Pbar Note 639

Phase Space Matching Errors

When Transferring a Beam From one Circular Accelerator to Another

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In a circular accelerator a particle with coordinates (x, q) moves in phase space such that the coordinates trace an ellipse that is conventionally written as:

$$bq^2 + 2axq + gx^2 = A^2 \quad [1]$$

If we make the following transformation of coordinates:

$$\begin{aligned} c &= x / \sqrt{b} \\ x &= \sqrt{b} \left(q + \frac{a}{b} x \right) \end{aligned} \quad [2]$$

the phase space trajectory is transformed into a circle:

$$c^2 + x^2 = A^2 \quad [3]$$

Equivalently, we can describe the particle trajectories in terms of A and

$$f = \text{atan} \left(\frac{x}{c} \right) \quad [4]$$

A beam will contain a distribution of amplitudes $r(A, f)$. The rms beam size is

$$s^2 = \int_0^{\infty} \int_0^{2\pi} r(A, f) d f d A, \quad [5]$$

and the beam emittance is:

$$e = 3s_A^2. \quad [6]$$

Now we consider the situation where a beam from one circular accelerator is transferred to another. The beta functions of the first accelerator propagated through the transfer line into the second accelerator will carry the subscript one while the beta functions at the same location in the second accelerator will carry the subscript 2.

We can transform the coordinates so that the phase space trajectories in the first accelerator will be circular. In this coordinate system the phase space trajectories of the

second accelerator will be elliptical and will be displaced from those of the first by a steering error $(\Delta x, \Delta \phi)$. The geometry is illustrated in the figure below.

In order to obtain an expression for the trajectories in the second accelerator in terms of the amplitude of the first accelerator we substitute for the position and angles the quantities:

$$\begin{aligned} x &= \sqrt{b_1} c \\ q &= (x - a_1 c) / \sqrt{b_1} \end{aligned} \quad [7]$$

into the phase space ellipse in the second accelerator. We obtain

$$b_2 \left(\frac{x - a_1 c}{\sqrt{b_1}} - \Delta q \right)^2 + 2 a_2 \left(\frac{x - a_1 c}{\sqrt{b_1}} - \Delta q \right) (\sqrt{b_1} c - \Delta x) + g_2 (\sqrt{b_1} c - \Delta x)^2 = A_2^2, \quad [8]$$

which can be simplified and rewritten as

$$b_r (x - \Delta x)^2 + 2 a_r (x - \Delta x) (c - \Delta c) + g_r (c - \Delta c)^2 = A_2^2 \quad [9]$$

where

$$\begin{aligned}
b_r &= b_2 / b_1 \\
a_r &= a_2 - \frac{a_1 b_2}{b_1} \\
g_r &= (1 + a_r^2) / b_r
\end{aligned} \tag{10}$$

and

$$\begin{aligned}
\Delta c &= \Delta x / \sqrt{b_1} \\
\Delta x &= \sqrt{b_1} \left(\Delta q + \frac{a_1}{b_1} \Delta x \right)
\end{aligned} \tag{11}$$

We can now compute the rms beam size in the second accelerator. It is easiest to compute the steering error term in rectangular coordinates:

$$\begin{aligned}
s_2^2 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A_2^2 r(\sqrt{c^2 + x^2}) dx dc \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (b_r x^2 + 2 a_r x c + g_r c^2) r(\sqrt{c^2 + x^2}) dx dc + b_r \Delta x^2 + 2 a_r \Delta x \Delta c + g_r \Delta c^2
\end{aligned} \tag{12}$$

A little algebra will show that the emittance growth depends only on the injection position and angle errors and the betafunctions of the second accelerator:

$$b_r \Delta x^2 + 2 a_r \Delta x \Delta c + g_r \Delta c^2 = b_2 \Delta q^2 + 2 a_2 \Delta q \Delta x + g_2 \Delta x^2 \tag{13}$$

