}^i\right) \text{ to insure that the total momentum be zero and a factor,}$$

$$U(E),$$

$$\left. \begin{aligned} U(E) &= 1, & (E - \sum_{i,m_i} \sqrt{(p_{m_i}^i)^2 + M_i^2}) &\geq 0 \\ U(E) &= 0 & \text{otherwise} \end{aligned} \right\} \quad (6)$$

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which restricts the integral to that region in momentum space up to the energy E . The first factor is simply

$$\frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} e^{i\vec{\lambda} \cdot \sum_{i,m_i} \vec{p}_{m_i}} d^3\vec{\lambda} \quad (7)$$

The second factor may be written

$$\frac{1}{2\pi i} \int_{-\infty-i\epsilon}^{\infty-i\epsilon} e^{i\alpha(E - \sum_{i,m_i} \sqrt{(\vec{p}_{m_i}^i)^2 + M_i^2})} \frac{d\alpha}{\alpha} \quad (8)$$

as may be seen by closing the contour above the real axis if the argument of the exponential is positive and below if it is negative. When these factors are inserted into (5) and the differentiation with respect to E is performed one finds for the number of states per unit energy corresponding to the emission of $n = \sum_i m_i$ particles of mass M_i

$$S_n = \frac{n^{n-1}}{(2\pi)^{3n+1}} \int d\alpha d^3\vec{\lambda} \left[e^{i\alpha(E - \sum_{i,m_i} \sqrt{(\vec{p}_{m_i}^i)^2 + M_i^2})} e^{i\vec{\lambda} \cdot \sum_{i,m_i} \vec{p}_{m_i}^i} \prod_{i,m_i} d^3\vec{p}_{m_i}^i \right] \quad (9)$$

Only one integral sign has been written in (9) in order to save space. It is to be understood that all variables are to be integrated from $-\infty$ to ∞

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with due regard to the factors ϵ_1, ϵ_2 occurring in the discontinuous factors. The momentum space integrals may now be evaluated. Equation (9) contains a product of n integrals of the form

$$I = \int e^{i[\vec{\lambda} \cdot \vec{p} - \alpha \sqrt{p^2 + M^2}]} d^3\vec{p}. \quad (10)$$

If $\vec{\lambda}$ is taken as the polar axis this may be written as

$$I = 2\pi \int_{-1}^1 dt \int_0^\infty p^2 dp e^{i[\lambda p t - \alpha \sqrt{p^2 + M^2}]} \quad (11)$$

The t integration may now be carried out and yields

$$I = \frac{2\pi}{i\lambda} \int_{-\infty}^\infty p dp e^{i(\lambda p - \alpha \sqrt{p^2 + M^2})}$$

or

$$I = -\frac{2\pi}{\lambda} \frac{d}{d\lambda} \int_{-\infty}^\infty dp e^{i(\lambda p - \alpha \sqrt{p^2 + M^2})} \quad (12)$$

It is convenient to introduce some changes of variable

$$p = M \sinh \theta, \quad \lambda_1 = M\lambda, \quad \alpha_1 = M\alpha, \quad (13)$$

after which one finds

$$I = -\frac{2\pi M^3}{\lambda_1} \frac{d}{d\lambda_1} \int_{-\infty}^\infty d\theta \cosh \theta e^{i(\lambda_1 \sinh \theta - \alpha_1 \cosh \theta)} \quad (14)$$

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Depending on the relative sizes of λ_1 and α_1 the argument of the exponential may be written in three different forms

$$(1) \quad \alpha_1 > \lambda_1 > 0 \quad (15)$$

$$(\lambda_1 \sinh \theta - \alpha_1 \cosh \theta) = -\sqrt{\alpha_1^2 - \lambda_1^2} \cosh(\theta - \phi_1)$$

$$\cosh \phi_1 = \alpha_1 / \sqrt{\alpha_1^2 - \lambda_1^2}$$

(2)

$$|\alpha_1| < \lambda_1$$

$$(\lambda_1 \sinh \theta - \alpha_1 \cosh \theta) = -\sqrt{\lambda_1^2 - \alpha_1^2} \sinh(\theta - \phi_2)$$

$$\sinh \phi_2 = \alpha_1 / \sqrt{\lambda_1^2 - \alpha_1^2}$$

(3)

$$-\alpha_1 > \lambda_1$$

$$(\lambda_1 \sinh \theta - \alpha_1 \cosh \theta) = \sqrt{\alpha_1^2 - \lambda_1^2} \cosh(\theta + \phi_3)$$

