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Abstract

A set of the electrostatic toroidal gyrokinetic Vlasov equation and the Poisson equation, which explicitly includes the polarization drift, is derived systematically by using Lie-transform method. The polarization drift is introduced in the gyrocenter equations of motion, and the corresponding polarization density is derived. Contrary to the wide-spread expectation, the inclusion of the polarization drift in the gyrocenter equations of motion does not affect the expression for the polarization density significantly. This is due to modification of the gyrocenter phase-space volume caused by the electrostatic potential [T. S. Hahm, Phys. Plasmas **3**, 4658 (1996)] .

I. INTRODUCTION

Nonlinear gyrokinetic formulations[1–7] have played an important role in progress towards understanding tokamak microturbulence.[8,9] Even after continuous research for almost three decades, there still remain some theoretical issues which deserve further clarification. For instance, in the standard gyrokinetic formulation, there is a polarization density term in the Poisson equation, while there is no polarization drift in the gyrokinetic Vlasov equation which is based on gyrocenter equations of motion.

It was widely believed that the gyrokinetic Poisson equation should be significantly changed if the polarization drift is explicitly included in the gyrocenter equations of motion. For instance, Sosenko, et al. kept the polarization drift in the quasi-particle equations, and found cancelations of leading order terms in the long wavelength limit in the Poisson equation.[10] Heikkinen, et al. applied this model to particle simulation of tokamak transport.[11]

In the present paper, we find that including the polarization drift explicitly in the gyrokinetic Vlasov equation does not significantly change the Poisson equation in the long wavelength limit. We note that Dimits[16] has reached the same conclusion on this issue independently. We adopt the Lie-transform method[13–15] as is done in the standard gyrokinetic theory, but the polarization drift is systematically included in the gyrocenter equations of motion by changing the definition of the gyrocenter. The definition of the guiding-center in this paper is the same as that in the standard gyrokinetic theory, whereas the definition of the gyrocenter is different. Various definitions of the guiding-center and the gyrocenter are summarized in Table I and Fig. 1. The different definition of the gyrocenter results in a different gyrocenter phase-space volume, which can affect the expression for the polarization density.[17] In many previous works, this effect was neglected.[10,11]

The principal results of this paper include:

1. A set of the electrostatic toroidal gyrokinetic Vlasov equation and the Poisson equation is systematically derived by using the Lie-transform method. The main difference from the standard gyrokinetic equations[1–7] is that the polarization drift in the gyrocenter equations of motion is introduced in this work.

2. We find that the polarization density remains almost the same in the long wavelength limit in homogeneous plasmas, although the polarization drift is introduced in the gyrocenter equations of motion. This is different from the expectation widely spread in the literature. The correct polarization density can only be derived by taking into account the change of the phase-space volume of gyrocenter coordinates due to the fluctuating $\mathbf{E} \times \mathbf{B}$ velocity.

3. The phase-space Lagrangian Lie-perturbation theory ensures that the gyrokinetic Vlasov-Poisson system has an exact energy invariant.

The remainder of this paper is organized as follows. In Sec. II, the results of a phase-space Lagrangian transformation from particle phase space to gyrocenter phase space are presented. The polarization drift is included in the gyrocenter equations of motion. We systematically derive the gyrokinetic Vlasov-Poisson system and the energy invariant in

Sec. III. In Sec. IV, we present discussion relevant to our work. We focus our attention on the systematic derivation of a set of gyrokinetic Vlasov equation and Poisson equation in the presence of polarization drift and on the clarification of the relationship between polarization drift and polarization density.

II. LIE-TRANSFORM FOR GYROCENTER PHASE SPACE WITH POLARIZATION DRIFT

In this section, we present the gyrocenter equations of motion including the polarization drift. The particle phase-space Lagrangian is given by

$$\gamma = \left(\frac{e}{c}\mathbf{A} + m\mathbf{v}\right)\cdot d\mathbf{x} - \left(\frac{1}{2}mv^2 + e\delta\phi\right)dt, \tag{1}$$

where $\mathbf{A}$ is the magnetic vector potential, $\mathbf{x}$ is the particle position, $\mathbf{v} = u\mathbf{b}+\mathbf{c}_\perp$ is the particle velocity, $\mathbf{b} \equiv \mathbf{B}/B$, $\mathbf{B} = \nabla \times \mathbf{A}$ is the inhomogeneous equilibrium magnetic field, and $\delta\phi$ is the fluctuating electrostatic potential. The standard nonlinear gyrokinetic ordering[1] consists of:

$$\omega/\Omega \sim k_\parallel \rho_i \sim e\delta\phi/T_i \sim \epsilon$$

and

$$k_\perp \rho_i \sim 1,$$

where ω and Ω are the characteristic fluctuation frequency and the ion cyclotron frequency, respectively, $k_\parallel$ and $k_\perp$ are the components of the wave vector in the parallel and perpendicular direction with respect to the magnetic field, $\rho_i = (T_i/m)^{1/2}/\Omega$ is the thermal ion gyroradius, and $\epsilon \ll 1$ is a small ordering parameter.