The emittance growth that occurs from the mismatch in betafunctions is most easily computed in terms of polar coordinates. We transform to polar coordinates with the substitution:

$$\begin{aligned}
c &= A_1 \cos f \\
x &= A_1 \sin f
\end{aligned} \tag{14}$$

The emittance growth from the beta function mismatch is easily calculated as shown below:

$$\begin{aligned}
& \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\mathbf{b}_r \mathbf{x}^2 + 2 \mathbf{a}_r \mathbf{x} \mathbf{c} + \mathbf{g}_r \mathbf{c}^2) r(\sqrt{\mathbf{c}^2 + \mathbf{x}^2}) d\mathbf{x} d\mathbf{c} \\
&= \int_0^{\infty} \int_0^p \left(\frac{\mathbf{g}_r + \mathbf{b}_r}{2} \right) A_1^2 \left[1 + \frac{\sqrt{(\mathbf{g}_r - \mathbf{b}_r)^2 + 4 \mathbf{a}_r^2}}{\mathbf{g}_r + \mathbf{b}_r} \cos 2(\mathbf{f} - \mathbf{f}_0) \right] r(A) d\mathbf{f} dA \\
&= \left(\frac{\mathbf{g}_r + \mathbf{b}_r}{2} \right) \mathbf{s}_1^2
\end{aligned} \tag{15}$$

In terms of the beta functions of the two machines:

$$\begin{aligned}
\left(\frac{\mathbf{g}_r + \mathbf{b}_r}{2} \right) &= \frac{1}{2} \left[\frac{\mathbf{b}_1}{\mathbf{b}_2} + \frac{\mathbf{b}_2}{\mathbf{b}_1} + \frac{(\mathbf{a}_2 \mathbf{b}_1 - \mathbf{a}_1 \mathbf{b}_2)^2}{\mathbf{b}_1 \mathbf{b}_2} \right] \\
&\geq 1
\end{aligned} \tag{16}$$

Finally, we should point out that steering errors consist of two components: the error for particles on the central momentum and the additional error from off momentum particles that occurs because of the dispersion mismatch.

$$\begin{aligned}
\Delta \mathbf{x} &= \Delta \mathbf{x}_0 + (\mathbf{h}_2 - \mathbf{h}_1) \frac{\Delta p}{p} \\
\Delta \mathbf{q} &= \Delta \mathbf{q}_0 + (\mathbf{h}'_2 - \mathbf{h}'_1) \frac{\Delta p}{p}
\end{aligned} \tag{17}$$

Putting it all together we have:

$$\begin{aligned}
\mathbf{e}_2 &= \frac{1}{2} \left[\frac{\mathbf{b}_1}{\mathbf{b}_2} + \frac{\mathbf{b}_2}{\mathbf{b}_1} + \frac{(\mathbf{a}_2 \mathbf{b}_1 - \mathbf{a}_1 \mathbf{b}_2)^2}{\mathbf{b}_1 \mathbf{b}_2} \right] \mathbf{e}_1 \\
&+ 3 \left[\mathbf{b}_2 \Delta \mathbf{q}_0^2 + 2 \mathbf{a}_2 \Delta \mathbf{q}_0 \Delta \mathbf{x}_0 + \mathbf{g}_2 \Delta \mathbf{x}_0^2 \right] \\
&+ 3 \left[\mathbf{b}_2 (\mathbf{h}'_2 - \mathbf{h}'_1)^2 + 2 \mathbf{a}_2 (\mathbf{h}'_2 - \mathbf{h}'_1)(\mathbf{h}_2 - \mathbf{h}_1) + \mathbf{g}_2 (\mathbf{h}_2 - \mathbf{h}_1)^2 \right] \left(\frac{\mathbf{s}_p^2 + \Delta p_o^2}{p^2} \right)
\end{aligned} \tag{18}$$

where $\mathbf{s}_p$ is the beam momentum spread and Δp_o is the momentum mismatch between the two accelerators.