$$\cosh \phi_3 = -\alpha_1 / \sqrt{\alpha_1^2 - \lambda_1^2}$$

The integrals over θ may be reduced to standard forms if the parameters α_1, λ_1 lie in either region 1 or in region 3. For $\alpha_1 > \lambda_1 > 0$ I becomes

$$I_1 = -\frac{2\pi M^3}{\lambda_1} \frac{d}{d\lambda_1} \int_{-\infty}^{\infty} \cosh \theta e^{-\sqrt{\alpha_1^2 - \lambda_1^2} \cosh(\theta - \phi_1)} d\theta \quad (16)$$

If a change of variable, $\theta' = \theta - \phi_1$, is made, Eq. 16 reduces to

$$I_1 = -\frac{2\pi M^3}{\lambda_1} \frac{d}{d\lambda_1} \frac{\alpha_1}{\sqrt{\alpha_1^2 - \lambda_1^2}} \int_{-\infty}^{\infty} d\theta' \cosh \theta' e^{-i\sqrt{\alpha_1^2 - \lambda_1^2} \cosh \theta'}$$

This is expressible in terms of the Hankel function $H_1^{(2)}$

$$I_1 = \frac{2\pi^2 M^3}{\lambda_1} \frac{d}{d\lambda_1} \frac{\alpha_1}{\sqrt{\alpha_1^2 - \lambda_1^2}} H_1^{(2)} \left(\sqrt{\alpha_1^2 - \lambda_1^2} \right) \quad (17)$$

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G. N. Watson, Bessel Functions, MacMillan, 1948, page 180, Eq. 11.

The differentiation with respect to λ_1 may be carried out by using the formula³

$$\frac{1}{z} \frac{d}{dz} \frac{H_1^{(k)}(z)}{z} = -\frac{H_2^{(k)}(z)}{z^2}$$

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Ibid, page 74, Eq. 10.

One finds

$$I_1 = 2\pi^2 \alpha_1 M^3 \frac{H_2^{(2)}(\sqrt{\alpha_1^2 - \lambda_1^2})}{\sqrt{\alpha_1^2 - \lambda_1^2}} \quad (18)$$

In region 3, $-\alpha_1 > \lambda_1$ the integral may be evaluated in the same manner.

It yields

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$$I_3 = -2\pi^2 \alpha_1^3 M^3 \frac{H_2^{(1)} \left(\sqrt{\alpha_1^2 - \lambda_1^2} \right)}{\alpha_1^2 - \lambda_1^2} \quad (19)$$

The integral in region 2, $|\alpha_1| < \lambda_1$,

$$I_2 = -\frac{2\pi M^3}{\lambda_1} \frac{d}{d\lambda_1} \int_{-\infty}^{\infty} d\theta \cosh \theta e^{i\sqrt{\lambda_1^2 - \alpha_1^2} \sinh(\theta - \phi_2)} \quad (20)$$

is not readily reducible to standard forms. It may be evaluated by observing that the integral, Equation (14), is an analytic function of α_1 and λ_1 and by appealing to the principle of analytic continuation. That choice of the phases of the square root $\sqrt{\alpha_1^2 - \lambda_1^2}$ may be made which joins the region 1 with the region 3. A careful examination⁴ of this points shows that

$$I_2 = 2\pi^2 \alpha_1^3 M^3 \frac{H_2^{(2)} \left(-i\sqrt{\lambda_1^2 - \alpha_1^2} \right)}{\alpha_1^2 - \lambda_1^2} \quad (21)$$

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We are grateful to Naurice Neuman who suggested this method of obtaining

$$I_2 \text{ Note } H_2^{(2)}(Z e^{-i\pi}) = -H_2^{(1)}(Z)$$

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Thus the integral I, Eq. (14), may be compactly represented as

$$I = 2\pi^2 \alpha_1^3 M^3 \frac{H_2^{(2)}(\sqrt{\alpha_1^2 - \lambda_1^2})}{\alpha_1^2 - \lambda_1^2} \quad (22)$$

if it is understood that the phase of the square root is chosen according to the following scheme

$$\left. \begin{array}{l} \sqrt{\alpha_1^2 - \lambda_1^2} \\ e^{-i\pi/2} \sqrt{\lambda_1^2 - \alpha_1^2} \\ e^{-i\pi} \sqrt{\alpha_1^2 - \lambda_1^2} \end{array} \right\} \begin{array}{l} \alpha_1 > \lambda_1 > 0 \\ |\alpha_1| < \lambda_1 > 0 \\ \alpha_1 < \lambda_1 < 0 \end{array} \quad (23)$$

