DISTRIBUTION OF THERMAL NEUTRONS
FROM FAST SOURCES IN EXPONENTIAL PILES

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INTRODUCTION

In the mathematical analysis of the exponential pile experiment, it is customary to assume that the extraneous sources emit only thermal neutrons. This assumption leads to an expression for the thermal flux which is rather inaccurate near the source plane, with the result that no measurements are made in this region. The effect of a reduction in the number of experimental points is to increase the error in the calculated value of the diffusion length. If the pile is loaded with slugs of high absorbing nonmultiplying material, the neutron attenuation may be so great and the range of measurement so small that any loss in points cannot be tolerated. It is possible to avoid this difficulty by adopting the more accurate assumption that the extraneous sources emit fast neutrons, say all at age zero, so that the derived flux expressions are comparatively accurate even in the neighborhood of the source plane. The procedure involves solving the age equation to determine the distributed thermal neutron source and then using ordinary one-group diffusion theory to find the thermal flux.

We assume that the pile is essentially homogeneous as far as slowing down properties are concerned. This is the case, for example, if the pile is fully loaded, or, in the case of a partially loaded pile, if the proportion of slug material in a lattice cell is small and if there is no absorption of neutrons above thermal energies by the slug material.

THE "METHOD OF IMAGES"

To avoid the rather difficult task of solving the age equation for a finite region, we resort to the so-called "method of images," which permits us to regard the pile as a part of an infinite region. The method involves postulating additional sources outside the pile in such a manner that the boundary conditions are still satisfied. For example, a pile consisting of the infinite half-space for which \( x > 0 \) (extrapolated boundary at \( x = 0 \)), with a unit source at \((a,0,0)\) could be replaced by an infinite pile with an additional
source of "anti-neutrons" (actually a sink) at \((-a,0,0)\). By symmetry the flux vanishes at \(x = 0\), so that, for \(x \geq 0\), the flux distributions in each of the two piles not only satisfy the same differential equation but also satisfy the same boundary conditions. According to a certain uniqueness theorem they must be identical. This device is rigorous only for thermal sources, since the extrapolation distance and, therefore, the location of the image point, is properly dependent on age. However, since the thermal flux distribution is only weakly dependent on the local properties of the source distribution, we can safely ignore this discrepancy.

In the case of the rectangular parallelepiped, images must be formed in each of the six faces. In addition, images of these images are required, the process continuing ad infinitum in each of the six directions. In order to determine the image positions, let us divide all space into rectangular parallelepipeds congruent to the original pile by extending the six plane sides to infinity and constructing an infinite number of planes parallel to these at proper distances. Then the sources within the individual image piles are positive or negative according as whether an even or odd number of reflections have been made in the sides (extended) of the basic pile. Each slab formed by extending just two sides of the original pile is then divided as a checkerboard.

![Diagram](image)

\[\text{Fig. 1}\]

A general rule of signs can be derived. If the coordinates of the center of the base of an image pile are \((i, j, a, l, c)\), the associated sign is \((-1)^{i+j+l}a^c\).

Let us illustrate with the two most common source patterns. First suppose that there is a single source at \((0,0,z_0)\), where \(0 \leq z_0 \leq c\). The images on the \(z\)-axis are located as in Figure 2.
These coordinates are of the form \( \epsilon z_0 \neq 2lc \), \( \epsilon = \pm 1 \), the plus sign for even images and the minus sign for odd. In general, images are located at points \((ia, ja, \epsilon z_0 \neq 2lc)\), \( i, j, l = 0, \pm 1, \pm 2, \ldots \), and the signs associated with the sources at these points are

\[ \epsilon(-1)^k \neq j. \]

Note that the case \( i = j = l = 0, \epsilon = \neq 1 \) gives the original source point.

Suppose next that unit sources are located at the four points

\[ \left( \pm \frac{a}{4}, \pm \frac{a}{4}, z_0 \right). \]

Then images are at

\[ \left( \pm \frac{a}{4} \neq ia, \pm \frac{a}{4} \neq ja, \epsilon z_0 \neq 2lc \right), \]

where

\[ i, j, l = 0, \pm 1, \pm 2, \ldots, \epsilon = \pm 1, \]

and the corresponding signs are

\[ \epsilon(-1)^i \neq j. \]
These results are summarized in the following table.

