Fermilab

$\bar{p}$ Note #265

Explicit Expression of the Electric
Field in the Kicker

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The analytical expression of the electric field $E(x,y)$ in the kicker region has been derived by Umstatter [1] in the complex form:

$$E_y(x,y) + i E_x(x,y) = \frac{C(1-t)}{\sqrt{(a-t)(b-t)}}, \quad (1)$$

with

$$t = \tanh^2 \left[\frac{\pi(x+iy)}{2h} \right], \quad (1a)$$

$$a = \tanh^{-2} \left(\frac{\pi l}{4h} \right), \quad (1b)$$

$$b = \tanh^{-2} \left[\frac{\pi(d+l/2)}{2h} \right], \quad (1c)$$

$$C = \frac{\pi\sqrt{b}}{2K(\sqrt{1-b/a})} \frac{V_k}{h}, \quad (1d)$$

where h , l , and d are geometrical parameters of the kicker and $K(\sqrt{1-b/a})$ is a complete elliptic integral.

It is trivial to find explicit representations of a real and imaginary part of t :

$$\begin{aligned} t &= \left[\tanh \left(\frac{x'}{2} + \frac{i}{2} y' \right) \right]^2 & x' &= \frac{\pi}{h} x, & y' &= \frac{\pi}{h} y \\ &= \left(\frac{\sinh x' + i \sin y'}{\cosh x' + \cos y'} \right)^2 \\ &= R_t + i I_t. \end{aligned} \quad (2)$$

Here, we set

$$R_t = \frac{\sinh^2 x' - \sin^2 y'}{(\cosh x' + \cos y')^2}, \quad (2a)$$

$$I_t = \frac{2 \sinh x' \sin y'}{(\cosh x' + \cos y')^2}. \quad (2b)$$

If the absolute values and the arguments of the complex functions of $E_y + iE_x$, $1-t$, $a-t$ and $b-t$ are introduced, then,

$$E_y + iE_x = E e^{i\theta}, \quad (3a)$$

$$1-t = R_1 e^{i\theta_1}, \quad (3b)$$

$$a-t = R_a e^{i\theta_a}, \quad (3c)$$

$$b-t = R_b e^{i\theta_b}. \quad (3d)$$

Substitutions of (2) into (3b) ~ (3d) yield

$$R_1 = [(1-R_t)^2 + I_t^2]^{1/2}, \quad (4a)$$

$$R_a = [(a-R_t)^2 + I_t^2]^{1/2}, \quad (4b)$$

$$R_b = [(b-R_t)^2 + I_t^2]^{1/2}, \quad (4c)$$

$$\cos \theta_1 = \frac{1-R_t}{R_1}, \quad \sin \theta_1 = \frac{-I_t}{R_1}, \quad (5a)$$

$$\cos \theta_a = \frac{a-R_t}{R_a}, \quad \sin \theta_a = \frac{-I_t}{R_a}, \quad (5b)$$

$$\cos \theta_b = \frac{b - R_t}{R_b}, \quad \sin \theta_b = \frac{-I_t}{R_b}. \quad (5c)$$

Now, substituting (3a) ~ (3d) into (1) and taking logarithm of both sides, we have

$$\log E + i\theta = \log C + \log R_1 - \frac{1}{2} \log R_a - \frac{1}{2} \log R_b + i \left(\theta_1 - \frac{\theta_a}{2} - \frac{\theta_b}{2} \right). \quad (6)$$

Equating real and imaginary parts of both sides in (6), we obtain

$$E = C R_1 R_a^{-\frac{1}{2}} R_b^{-\frac{1}{2}}, \quad (7a)$$

$$\theta = \theta_1 - \frac{\theta_a}{2} - \frac{\theta_b}{2}. \quad (7b)$$

Then we can write $E_x(x,y)$ and $E_y(x,y)$ in the following forms:

$$\begin{aligned} E_x(x,y) &= E \sin \theta \\ &= E \sin \left[\frac{1}{2} (2\theta_1 - \theta_a - \theta_b) \right] \\ &= \frac{E}{\sqrt{2}} [1 - \cos (2\theta_1 - \theta_a - \theta_b)]^{1/2} \\ &= \frac{E}{\sqrt{2}} [1 - (2\cos^2 \theta_1 - 1)(\cos \theta_a \cos \theta_b \\ &\quad - \sin \theta_a \sin \theta_b) - 2\sin \theta_1 (\cos \theta_1 (\sin \theta_a \cos \theta_b + \cos \theta_a \sin \theta_b))]^{1/2}, \end{aligned} \quad (8a)$$

similarly,

$$\begin{aligned} E_y(x,y) &= \frac{E}{\sqrt{2}} [1 + (2\cos^2 \theta_1 - 1)(\cos \theta_a \cos \theta_b - \sin \theta_a \sin \theta_b) \\ &\quad + 2\sin \theta_1 (\cos \theta_1 (\sin \theta_a \cos \theta_b + \cos \theta_a \sin \theta_b))]^{1/2} \end{aligned} \quad (8b)$$

Finally, substituting (5a) ~ (5c) into (8a) and (8b), we have

$$\begin{aligned} E_x(x,y) &= \frac{E}{\sqrt{2}} \left\{ 1 - \frac{(1-R_t)^2 - I_t^2}{R_1^2 R_a R_b} [(a-R_t)(b-R_t) - I_t^2] \right. \\ &\quad \left. - \frac{2(1-R_t)I_t^2}{R_1^2 R_a R_b} (a+b-2R_t) \right\}^{1/2}, \end{aligned} \quad (9a)$$

$$\bar{E}_y(x, y) = \frac{E}{\sqrt{2}} \left\{ 1 + \frac{(1-R_t)^2 - I_t^2}{R_1^2 R_a R_b} \left[(a-R_t)(b-R_t) - I_t^2 \right] + \frac{2(1-R_t)I_t^2}{R_1^2 R_a R_b} (a+b-2R_t) \right\}^{1/2}, \quad (9b)$$

where

$$E = \frac{c \left[(1-R_1)^2 + I_t^2 \right]^{1/2}}{\left[(a-R_t)^2 + I_t^2 \right]^{1/4} \left[(b-R_t)^2 + I_t^2 \right]^{1/4}}. \quad (9c)$$

Here, we consider the case that values of a and b are relatively near to unity. For simplicity, we set

$$\begin{aligned} a &= 1 + \epsilon, \\ b &= 1 + \delta. \end{aligned} \quad (\epsilon, \delta \ll 1)$$

Expanding right-hand sides of (9a) and (9b) with respect to ϵ & δ , we obtain

$$E_x(x, y) = O(\Delta^2), \quad (10a)$$

$$E_y(x, y) = c \left[1 - \frac{1-R_t}{2R_1^2} \Delta \right] + O(\Delta^2), \quad (10b)$$

with $\Delta = \epsilon + \delta$.

As a special case, we consider the electronic field on the y axis, that is, $x = 0$. Then,

$$\frac{1-R_t}{2R_1^2} = \frac{1}{2} \cos^2 \frac{y'}{2} \quad \text{or} \quad \frac{1}{2} \cos^2 \frac{\pi y}{2h}, \quad (11)$$

The expression (10b) with (11) agrees to the result obtained previously [2].

REFERENCES

- [1] H. Umstatter, unpublished (1982)
- [2] K. Takayama, "Effects of Non-Linear Electrode on Transverse Cooling", in Proceedings of Cooling Workshop (1983) and p Note 264.