If one now expresses α_1, λ_1 in terms of α, λ and substitutes expressions of the type 22 for each mass M_i into Eq. (9), the expression for the number of states of the required type per unit energy, one finds

$$S_n = \frac{\prod_i M_i^{2m_i} \Omega^{n-1}}{2^{2n-1} \pi^n} \int_{-\infty-i\epsilon}^{\infty-i\epsilon} d\alpha \int_0^\infty \frac{\lambda^2 d\lambda}{(\alpha^2 - \lambda^2)^n} e^{i\alpha E} \prod_i \left[H_2^{(2)}(M_i \sqrt{\alpha^2 - \lambda^2}) \right]^{m_i} \quad (24)$$

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Before examining S_n in great detail it is interesting to evaluate it in the extreme relativistic limit where the masses of all particles concerned are very small compared to E . We may in this case expand $H_2^{(2)}$ about the origin and retain only the first term⁵

$$H_2^{(2)}(z) \cong \frac{4i}{\pi z^2} \quad (25)$$

⁵ G. N. Watson, Theory of Bessel Functions, p.84.

An examination of the restrictions 23 shows that regardless of the relative values of α, λ

$$H_2^{(2)}(M_1 \sqrt{\alpha^2 - \lambda^2}) \cong \frac{4i}{M_1^2 \pi (\alpha^2 - \lambda^2)} \quad (26)$$

If this is substituted into Eq. (24) one finds

$$S_n = \frac{2 i^{n-1}}{\pi^{2n}} \int_{-\infty-i\epsilon}^{\infty-i\epsilon} d\alpha \alpha^n \int_0^\infty d\lambda \lambda^2 \frac{e^{i\alpha E}}{(\alpha^2 - \lambda^2)^{2n}} \quad (27)$$

a result more easily obtained by integrating Eq. (11) directly, after setting $M_1 = 0$, and then substituting into the expression 9 for S_n . S_n may be easily evaluated by applying the method of residues.

$$S_n = \frac{2 i^{n-1}}{\pi^{2n}} \int_{-\infty-i\epsilon}^{\infty-i\epsilon} \alpha^n e^{i\alpha E} d\alpha \int_{-\infty}^\infty d\lambda \frac{\lambda^2}{(\alpha^2 - \lambda^2)^{2n}} \quad (28)$$

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The integral over λ may be evaluated by closing the contour above the real axis. There are poles of order $2n$ at $\lambda = \pm \alpha$; therefore if $\epsilon_2 > \epsilon_1$ only the pole at $\lambda = -\alpha$ will be encircled. One may write

$$I = \int_{-\infty}^{\infty} \frac{d\lambda \lambda^2}{(\lambda^2 - \alpha^2)^{2n}} = \int_{-\infty}^{\infty} \frac{d\lambda}{(\lambda^2 - \alpha^2)^{2n-1}} + \alpha^2 \int_{-\infty}^{\infty} \frac{d\lambda}{(\lambda^2 - \alpha^2)^{2n}} \quad (29)$$

Consider a typical integral about $\lambda = -\alpha$.

$$\oint \frac{d\lambda}{(\lambda - \alpha)^m (\lambda + \alpha)^m}$$

By Cauchy's theorem this is

$$I = \frac{2\pi i}{(m-1)!} \frac{d^{m-1}}{d\lambda^{m-1}} \frac{1}{(\lambda - \alpha)^m} \bigg|_{\lambda = -\alpha}$$

$$I = 2\pi i \frac{(-1)^n (2m-1)!}{[(m-1)!]^2 2^{2m-1} \alpha^{2m-1}} \quad (30)$$

After applying Eq. (30) to the integrals in 29 one finds

$$I = \frac{-\pi i (4n-3)!}{2^{4n-3} [(2n-1)!]^2} \frac{(2n-1)}{\alpha^{4n-3}} \quad (31)$$

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The remaining integral over α in the expression for S_n may be similarly evaluated by closing the contour above the real axis. One finds

$$\int_{-\infty}^{\infty} \frac{e^{i\alpha E}}{\alpha^{3n-3}} d\alpha = \frac{2\pi i (iE)^{3n-4}}{(3n-4)!} \quad (32)$$