Starting from the unperturbed phase-space Lagrangian of a charged particle, one can perform the Lie perturbation analysis as described in Refs. 13–15, to obtain the unperturbed guiding-center phase-space Lagrangian,

$$\Gamma_0 = \left(\frac{e}{c}\mathbf{A} + mU\mathbf{b}\right)\cdot d\mathbf{R} + \frac{\mu B}{\Omega}d\theta - \left(\mu B + mU^2/2\right)dt. \tag{2}$$

Here, $(\mathbf{R}, U, \mu, \theta)$ are guiding-center phase-space coordinates: $\mathbf{R}$ denotes the guiding-center position, U is the parallel guiding-center velocity, μ is the guiding-center magnetic moment, and θ is the guiding-center gyro-phase angle. All the definitions are the same as those of the standard modern gyrokinetics[7] so far. Catto has used a preliminary transformation from particle to guiding-center variables for the linear gyrokinetic theory.[18] This remains to be useful for the nonlinear derivation as well.

Next, the gyrocenter phase-space transformation is needed to remove the gyro-phase angle dependence contained in the fluctuation. The perturbed guiding-center phase-space Lagrangian is:

$$\Gamma_1 = -e\delta\phi(\mathbf{R} + \boldsymbol{\rho},\, t)dt \equiv -e\delta\phi_{gc}dt, \tag{3}$$

where $\boldsymbol{\rho} = \mathbf{b} \times \mathbf{c}_\perp/\Omega$. The lowest-order gyrocenter phase-space Lagrangian has the same expression as the corresponding expression in standard modern gyrokinetics,

$$\overline{\Gamma}_0 = \left(\frac{e}{c}\mathbf{A} + m\overline{U}\mathbf{b}\right) \cdot d\overline{\mathbf{R}} + \frac{\overline{\mu}B}{\Omega}d\overline{\theta} - \left(\overline{\mu}B + \frac{1}{2}m\overline{U}^2\right) dt. \tag{4}$$

"$\overline{}$" in this paper means a physical quantity in gyrocenter phase space.

Our derivation deviates from the standard modern gyrokinetic theory in the following. For the first-order gyrocenter phase-space Lagrangian, we choose the generators and gauge function of Lie transformation such that

$$\overline{\Gamma}_1 = m\delta\mathbf{u}_E \cdot d\overline{\mathbf{R}} - \overline{H}_1 dt, \tag{5}$$

where $\delta\mathbf{u}_E = c\mathbf{b} \times \overline{\nabla}\langle\delta\phi_{gc}\rangle/B$ and $\overline{H}_1$ is gyro-phase angle independent. Here, introduction of the fluctuating $\mathbf{E} \times \mathbf{B}$ velocity results in a different definition of the gyrocenter position. In this way, the polarization drift can explicitly be included in the gyrocenter equations of motion, as can be seen shortly. According to the nonlinear gyrokinetic ordering, the first-order Hamiltonian is given by

$$\overline{H}_1 = e\langle \delta\phi_{gc}\rangle, \tag{6a}$$

and the generators are given by

$$\begin{aligned}
G_1^{\mathbf{R}} &= \frac{1}{B^*_{0\parallel}}\left(\frac{mc}{e}\mathbf{b}\times\delta\mathbf{u}_E - \frac{c}{e}\mathbf{b}\times\overline{\nabla}S_1 - \frac{\mathbf{B}_0^*}{m}\frac{\partial S_1}{\partial \overline{U}}\right),\\
G_1^{U} &= \frac{\mathbf{B}_0^*}{mB^*_{0\parallel}}\cdot\left(\overline{\nabla}S_1 - m\delta\mathbf{u}_E\right),\\
G_1^{\mu} &= \frac{e}{mc}\frac{\partial S_1}{\partial\overline{\theta}},\\
G_1^{\theta} &= -\frac{e}{mc}\frac{\partial S_1}{\partial\overline{\mu}},
\end{aligned} \tag{6b}$$

and the gauge function is

$$S_1 = \frac{e}{\Omega}\int d\overline{\theta}\,\widetilde{\delta\phi}_{gc}. \tag{6c}$$