<table>
<thead>
<tr>
<th>Source Locations</th>
<th>Associated Signs</th>
</tr>
</thead>
<tbody>
<tr>
<td>((ia, ja, \epsilon z_o \neq 2\lambda c))</td>
<td>(\epsilon(-1)^i \neq j)</td>
</tr>
<tr>
<td>((\pm a/4 \neq ia, \pm a/4 \neq ja, \epsilon z_o \neq 2\lambda c))</td>
<td>(\epsilon(-1)^i \neq j)</td>
</tr>
</tbody>
</table>

To generalize somewhat upon the preceding discussion, suppose that sources of fast neutrons are located at points \(\vec{r}_n\), \(n = 1, 2, 3, \ldots\), with source strengths \(\epsilon_n S\), where \(\epsilon_n = \pm 1\). Then the slowing down density due to the \(n\)th source is

\[
\frac{\epsilon_n S}{(4\pi\tau)^{3/2}} e^{-\frac{|\vec{r} - \vec{r}_n|^2}{4\tau}}
\]

and the total slowing down density, considering all sources, is given by

\[
q(\vec{r}, \tau) = \frac{S}{(4\pi\tau)^{3/2}} \sum_{n=1}^{\infty} \epsilon_n e^{-\frac{|\vec{r} - \vec{r}_n|^2}{4\tau}}
\]

This series converges very rapidly, so that images at points \(\vec{r}_n\) a few "lattice distances" away from the basic parallelepiped are insignificant. The symmetrical arrangement of the source and image points plus the fact that, for exponential experiments, measurements are made only at points \(\vec{r} = (0, 0, z)\) permits the collection of the terms of the series into groups of up to 8 terms having identical values. Thus the infinite form of (3) does not lead to a prohibitive amount of calculation.

**DIFFUSION THEORY**

**Part 1. Single Region Pile**

From ordinary one-group theory, the thermal flux is the solution to

\[
(\nabla^2 - k^2) \phi = -\frac{q(\vec{r}, \tau)}{D}
\]

where $K$ is the reciprocal of the diffusion length, $D$ is the diffusion constant and $\tau$ is understood to be thermal age. The solution to (2) for a single fast source of unit strength is

\[ \frac{e^{K^2 \tau}}{8\pi DR} \int e^{-Kr} \text{erfc} \left( K\sqrt{\tau} - \frac{r}{2\sqrt{\tau}} \right) - e^{Kr} \text{erfc} \left( K\sqrt{\tau} + \frac{r}{2\sqrt{\tau}} \right) J, \]

where $r$ is the distance from the source to the field point. If, as in the preceding section, there are sources at points $\hat{r}_n$, $n = 1, 2, 3, \ldots$, with strengths $\epsilon_n S$, where $\epsilon_n = \pm 1$, the thermal flux is given by

\[ \phi(\hat{r}) = \frac{e^{K^2 \tau}}{8\pi D} \sum_{n=1}^{\infty} \frac{\epsilon_n}{|r - \hat{r}_n|} \int e^{-K |\hat{r} - \hat{r}_n|} \text{erfc} \left( K\sqrt{\tau} - \frac{|\hat{r} - \hat{r}_n|}{2\sqrt{\tau}} \right) \]

\[ - e^{K |\hat{r} - \hat{r}_n|} \text{erfc} \left( K\sqrt{\tau} + \frac{|r - \hat{r}_n|}{2\sqrt{\tau}} \right) J. \]

(3)

**Part 2: Two Region Pile**

Suppose now that the pile is loaded only in the region $s < z < c$. Then the region $0 < z < s$ is essentially a reflector. If we assume that, to an acceptable degree of approximation, the two regions are identical as far as the slowing-down of the source neutrons is concerned, the thermal source term is the same for both regions. Letting subscripts 1 and 2 refer respectively to the regions $0 < z < s$ and $s < z < c$, the flux must satisfy

\[ (\nabla^2 - B_i) \phi_i = -\frac{q(\hat{r}, \tau)}{D} \quad \text{,} \quad i = 1, 2, \]

(4)

where, for cases pertinent to this discussion, both $B_1$ and $B_2$ will be negative.