The expression for the density of states per unit energy in the extreme relativistic limit is therefore

$$S_n = \frac{\Omega^{n-1} (2n-1)(4n-3)!}{\pi^{2n-2} 2^{4n-4} [(2n-1)!]^2} \frac{E^{3n-4}}{(3n-4)!} \quad (33)$$

This is to be compared with the corresponding formula, of Fermi's paper⁶,

⁶ Ibid, page 576, Eq. 13.

neglecting momentum conservation.

$$S_n = \frac{\Omega^{n-1}}{\pi^{2n}} \frac{E^{3n-1}}{(3n-1)!}$$

III. Approximate Evaluation of Phase Space Integral

It is very difficult to proceed further in the evaluation of the phase space integral when the masses of the particles concerned is not zero. The difficulty arises mainly out of the fact that the Hankel functions have a branch point at the origin. In order to obtain some approximate formulae and numerical results one may arbitrarily divide particles into two classes: (1) pions and (2) heavy mesons and nucleons. The pions because of their small mass will be treated as being extremely relativistic; the other particles may be taken as non-relativistic. If we now return to Eq. (12) of the previous section, it may be written

$$\begin{aligned}
 I_{ER} &= -\frac{2\pi}{\lambda} \frac{d}{d\lambda} \int_{-\infty}^{\infty} dp e^{i[\lambda p - \alpha(|p| + M)]} && \text{(Extreme Relativistic)} \\
 I_{NR} &= -\frac{2\pi}{\lambda} \frac{d}{d\lambda} e^{-i\alpha M} \int_{-\infty}^{\infty} dp e^{i(\lambda p - \alpha \frac{p^2}{2M})} && \text{(Non-Relativistic)}
 \end{aligned}
 \tag{34}$$

Both integrals are easily evaluated. The first may be written

$$I_{ER} = -\frac{4\pi}{\lambda} \frac{d}{d\lambda} \int_0^{\infty} e^{-i\alpha p} \cos \lambda p \, dp = \frac{4\pi}{\lambda} \frac{d}{d\lambda} \frac{i\alpha e^{-i\alpha M}}{(\lambda^2 - \alpha^2)}
 \tag{35}$$

After performing the differentiation with respect to λ one finds

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$$I_{ER} = \frac{8\pi i \alpha}{(\lambda^2 - \alpha^2)^2} e^{-i\alpha M}$$

The other integral may be evaluated by completing the square in the exponent.

$$I_{NR} = -\frac{2\pi}{\lambda} \frac{d}{d\lambda} e^{-i\alpha M} e^{\frac{i M \lambda^2}{2\lambda}} \int_{-\infty}^{\infty} e^{-i \frac{\alpha}{2M} p'^2} dp'$$

$$I_{NR} = -\frac{2\pi}{\lambda} \frac{d}{d\lambda} e^{-i\alpha M} e^{\frac{i M \lambda^2}{2\alpha}} \pi^{\frac{1}{2}} \left(\frac{2M}{\alpha}\right)^{\frac{1}{2}} e^{-i\pi/4}$$

Finally one obtains

$$I_{NR} = -\pi^{3/2} e^{i\pi/4} \left(\frac{2M}{\alpha}\right)^{3/2} e^{-i\alpha M} e^{\frac{i M \lambda^2}{2\alpha}} \quad (36)$$

If the first class of particles, $i = 1$, is taken to represent pions and the others to represent heavy mesons and nucleons, one obtains the following expression for S_n :

$$S_n = C_n \int_{-\infty-i\epsilon}^{\infty-i\epsilon} \frac{d\alpha d^3\lambda}{(\lambda^2 - \alpha^2)^{2m_1}} e^{i\alpha(E - \sum m_i M_i)} \frac{e^{i \sum \frac{m_i M_i \lambda^2}{2\alpha}}}{\alpha^{\frac{1}{2}(3n - 5m_1)}} \quad (37)$$

The constant C_n is

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$$C_n = \frac{\Omega^{n-1} \prod_i M_i^{3m_i} (-1)^{n-m_i} e^{i \frac{\pi}{4} (n+m_i)}}{2^{\frac{3n}{2} - \frac{3m_i}{2} + 1} \pi^{\frac{3n}{2} + \frac{m_i}{2} + 1}} \quad (38)$$