Here, $\mathbf{B}_0^* = \nabla\times A + mc/eU\nabla\times\mathbf{b}$, $B^*_{0\parallel} = \mathbf{b}\cdot\mathbf{B}_0^*$, $\widetilde{\delta\phi}_{gc} = \delta\phi_{gc} - \langle\delta\phi_{gc}\rangle$, and the bracket means gyro-phase angle average, i.e, $\langle\delta\phi_{gc}\rangle \equiv (2\pi)^{-1}\oint d\theta\delta\phi(\mathbf{R}+\boldsymbol{\rho}, t)$, for instance. The main difference from the standard modern gyrokinetics is that there is an additional term, $mc\mathbf{b}\times\delta\mathbf{u}_E/(eB^*_{0\parallel})$, in $G_1^{\mathbf{R}}$. This indicates that the definition of gyrocenter position, $\overline{\mathbf{R}}$, is different from that of standard modern gyrokinetics. The difference is illustrated in Table I and Fig. 1.

The second-order perturbation analysis is quite similar to that of Ref. 4, but the result is not exactly the same due to the different first-order gyrocenter phase-space Lagrangian adopted in this work. The second-order terms should be kept for energy conservation up to $O(\epsilon^2)$ as discussed in Ref. 3 and further detail in a recent review article.[7] Finally, the total gyrocenter phase-space Lagrangian can be written as

$$\Gamma = \left(\frac{e}{c}\mathbf{A} + m\overline{U}\mathbf{b} + m\delta\mathbf{u}_E\right)\cdot d\overline{\mathbf{R}} + \frac{\overline{\mu}B}{\Omega}d\overline{\theta} - \left(\overline{\mu}B + \frac{1}{2}m\overline{U}^2 + e\delta\Psi_{gy}\right)dt. \tag{7}$$

Here, the effective gyrocenter perturbation potential is

$$\delta\Psi_{gy} = \langle\delta\phi_{gc}\rangle - \frac{e}{2B}\frac{\partial}{\partial\mu}\langle\widetilde{\delta\phi}_{gc}^2\rangle + \frac{mc^2}{eB^2}|\overline{\nabla}_\perp\langle\delta\phi_{gc}\rangle|^2. \tag{8}$$

The Euler-Lagrangian equations corresponding to the gyrocenter phase-space Lagrangian[4] are given by

$$\frac{e}{c}\frac{d\overline{\mathbf{R}}}{dt}\times\mathbf{B}^* - m\mathbf{b}\frac{d\overline{R}}{dt} = \overline{\mu}\overline{\nabla}B + e\overline{\nabla}\delta\Psi_{gy} + m\frac{\partial}{\partial t}\delta\mathbf{u}_E, \tag{9}$$

and

$$m\mathbf{b} \cdot \frac{d\overline{\mathbf{R}}}{dt} = m\overline{U}. \tag{10}$$

Here,

$$\mathbf{B}^* = \mathbf{B}_0^* + mc/e\overline{\nabla} \times \delta\mathbf{u}_E, \tag{11}$$

and

$$B_\parallel^* = \mathbf{b} \cdot \mathbf{B}^* \simeq B_{0\parallel}^* + \frac{mc^2}{e}\overline{\nabla}_\perp \cdot \left(\frac{1}{B}\overline{\nabla}_\perp \langle\delta\phi_{gc}\rangle\right). \tag{12}$$

The second term associated with the potential fluctuation in $B_\parallel^*$ is new feature which was absent in the standard gyrokinetic theory. It results from the fluctuating $\mathbf{E} \times \mathbf{B}$ drift which is introduced in the gyrocenter phase-space Lagrangian.

Equations (9)-(10) can be decomposed into the following gyrocenter equations of motion which look more familiar:

$$\frac{d\overline{\mathbf{R}}}{dt} = \overline{U}\frac{\mathbf{B}^*}{B_\parallel^*} + \frac{c\mathbf{b}}{eB_{0\parallel}^*} \times \left(\overline{\mu}\overline{\nabla}B + e\overline{\nabla}\delta\Psi_{gy}\right) - \frac{c}{\Omega B_\parallel^*}\frac{\partial}{\partial t}\overline{\nabla}_\perp\langle\delta\phi_{gc}\rangle, \tag{13}$$

$$\frac{d\overline{U}}{dt} = -\frac{\mathbf{B}^*}{B_\parallel^*} \cdot \left(\overline{\mu}\overline{\nabla}B + e\overline{\nabla}\delta\Psi_{gy} + \frac{\partial}{\partial t}\delta\mathbf{u}_E\right). \tag{14}$$

The last term in Equation (13) is the polarization drift. It can be attributed to the introduction of the fluctuating $\mathbf{E} \times \mathbf{B}$ velocity in the gyrocenter phase-space Lagrangian.