Recalling the form of the solution of the homogeneous equation from the standard treatment assuming point thermal sources, let us attempt to find a similar expression for the flux in each region. Thus assume

\[ \phi_i = \sum_{k, m \text{ odd}} A_{i km}(z) \cos \frac{\pi x}{a} \cos \frac{m \pi y}{a} \quad , \quad i = 1, 2, \]

(5)

2. See HW-31886, Neutron Flux Distribution from a Point Source in an Infinite Medium, this author, or Wallace, P.R. and LeCaine, Jr., Elementary Approximations in the Theory of Neutron Diffusion, M.T.-12, National Research Council of Canada.
where, at present, the $A_{ikm}(z)$ are unknown. It should be noted that the boundary conditions at $x,y = \pm a$ are satisfied identically. The $A_{ikm}$ will be chosen so that the $\phi_i$ satisfy the differential equations and so that the boundary conditions at $z = 0$, $z = s$, and $z = c$ are also satisfied.

Before substituting (5) into (4), we expand the right member of (4) in a series of the same type as (5). Referring to (1), we have

$$\frac{-q(r, \zeta)}{D} = \frac{-S}{D(4\pi c)^{3/2}} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\xi_n}{n} \left( -\frac{(x-x_n)^2}{4\pi c} - \frac{(y-y_n)^2}{4\pi c} - \frac{(z-z_n)^2}{4\pi c} \right).$$

Expanding the $x$ and $y$ parts of each term separately in Fourier cosine series (the sine terms vanish because of symmetry) and reversing the order of summation,

$$\frac{-q(r, \zeta)}{D} = \sum_{k, m \text{ odd}} \left\{ \sum_{n=1}^{\infty} F_{km} \frac{(z-z_n)^2}{4\pi c} \right\} \cos \frac{k\pi x}{a} \cos \frac{m\pi y}{a},$$

where

$$F_{km} = \frac{-S}{D(4\pi c)^{3/2}} \cdot \frac{16}{a^2} \int_{0}^{2\pi} \int_{0}^{\phi} \int_{0}^{2\phi} \frac{(x-x_n)^2 + (y-y_n)^2}{4\pi c} \cos \frac{k\pi x}{a} \cos \frac{m\pi y}{a} \, dx \, dy \,.$$

This result is obtained most directly by multiplying both members of (6) by \[
\cos \frac{k\pi x}{a} \cos \frac{m\pi y}{a},
\]
integrating from 0 to $a$ with respect to $x$ and $y$ and solving for the only remaining coefficient on the right, $F^n_{k'm'}$.

Substituting (5) and (6) into (4),

$$\sum_{k, m \text{ odd}} \left\{ \int \left( -\frac{k^2 + m^2}{a^2} \eta^2 \right) d\eta \right\} A_{ikm}(z) - A''_{ikm}(z) \right\} \cos \frac{k\pi x}{a} \cos \frac{m\pi y}{a}$$

$$= \sum_{k, m \text{ odd}} \left\{ \sum_{n=1}^{\infty} F_{km} \frac{(z-z_n)^2}{4\pi c} \right\} \cos \frac{k\pi x}{a} \cos \frac{m\pi y}{a}.$$
This equation implies that

\[ A_{ikm}^{n}(z) - \gamma_{ikm}^{2} A_{ikm}(z) = \sum_{n=1}^{\infty} F_{n k m} e^{-\frac{(z - z_{n})^{2}}{4 \tau}}, \]  

(8)

where

\[ \gamma_{ikm}^{2} = \frac{(k^{2} - m^{2}) c^{2}}{a^{2}} - B_{i}. \]  

(9)

Since, in general, the \( B_{i} \) are negative, the \( \gamma_{ikm}^{2} \) are all positive.