The integral over λ in Eq. (37) may be evaluated in part to yield

$$\Lambda(m_i, \alpha) = 4\pi \int_0^\infty \frac{e^{i \frac{\lambda^2}{2\alpha} \sum m_i M_i}}{(\lambda^2 - \alpha^2)^{2m}} \lambda^2 d\lambda \quad (39)$$

Since only small values of α contribute to Eq. (37), the exponential oscillates rapidly and only small values of λ contribute to Λ . If one sets $\lambda = 0$ in the denominator of (39) the integral may be evaluated and yields

$$\Lambda(m_i, \alpha) \cong \frac{\Lambda(0, \alpha)}{\alpha^{4m_i}} = \frac{-(2\pi)^{3/2} e^{-i\pi/4}}{(\sum m_i M_i)^{3/2}} \frac{1}{\alpha^{4m_i - 3/2}} \quad (40)$$

This evaluation of (39) means that the momentum carried away by the pions has been neglected. If this result is substituted into Eq. (37) one finds

$$S_n = - \left(\frac{2\pi}{\sum m_i M_i} \right)^{3/2} e^{-i \frac{\pi}{4}} C_n \int_{-\infty - i\epsilon}^{\infty - i\epsilon} \frac{d\alpha e^{i\alpha(E - \sum m_i M_i)}}{\alpha^{\frac{1}{2}(3n + 3m_i - 3)}} \quad (41)$$

This integral may be easily evaluated in the case when $\frac{3}{2}(n + m_i - 1)$ is an integer since the contour may be closed above the real axis and thus yields

the residue at $\alpha = 0$. When $\frac{3}{2}(m_1 + n - 1)$ is not integral, we may infer the same result by observing that S_n is an analytic function of E and applying the principle of continuation. Thus, one finds

$$S_n = - \left(\frac{2\pi}{\sum m_i M_i} \right)^{3/2} e^{-i\pi/4} C_n \frac{2\pi i \left[i(E - \sum m_i M_i) \right]^{\frac{1}{2}(3n + 3m_1 - 5)}}{\Gamma\left(\frac{3}{2}(n + m_1 - 1)\right)} \quad (42)$$

Here Γ is the ordinary Γ function.

If one now substitutes, in this expression, the Eq. (38) for C_n one finds

$$S_n = \frac{\Omega^{n-1} \prod_i M_i^{3m_i} (E - \sum m_i M_i)^{\frac{1}{2}(3n + 3m_1 - 5)}}{\left(\sum m_i M_i \right)^{3/2} 2^{\frac{1}{2}(3n - 3m_1 - 1)} \pi^{\frac{1}{2}(3n + m_1 - 3)} \Gamma\left(\frac{3}{2}n + \frac{3}{2}m_1 - \frac{3}{2}\right)} \quad (43)$$

IV. Applications

The results of the foregoing sections may be applied to compute the probabilities of various nuclear processes. If the total cross section is assumed to be $\pi(\hbar/\mu c)^2$ that part of the cross section leading to the production of m_i particles of mass M_i would be

$$\sigma(m'_1, \dots, m'_l) = \frac{S(m'_1, \dots, m'_l)}{\sum_{m_1 \dots m_l} S(m_1, \dots, m_l)} \sigma_T$$

This formula is not quite right however since in the foregoing considerations the indistinguishability of the particles formed has been neglected along with other requirements such as charge and angular momentum conservation. Besides these conservation laws some conservation law must be applied regarding the number of nucleons involved in a reaction; otherwise any nuclear reaction would lead to the formation of large numbers of particles of small mass since the phase space available in this case is overwhelmingly large. One must either assume that the number of nucleons involved in a reaction is conserved or that there is some other restrictive relationship. If one believes that nucleons can be produced in pairs this requirement means that the difference between the number of nucleons and anti-nucleons formed must be a constant

$$N - N' = \text{constant} \quad (44)$$

This assumption will be made in carrying out calculations in this paper.

Although conservation of charge and angular momentum have been ignored the fact that particles may be indistinguishable will be taken into

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account by dividing the statistical weight factors by appropriate factorials which reduce them because the particles may be indistinguishable. Thus, the cross sections reported will be calculated by writing

$$\sigma(m'_1, \dots, m'_\ell) = \frac{\frac{S(m'_1, \dots, m'_\ell)}{m'_1! \dots m'_\ell!}}{\sum_{m_1 \dots m_\ell} \frac{S(m_1, \dots, m_\ell)}{m_1! \dots m_\ell!}} \quad (45)$$

In the tables calculations have been made for cases in which pions have been formed together with nucleons and a possible particle of mass 967 times that of the electron.