III. GYROKINETIC VLASOV-POISSON EQUATIONS AND ENERGY CONSERVATION LAW

We have derived the gyrocenter equations of motion with the polarization drift in the previous section. With Equations (13) and (14), the gyrokinetic Vlasov equation for the gyrocenter distribution function $\overline{F}(\overline{\mathbf{R}}, \overline{\mu}, \overline{U}, t)$ can be written as

$$\frac{\partial\overline{F}}{\partial t} + \frac{d\overline{\mathbf{R}}}{dt} \cdot \overline{\nabla F} + \frac{d\overline{U}}{dt}\frac{\partial\overline{F}}{\partial\overline{U}} = 0. \tag{15}$$

Here, $d\overline{\mu}/dt \equiv 0$ and $\partial\overline{F}/\partial\overline{\theta} \equiv 0$ have been used. It has been recognized that the gyrokinetic Vlasov equation, the gyrokinetic Poisson equation, and the corresponding energy invariant are three important pillars of nonlinear gyrokinetic theory.[7] Therefore, it's desirable to treat them on an equal footing. A field theoretical variational method was introduced in deriving systematically the gyrokinetic Vlasov-Maxwell system.[19,20] In the present paper, we express the particle charge density in Poisson equation and corresponding energy invariant in terms of the gyrocenter distribution function by a pull-back transformation from the gyrocenter phase space to the particle phase space. Then, we evaluate them explicitly from the most general formal expression of the pull-back transformation. The polarization density in the gyrokinetic Poisson equation will be discussed in details.

Poisson equation in particle phase space is given by

$$\nabla^2 \delta\phi(\mathbf{x},\, t) = -4\pi e \left[n_i(\mathbf{x},\, t) - n_e(\mathbf{x},\, t) \right]. \tag{16}$$

In this paper, we do not specify the electron dynamics, so the electron density can be written in a primitive form

$$n_e(\mathbf{x},\, t) = \int d^3\mathbf{x} d^3\mathbf{v} f_e(\mathbf{x},\, \mathbf{v},\, t). \tag{17}$$

The ion particle density written in terms of the gyrocenter distribution function via a pull-back transformation is given by

$$\begin{aligned} n_i(\mathbf{x},\, t) &= \int d^3\mathbf{v} f_i(\mathbf{x},\, \mathbf{v},\, t) \\ &= \int d^3\mathbf{v} d^3\mathbf{x}' f_i(\mathbf{x}',\, \mathbf{v},\, t) \delta^3(\mathbf{x}' - \mathbf{x}) \\ &= \int d^6 Z F_i(Z) \delta^3(\mathbf{R} + \boldsymbol{\rho} - \mathbf{x}) \\ &= \int d^6 \overline{Z}\, \overline{F}_i(\overline{Z})\, T_{gy}^{-1} \delta^3(\overline{\mathbf{R}} + \overline{\boldsymbol{\rho}} - \mathbf{x}). \end{aligned} \tag{18}$$

Here, $d^6\overline{Z} = \left| \partial(\mathbf{v},\, \mathbf{x})/\partial \overline{Z} \right| d^3\overline{R} d\overline{U} d\overline{\mu} d\overline{\theta}$, and $\left| \partial(\mathbf{v},\, \mathbf{x})/\partial \overline{Z} \right| \simeq B^*_{0\|}/m + c^2 \overline{\nabla}_\perp \cdot \left(\frac{1}{B} \overline{\nabla}_\perp \langle \delta\phi_{gc} \rangle \right)/e$ is the Jacobian for transformation from the particle phase space to the gyrocenter phase space. As discussed in Section II, the second term makes the gyrocenter phase-space volume different from that of the standard gyrokinetic theory. The gyrocenter phase-space volume is modified due to the different definition of gyrocenter position used in this work. Equation (18) contains contributions from two parts. One part is the gyro-averaged gyrocenter density

$$\overline{N}_i(\mathbf{x},\, t) = \int \frac{B^*_{0\|}}{m} d^3\overline{R} d\overline{U} d\overline{\mu} d\overline{\theta}\, \overline{F}_i(\overline{Z}) \delta^3 \left(\overline{\mathbf{R}} + \overline{\boldsymbol{\rho}} - \mathbf{x} \right), \tag{19}$$