Let us first solve the equation

\[ A^{n}(z) - \gamma^{2} A(z) = e^{-\frac{(z - z_{n})^{2}}{4 \tau}}, \]  

(10)

and use that result to obtain the solution of (8). The general solution of the homogeneous equation is

\[ \alpha_{e}^{z} \neq \beta e^{-\gamma z} \]  

where \( \alpha \) and \( \beta \) are arbitrary constants. Following the method of variation of parameters, we set

\[ A(z) = \alpha(z) e^{\gamma z} + \beta(z) e^{-\gamma z}. \]  

(11)

Then

\[ A'(z) = \gamma \alpha(z) e^{\gamma z} - \gamma \beta(z) e^{-\gamma z} + \alpha'(z) e^{\gamma z} - \beta'(z) e^{-\gamma z}. \]  

(12)

Let

\[ \alpha'(z) e^{\gamma z} + \beta'(z) e^{-\gamma z} = 0. \]  

(13)

Then

\[ A''(z) = \gamma^{2} \alpha(z) e^{\gamma z} - \gamma^{2} \beta(z) e^{-\gamma z} + \gamma \alpha'(z) e^{\gamma z} - \gamma \beta'(z) e^{-\gamma z}. \]  

(14)

Substituting (11) and (14) into (10),

\[ \gamma \alpha'(z) e^{\gamma z} - \gamma \beta'(z) e^{-\gamma z} = e^{-\frac{(z - z_{n})^{2}}{4 \tau}}. \]  

(15)
Solving (13) and (1') simultaneously,

\[
\begin{align*}
\alpha'(z) &= \frac{1}{2y} e^{-\frac{(z - z_n)^2}{4\tau}} - yz, \\
\beta'(z) &= -\frac{1}{2y} e^{-\frac{(z - z_n)^2}{4\tau}} + yz
\end{align*}
\]

Integrating,

\[
\begin{align*}
\alpha(z) &= \frac{1}{2y} \int_A^z e^{-\frac{(z' - z_n)^2}{4\tau}} - yz' \\
\beta(z) &= -\frac{1}{2y} \int_B^z e^{-\frac{(z' - z_n)^2}{4\tau}} + yz'
\end{align*}
\]

where A and B are constants to be determined.

Let \( z' - z_n = u \) in the above integrals:

\[
\begin{align*}
\alpha(z) &= \frac{e^{-\gamma_{zn}}}{2y} \int_{A - z_n}^{z - z_n} e^{-\frac{u^2}{4\tau}} - Yu \\
\beta(z) &= \frac{e^{-\gamma_{zn}}}{2y} \int_{B - z_n}^{z - z_n} e^{-\frac{u^2}{4\tau}} + Yu
\end{align*}
\]

Completing the squares in the exponents,

\[
\begin{align*}
\alpha(z) &= \frac{1}{2y} e^{-\gamma_{zn}} \int_{A - z_n}^{z - z_n} e^{-\left(-\frac{u}{2\sqrt{\tau}} + \gamma_{zn}\right)^2} \\
\beta(z) &= \frac{-1}{2y} e^{-\gamma_{zn}} \int_{B - z_n}^{z - z_n} e^{-\left(-\frac{u}{2\sqrt{\tau}} - \gamma_{zn}\right)^2}
\end{align*}
\]

Since

\[
\frac{d}{du} \text{erf} \left( \frac{u}{2\sqrt{\tau}} \pm \gamma\sqrt{\tau} \right) = \frac{2}{\sqrt{\pi}} e^{-\left(\frac{u}{2\sqrt{\tau}} \pm \gamma\sqrt{\tau}\right)^2}
\]

\[
\frac{1}{2\sqrt{\tau}}
\]
we have
\[ \chi(z) = \sqrt{\pi \tau} \frac{e^{\gamma^2 \tau}}{2} \frac{e^{-\gamma z_n}}{\operatorname{erf}(\frac{u}{2 \sqrt{\gamma \tau}}) - \gamma \sqrt{\tau}} \left( \frac{z - z_n}{A - z_n} \right), \]
\[ \beta(z) = -\sqrt{\pi \tau} \frac{e^{\gamma^2 \tau}}{2} \frac{e^{-\gamma z_n}}{\operatorname{erf}(\frac{u}{2 \sqrt{\gamma \tau}}) + \gamma \sqrt{\tau}} \left( \frac{z - z_n}{B - z_n} \right). \]  