The characteristic feature of all high energy processes is that elastic scattering of high energy nucleon becomes less important as the energy becomes higher.

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TABLE I

Probabilities for emission of m_1 , m_2 , m_3 particles of mass $M_1 = 176$, $M_2 = 967$, $M_3 = 1848$ (in units of the electron mass) at the indicated energy E (center of mass system) are of distinguishable particles.

m_1	m_2	m_3	2.5 Bev.	3.9 Bev.	4.71 Bev.
0	0	2	.5846	.007388	.0008
1			.4127	.2302	.0647
2			.0027	.3679	.2957
3			0	.09187	.2547
4				.00503	.0601
5				.00007	.0046
0	1	2		.05910	.0111
1				.1391	.0936
2				.03006	.0954
3				.0010	.0207
4				.000005	.0012
0	2	2		.0562	.0288
1				.00942	.0365
2				.00013	.0071
3				0	.0003
0	3	2		.00248	.0148
1	3	2		.000005	.0024
0	4	2			.0008
0	0	4			.0063
1	0	4			0
0	1	4			.0001
1	1	4			0

TABLE II.

Probabilities for emission of m_1, m_2, m_3 particles of mass $M_1 = 176, M_2 = 967, M_3 = 1848$ (in units of the electron mass) at the indicated energy E (center of mass system) are of undistinguishable particles.

m_1	m_2	m_3	2.5 Bev.	3.9 Bev.	4.71 Bev.
0	0	2	.5854	.0108	.0018
1			.4132	.3367	.1412
2			.0014	.2690	.3228
3			0	.0224	.0927
4				.0003	.0055
5				0	0
0	1	2		.08642	.0242
1				.2035	.2042
2				.02198	.1042
3				.00024	.0075
4				0	.0001
0	2	2		.0411	.0315
1				.0069	.0398
2				.0001	.0039
3				0	0
0	3	2		.0006	.0054
1	3	2		0	.0009
0	4	2			.0001
0	0	4			.0138
1	0	4			.0001
0	1	4			.0003
1	1	4			0

V. Conclusion

The calculations of Fermi have been extended to include the possible production (singly) of a particle of mass 967 m. The tables may be used to estimate the effect on this probability of assuming it to be produced only in pairs. Some attempt has been made to include the pion rest mass into the calculations in a better way than writing $E = P$ by choosing $E = \mu + P$. This is especially important where the multiplicity is high and the momenta of individual pions is small since it insures zero probability for production of more pions than are energetically possible. Thus, the multiplicities indicated in the tables are generally lower than those listed in Fermi's paper. Thus, the probabilities are roughly taken to be proportional to

$$P_n \propto (E - n\mu)^{3n}$$

rather than

$$P_n \propto E^{3n}.$$

Finally, it is important to notice that the Fermi theory is inconsistent in some respects. The most important of these is that classically one cannot conserve momentum of particles in a box of finite volume since light particles may reach its walls before heavier ones. Since the box is an imaginary one (unlike the boundary of a nucleus) it makes no sense to deal with the dynamics of the box. This inconsistency makes it impossible to use the present method and to deal correctly with the requirement that the center of gravity of the particles remain at rest, an effect which introduces drastic modifications in the case of relativistic particles and to some extent clears up the question of the ad hoc insertion of the Lorentz contraction factor.

These questions will be discussed in some detail in a later paper⁷.

⁷ Lepore, Neuman, and Stuart, Abstract to the New York meeting of the American Physical Society, January 29, 1954, UCRL Report-2399.

Although no attempt has been made to deal with the important questions of conservation of angular momentum and isotopic spin⁸, the effect of possible

⁸ Enrico Fermi, Phys. Rev. 92, 452 (1953)

indistinguishability of the particles produced has been studied in the second table. However, in view of the other omissions it seems reasonable to base comparisons with experimental data with the figures of the first table since the fact that pions of different charges (+, -, 0) will be produced tends to diminish the effects of indistinguishability.

The importance of a treatment of the phase integrals which properly includes pion momentum conservation is emphasized by comparing the formula for S_n in the extreme relativistic approximation where pion momentum conservation has been rigorously included with the corresponding formula of Fermi's paper in which it has been neglected. It seems reasonable that a different treatment of Eq. (39) may be devised to properly include it in cases of practical interest.