and the other part is the polarization density

$$\begin{aligned} n_{pol}(\mathbf{x},\,t) \;=\; & -\int \frac{B_{\|}^{*}}{m} d^3\overline{R} d\overline{U} d\overline{\mu} d\overline{\theta}\, \overline{F}_i(\overline{Z})\, \left(G_1^{\mathbf{R}} \cdot \overline{\nabla} + G_1^{\mu}\partial_{\overline{\mu}} + G_1^{\theta}\partial_{\overline{\theta}} \right) \delta^3(\overline{\mathbf{R}} + \overline{\boldsymbol{\rho}} - \mathbf{x}) \\ & + \int \frac{B_{0\|}^{*}}{m} d^3\overline{R} d\overline{U} d\overline{\mu} d\overline{\theta}\, \overline{F}_i(\overline{Z})\, \frac{mc^2}{eB_{0\|}^{*}} \nabla_{\perp} \cdot \left(\frac{1}{B} \nabla_{\perp} \langle \delta\phi_{gc} \rangle \right) \delta^3(\overline{\mathbf{R}} + \overline{\boldsymbol{\rho}} - \mathbf{x}). \quad (20) \end{aligned}$$

Here, $\overline{N}_i$ does not explicitly include the potential fluctuation. We keep all the potential fluctuation associated terms in the polarization density. The second term of n_{pol} is attributed to the modification of the gyrocenter phase-space volume due to $\langle \delta\phi_{gc} \rangle$.[17] The time-dependent, self-generated $\mathbf{E} \times \mathbf{B}$ zonal flows[8,21,22] also contribute to this term. More recently, the modification of Jacobian due to the equilibrium $\mathbf{E} \times \mathbf{B}$ drift was avoided by moving it into the Hamiltonian part in Ref. 12 rather than keeping it in the sympletic part of the Lagrangian.[17,23] Combining Equations (17), (19) and (20), the gyrokinetic Poisson equation is written as

$$\nabla^2 \delta\phi(\mathbf{x},\,t) = -4\pi e \left[\overline{N}_i(\mathbf{x},\,t) + n_{pol}(\mathbf{x},\,t) - n_e(\mathbf{x},\,t) \right]. \quad (21)$$

The global gyrokinetic Vlasov-Poisson energy invariant is obtained by the Noether's theorem and integration over space, as described in Eq. (50) of Ref. 20, and Eq. (199) of Ref. 7:

$$\begin{aligned} E \;=\; & \int \frac{d^3x}{8\pi} \left(|\nabla\delta\phi|^2 + B^2 \right) + \int d^6 z f_e \frac{1}{2} m_e v^2 \\ & + \int \frac{B_{\|}^{*}}{m} d^3\overline{R} d\overline{U} d\overline{\mu} d\overline{\theta}\, \overline{F}_i(\overline{Z}) \left[\overline{\mu} B + \frac{1}{2} m \overline{U}^2 + e\delta\Psi_{gy} - e\langle T_{gy}^{-1} \delta\phi_{gc} \rangle \right] \\ =\; & \int \frac{d^3x}{8\pi} \left(|\nabla\delta\phi|^2 + B^2 \right) + \int d^6 z f_e \frac{1}{2} m_e v^2 \\ & + \int \frac{B_{\|}^{*}}{m} d^3\overline{R} d\overline{U} d\overline{\mu} d\overline{\theta}\, \overline{F}_i(\overline{Z}) \left[\overline{\mu} B + \frac{1}{2} m \overline{U}^2 + \frac{e^2}{2B} \partial_{\overline{\mu}} \langle \widetilde{\delta\phi}_{gc}^{2} \rangle \right]. \quad (22) \end{aligned}$$

So far, the gyrokinetic Vlasov-Poisson system and the corresponding energy invariant are presented formally. Next, we consider two limiting cases and evaluate them explicitly.

Case I. In the long wavelength limit, i.e., $k_{\perp}\rho_i \ll 1$, the gyrokinetic Vlasov equation is

$$\frac{\partial \overline{F}_i}{\partial t} + \frac{d\overline{\mathbf{R}}}{dt} \cdot \overline{\nabla} \overline{F}_i + \frac{d\overline{U}}{dt} \frac{\partial \overline{F}_i}{\partial \overline{U}} = 0. \quad (23)$$

The corresponding gyrocenter equations of motion,

$$\frac{d\overline{\mathbf{R}}}{dt} = \overline{U}\frac{\mathbf{B}^*}{B_\parallel^*} + \frac{c\mathbf{b}}{eB_{0\parallel}^*} \times \left(\overline{\mu}\overline{\nabla}B + e\overline{\nabla}\delta\Psi_{gy}\right) - \frac{c}{\Omega B_\parallel^*}\frac{\partial}{\partial t}\overline{\nabla}_\perp\langle\delta\phi_{gc}\rangle, \tag{24}$$

$$\frac{d\overline{U}}{dt} = -\frac{\mathbf{B}^*}{B_\parallel^*} \cdot \left(\overline{\mu}\overline{\nabla}B + e\overline{\nabla}\delta\Psi_{gy} + \frac{\partial}{\partial t}\delta\mathbf{u}_E\right). \tag{25}$$