Substituting (16) into (11)

\[ A(z) = \sqrt{\pi \tau} \frac{e^{\gamma^2 \tau}}{2} \left\{ \frac{e^{-\gamma(z - z_n)}}{\operatorname{erf}(\frac{u}{2 \sqrt{\gamma \tau}}) - \gamma \sqrt{\tau}} \right\} \left( \frac{z - z_n}{A - z_n} \right), \]
\[ - e^{-\gamma(z - z_n)} \frac{e^{-\gamma z_n}}{\operatorname{erf}(\frac{u}{2 \sqrt{\gamma \tau}}) + \gamma \sqrt{\tau}} \left( \frac{z - z_n}{B - z_n} \right). \]  

This is the general solution of equation (10). However, for later convenience we add to it a form of the general solution of the homogeneous equation. Thus

\[ A(z) = E \sinh \gamma(z - z') \neq p(z), \]

(17)

where \( E \) and \( z' \) are arbitrary constants and \( p(z) \) is the right member in the preceding equation.

Although (17) seems to contain too many arbitrary constants \( (E, z', A, B) \), they are not all independent. In fact, we take \( A = B = z' \).

Refer once again to (8). Its solution can now be written as

\[ A_{ikm}(z) = E_{ikm} \sinh \gamma_{ikm}(z - z') \neq p_{ikm}(z), \]

(18)

where

\[ p_{ikm}(z) = \sqrt{\pi \tau} \frac{e^{\gamma_{ikm}^2 \tau}}{2 \gamma_{ikm}} \sum_{n=1}^{\infty} f_{ikm}^{n} \left\{ e^{-\gamma_{ikm}(z - z_n)} \frac{e^{-\gamma_{ikm} z_n}}{\operatorname{erf}(\frac{u}{2 \sqrt{\gamma_{ikm} \tau}}) - \gamma_{ikm} \sqrt{\tau}} \right\} \left( \frac{z - z_n}{z' - z_n} \right), \]

(19)

where \( E \) and \( z' \) remain as undetermined constants.
Since the flux vanishes at \( z = 0 \) and at \( z = c \) for all \( x, y \), equation (5) implies that:

\[
A_{1km}(0) = 0 \quad , \quad A_{2km}(c) = 0 \quad , \quad \text{all } k, m.
\]

These conditions are satisfied if, in (18) and (19) we set \( z_i = 0 \) for \( i = 1 \) and \( z_i = c \) for \( i = 2 \). Thus, suppressing for the sake of brevity the subscripts \( km \),

\[
A_1(z) = E_1 \sinh \gamma_1 z \neq p_1(z) \quad , \quad 0 \leq z \leq s \quad , \quad (20)
\]
\[
A_2(z) = E_2 \sinh \gamma_2 (z - c) \neq p_2(z) \quad , \quad s \leq z \leq c \quad ; \quad (21)
\]

where

\[
p_1(z) = \frac{\sqrt{\eta \gamma}}{2} \sum_{n=1}^{\infty} \gamma_1 \sum_{n=1}^{\infty} F_n .
\]
\[
\left\{ \begin{array}{l}
\gamma_1 (z - z_n) \left[ \text{erf} \left( \frac{z - z_n}{2 \sqrt{\tau}} \right) - \gamma_1 \sqrt{\tau} \right] - \gamma_1 (z - z_n) \left[ \text{erf} \left( \frac{z - z_n}{2 \sqrt{\tau}} \right) - \gamma_1 \sqrt{\tau} \right] \\
- \gamma_1 (z - z_n) \left[ \text{erf} \left( \frac{z - z_n}{2 \sqrt{\tau}} \right) - \gamma_1 \sqrt{\tau} \right] - \gamma_1 (z - z_n) \left[ \text{erf} \left( \frac{z - z_n}{2 \sqrt{\tau}} \right) - \gamma_1 \sqrt{\tau} \right]
\end{array} \right\} , \quad (22)
\]