Here, the effective gyrocenter perturbation potential in the long wavelength limit is reduced to

$$\delta\Psi_{gy} = \langle\delta\phi_{gc}\rangle + \frac{1}{2e}m|\delta\overline{\mathbf{u}}_E|^2, \tag{26}$$

where $\delta\overline{\mathbf{u}}_E = c\mathbf{b}\times\overline{\nabla}\,\overline{\delta\phi}/B$, with $\overline{\delta\phi} \equiv \delta\phi(\overline{\mathbf{R}},\, \overline{\mu},\, \overline{U},\, t)$.

Next, we calculate the polarization density in the long wavelength limit. The first term of Equation (20) can be written as

$$\begin{aligned} & -\int \frac{B_\parallel^*}{m} d^3\overline{R}d\overline{U}d\overline{\mu}d\overline{\theta}\,\overline{F}_i(\overline{Z})\langle G_1^R\overline{\nabla} + G_1^\mu\partial_{\overline{\mu}}\overline{\boldsymbol{\rho}}\cdot\overline{\nabla} + G_1^\theta\partial_{\overline{\theta}}\overline{\boldsymbol{\rho}}\cdot\overline{\nabla}\rangle\delta^3\left(\overline{\mathbf{R}}-\mathbf{x}\right) \\ = {} & \nabla\cdot\int \frac{B_\parallel^*}{m} d\overline{U}\,d\overline{\mu}\,d\overline{\theta}\,\overline{F}_i(\overline{Z})\langle G_1^R + G_1^\mu\partial_{\overline{\mu}}\overline{\boldsymbol{\rho}} + G_1^\theta\partial_{\overline{\theta}}\overline{\boldsymbol{\rho}}\rangle, \end{aligned} \tag{27}$$

where

$$\begin{aligned} \langle G_1^R + G_1^\mu\partial_{\overline{\mu}}\overline{\boldsymbol{\rho}} + G_1^\theta\partial_{\overline{\theta}}\overline{\boldsymbol{\rho}}\rangle &= \frac{mc}{eB_{0\parallel}^*}\mathbf{b}\times\delta\mathbf{u}_E + \frac{e}{B}\partial_{\overline{\mu}}\langle\widetilde{\delta\phi}_{gc}\overline{\boldsymbol{\rho}}\rangle \\ &= -\frac{mc^2}{eBB_{0\parallel}^*}\nabla_\perp\delta\phi + \frac{mc^2}{eB^2}\nabla_\perp\delta\phi \\ &\simeq 0. \end{aligned} \tag{28}$$

It is shown that the first term vanishes up to $O(k_\perp^2\rho_i^2)$. This is due to cancelation of the contributions from G_1^μ and G_1^θ (the usual polarization density) and that from $G_1^{\mathbf{R}}$ which is introduced to get polarization drift explicitly in the gyrocenter equations of motion. This lowest order cancelation has led to an expression of polarization density which scales like $k_\perp^4\rho_i^4$ in the long wavelength limit, in the literature.[10,11] However, we find that the polarization density comes from the second term associated with modification of the gyrocenter phase-space volume with the following expression:

$$n_{pol}(\mathbf{x},\, t) \simeq \frac{mc^2N_i}{eB}\nabla_\perp\cdot\left(\frac{1}{B}\nabla_\perp\delta\phi\right), \tag{29}$$

where

$$N_i(\mathbf{x},\, t) = \int \frac{B_{0\parallel}^*}{m} d^3\overline{R}d\overline{U}d\overline{\mu}d\overline{\theta}\,\overline{F}_i(\overline{Z})\delta^3\left(\overline{\mathbf{R}}-\mathbf{x}\right). \tag{30}$$

Therefore, the gyrokinetic Poisson equation in the long wavelength limit is given by

$$\nabla^2 \delta\phi(\mathbf{x},\, t) = -4\pi e \left[\overline{N}_i(\mathbf{x},\, t) + \frac{mc^2 N_i}{eB} \nabla_\perp \cdot \left(\frac{1}{B} \nabla_\perp \delta\phi \right) - n_e(\mathbf{x},\, t) \right]. \tag{31}$$

Contrary to the conventional wisdom, the gyrokinetic Poisson equation does not change significantly although the polarization drift is included in the gyrokinetic Vlasov equation. For weak variation of N_i and B, the polarization density in the presence of polarization drift, i.e., Eq. (29) is almost identical to the standard expression (for instance, Eq. (31) of Ref. 24)

$$\frac{mc^2}{e} \nabla_\perp \cdot \left(\frac{N_i}{B^2} \nabla_\perp \delta\phi \right).$$

The importance of modification of gyrocenter phase-space volume to the polarization density[17] was also noted in Dimits' work.[16]

For the energy invariant, the last term of Eq. (22) in the long wavelength limit is reduced to

$$\frac{1}{2} m \left| \delta \overline{\mathbf{u}}_E \right|^2 \tag{32}$$

Therefore, the energy invariant in the long wavelength limit can be obtained by substituting Eq. (32) to Eq. (22)

$$\begin{aligned} E \; = & \int \frac{d^3x}{8\pi} \left(|\nabla \delta\phi|^2 + B^2 \right) + \int d^6 z f_e \frac{1}{2} m_e v^2 \\ & + \int \frac{B_\parallel^*}{m} d^3\overline{R} d\overline{U} d\overline{\mu} d\overline{\theta}\, \overline{F}_i(\overline{Z}) \left[\overline{\mu} B + \frac{1}{2} m \overline{U}^2 + \frac{1}{2} m \left| \delta \overline{\mathbf{u}}_E \right|^2 \right]. \end{aligned} \tag{33}$$

It has the same appearance as the standard one,[4] but with a different gyrocenter phase-space volume in the ion kinetic energy expression.

Case II. Now, we consider an arbitrary $k_\perp \rho_i$ but with an assumption of Maxwellian ion gyrocenter distribution function in $\overline{\mu}$, i.e., $\overline{F} \propto \exp[-\overline{\mu} B/T_i]$.

In this case, we calculate the polarization density, Eq. (20), by using Fourier expansion. After an integration in phase space, the contribution from G_1^R term on the first line of Equation (20) can be written as

$$\sum_k \exp(i\mathbf{k} \cdot \mathbf{x}) N_i b \Gamma_0(b) \frac{e\delta\phi_k}{T_i}. \tag{34}$$

Here, $\Gamma_0(b) = I_0(b)e^{-b}$, and I_0 is the 0th order modified Bessel function, where $b = k_\perp^2\rho_i^2$. The second line of Equation (20) coming from modification of the gyrocenter phase-space volume is reduced to

$$-\sum_k \exp(i\mathbf{k}\cdot\mathbf{x})N_i b\Gamma_0(b)\frac{e\delta\phi_k}{T_i}, \tag{35}$$

which cancels the contribution from G_1^R term. The variation of B and N_i is neglected in this limiting case. Then, the contribution from remaining G_1^μ and G_1^θ in the first line of Equation (20) turns out to be the same as the standard polarization density

$$-\sum_k \exp(i\mathbf{k}\cdot\mathbf{x})N_i(1-\Gamma_0(b))\frac{e\delta\phi_k}{T_i}. \tag{36}$$

Therefore, the Poisson equation in this limit can be given by

$$\nabla^2\delta\phi(\mathbf{x},\,t) = -4\pi e\left[\overline{N}_i(\mathbf{x},\,t) - n_e(\mathbf{x},\,t) - \sum_k \exp(i\mathbf{k}\cdot\mathbf{x})N_i(1-\Gamma_0(b))\frac{e\delta\phi_k}{T_i}\right]. \tag{37}$$

The corresponding energy invariant for this system can be written as

$$\begin{aligned} E \;=\; & \int\frac{d^3x}{8\pi}\left(|\nabla\delta\phi|^2 + B^2\right) + \int d^6z f_e\frac{1}{2}m_e v^2 \\ & + \int\frac{B_\parallel^*}{m}d^3\overline{R}d\overline{U}d\overline{\mu}d\overline{\theta}\,\overline{F}_i(\overline{Z})\left[\overline{\mu}B + \frac{1}{2}m\overline{U}^2\right] + \sum_k\frac{e^2}{2T_i}N_i(1-\Gamma_0)|\delta\phi_k|^2. \end{aligned} \tag{38}$$

Once again, the energy invariant in this case also has the same expression as the standard one but with the modification of the ion gyrocenter phase-space volume which is related to the polarization density. Note that polarization drift does not appear explicitly in this expression. Both the gyrokinetic Poisson equation and the energy invariant in this case can reduce to Eqs. (31) and (33) for the long wavelength limit (for weak variation of N_i and B) by expanding the modified Bessel function Γ_0 for small b.

IV. CONCLUSION

In the present paper, via introducing the fluctuating $\mathbf{E}\times\mathbf{B}$ velocity in the first-order perturbed gyrocenter phase-space Lagrangian, we have derived a set of electrostatic toroidal gyrokinetic Vlasov equation which includes the polarization drift explicitly. This results in a different definition of gyrocenter position and consequently the gyrocenter phase-space volume is modified. The corresponding polarization density has been derived as well in which the modification of phase-space volume is taken into account. We find that the polarization density does not change significantly in the long wavelength limit. This is contrary to the widely spread expectation.