and

\[
p_2(z) = \frac{\sqrt{\eta \gamma}}{2} \sum_{n=1}^{\infty} \gamma_2 \sum_{n=1}^{\infty} F_n .
\]
\[
\left\{ \begin{array}{l}
\gamma_2 (z - z_n) \left[ \text{erf} \left( \frac{z - z_n}{2 \sqrt{\tau}} \right) - \gamma_2 \sqrt{\tau} \right] - \gamma_2 (z - z_n) \left[ \text{erf} \left( \frac{z - z_n}{2 \sqrt{\tau}} \right) - \gamma_2 \sqrt{\tau} \right] \\
- \gamma_2 (z - z_n) \left[ \text{erf} \left( \frac{z - z_n}{2 \sqrt{\tau}} \right) - \gamma_2 \sqrt{\tau} \right] - \gamma_2 (z - z_n) \left[ \text{erf} \left( \frac{z - z_n}{2 \sqrt{\tau}} \right) - \gamma_2 \sqrt{\tau} \right]
\end{array} \right\} , \quad (23)
\]

Boundary conditions at the interface \( z = s \) are

\[
\phi_1(x,y,s) = \phi_2(x,y,s) \quad ; \quad D_1 \frac{\partial}{\partial z} \phi_1(x,y,s) = D_2 \frac{\partial}{\partial z} \phi_2(x,y,s) .
\]
(In some cases, it is reasonably accurate to take \( D_1 = D_2 \), which could then be canceled out completely.) Since (5) holds in particular for \( z = s \),

\[
A_{1km}(s) = A_{2km}(s) \quad ; \quad D_1 \frac{\partial}{\partial z} A_{1km}(s) = D_2 \frac{\partial}{\partial z} A_{2km}(s)
\]

Then, from (20) and (21),

\[
\begin{align*}
E_1 \sinh \gamma_1 s - E_2 \sinh \gamma_2 (s - c) &= -p_1(s) \neq p_2(s) \\
E_1 D_1 \frac{\delta}{\delta z} \sinh \gamma_1 s - E_2 D_2 \frac{\delta}{\delta z} \cosh \gamma_2 (s - c) &= -p_1'(s) \neq p_2'(s)
\end{align*}
\]

(24)

The equations derived above should be used according to the following outline.

1. For a choice of \( B_1, B_2 \), find the \( \gamma_{1km} \) and \( \gamma_{2km} \) from (9).
2. Calculate the \( F_{km} \) from (7), referring to Table 1 for the image positions and signs.
3. Calculate the \( p_{1km}(s), p_{2km}(s), p_{1km}'(s) \), and \( p_{2km}'(s) \) from (19).
4. Solve equations (24) simultaneously for \( E_{1km} \) and \( E_{2km} \).
5. Calculate \( A_{1km}(z) \) from (20) and \( A_{2km}(z) \) from (21) for each coordinate \( z \) corresponding to an experimental point.
6. Calculate the flux from (5) and compare with the experimental value of the flux at each experimental point. The magnitude of the error indicates the appropriateness of the choice of \( B_2(B_1 \), of course, is a known quantity).
7. Repeat the procedure, using trial and error or the method of least squares to refine the value of the buckling.
8. If a value for the extrapolation distance is also to be found, the dimensions \( a \) and \( c \) of the pile can be chosen along with \( B_2 \) and all three parameters can be determined by the above procedure.

It should be remarked that the quantities \( p_1' \) and \( p_2' \), which are needed in (24) present no additional problem. Due to fortuitous cancellation, they differ from the expressions for \( p_1 \) and \( p_2 \) of (22) and (23) only in two minor respects. If we eliminate the \( \gamma_1 \) in the leading coefficients

\[
\frac{\sqrt{\gamma_1}}{2 \gamma_1}
\]

and change the sign between the square brackets to plus, \( p_1 \) is converted to \( p_1' \).