We also consider the arbitrary $k_\perp \rho_i$ case with an assumption of Maxwellian gyrocenter distribution function. The same polarization density as that of the standard gyrokinetic theory is obtained. It can be reduced to the expression for the polarization density in the long wavelength limit by expanding the modified Bessel function. In conclusion, gyrokinetic Poisson equation in the presence of polarization drift is almost identical to that of the standard gyrokinetic theory. This can be attributed to the modification of the Jacobian for transformation from the particle phase space to the gyrocenter phase space[17] which was ignored in previous works.[10–12]

Including the polarization drift explicitly in the gyrocenter equation of motion does not change significantly neither the gyrocenter equation of motion nor the energy invariant. The phase-space Lagrangian Lie-perturbation theory ensures that the gyrokinetic Vlasov-Poisson system has an exact energy conservation law. We present the gyrokinetic Vlasov equation and Poisson equation and the corresponding energy invariant in two limiting cases.

The appearance of polarization drift containing a derivative with respect to time puts more demand on the numerical scheme to be used. On the other hand, the familiar Laplacian operator in the gyrokinetic Poisson equation seems to suggest that the situation is not as bad[25] as that implied by Refs. 10,11.

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TABLE I: Definitions of guiding center and gyrocenter in standard gyrokinetic theory and this work.

	Guiding Center	Gyrocenter
Standard	$\mathbf{R} = \mathbf{x} - \boldsymbol{\rho}$	$\overline{\mathbf{R}} = \mathbf{x} - \boldsymbol{\rho} - \frac{1}{B^*_{0\parallel}} \left(\frac{c}{e} \mathbf{b} \times \overline{\nabla} S_1 + \frac{\mathbf{B}^*_0}{m} \frac{\partial S_1}{\partial \overline{U}} \right)$
This work	$\mathbf{R} = \mathbf{x} - \boldsymbol{\rho}$	$\overline{\mathbf{R}} = \mathbf{x} - \boldsymbol{\rho} + \frac{mc}{eB^*_{0\parallel}} \mathbf{b} \times \delta \mathbf{u}_E - \frac{1}{B^*_{0\parallel}} \left(\frac{c}{e} \mathbf{b} \times \overline{\nabla} S_1 + \frac{\mathbf{B}^*_0}{m} \frac{\partial S_1}{\partial \overline{U}} \right)$

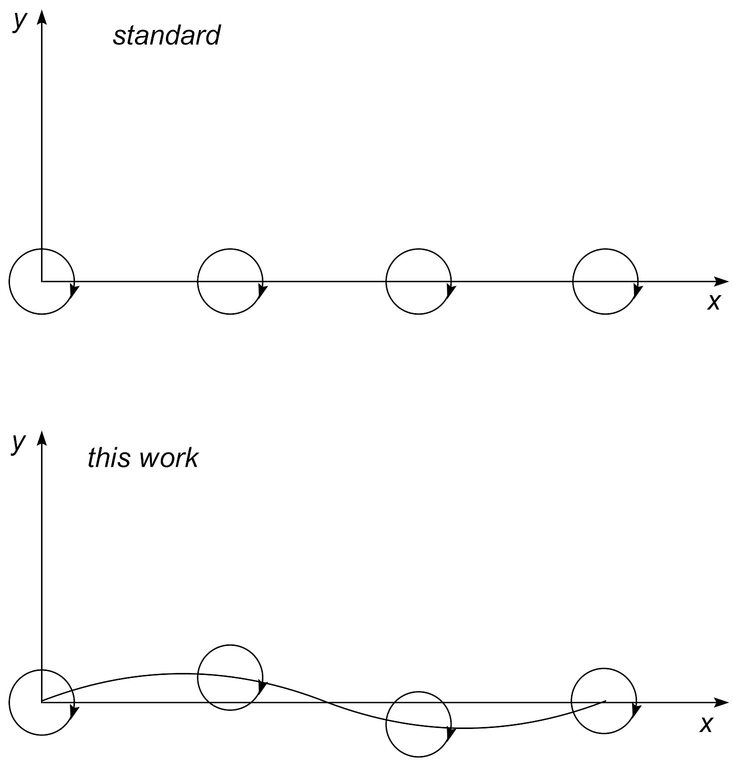

FIG. 1: Definitions of gyrocenter in standard gyrokinetic theory and this work for uniform magnetic field. $\phi = \langle \phi \rangle$ and a single mode $\phi \propto \cos(kx)$ are assumed for illustration.

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