COMPACTNESS AND EQUIVALENT NOTIONS

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COMPACTNESS AND EQUIVALENT NOTIONS

THESIS

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CHAPTER I

COMPACTNESS

One of the classic theorems concerning the real numbers states that every open cover of a closed and bounded subset of the real line contains a finite subcover. Compactness is an abstraction of that notion, and there are several ideas concerning it which are equivalent and many which are similar. The purpose of this paper is to synthesize the more important of these ideas. This synthesis is accomplished by demonstrating either situations in which two ordinarily different conditions are equivalent or combinations of two or more properties which will guarantee a third.

The definition of the above mentioned open cover is a proper beginning.

Definition 1.1. Let \((S,U)\) be a topological space. Then we say a collection of sets \(C\) covers \(S\) if \(S\) is a subset of the union of \(C\). By an open cover we shall mean a collection of open sets which covers \(S\). By a subcover of an open cover \(C\) we shall mean a subset of \(C\) which covers \(S\).

Example 1.1. Let \(S\) be the set of real numbers and \(C = \{(a,a+1):a = 0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \pm 2, \ldots\}\). \(C\) is a minimal cover of \(R\); i.e., there is no proper subcover of \(C\).
We now come to a precise definition of compactness in a general setting.

**Definition 1.2.** A topological space \((S,U)\) is compact if each open cover of \(S\) has a finite subcover.

**Example 1.2.** The Reals with the usual topology is not compact, as can be seen from Example 1.1. The Heine-Borel theorem, which states that any closed and bounded subset of the reals is compact, can be found in any book on real analysis. The necessity of boundedness is seen by Example 1.1, and the insufficiency of boundedness is shown by: \(E = (0,1),\) \(C = \{\left(\frac{1}{n}, 1\right) : n \text{ is a positive integer}\};\) \(C\) covers \(E\) but has no finite subcover for \(E.\)

The simplest equivalence to compactness concerns the following property of some collections of sets.

**Definition 1.3.** A collection of sets has the finite intersection property (f.i.p.) if the intersection of each finite subcollection is nonempty.

**Theorem 1.1.** In order that a topological space \((S,U)\) be compact it is necessary and sufficient that each collection of closed sets with f.i.p. have a nonvoid intersection.

**Proof.** Necessity: Let \((S,U)\) be a compact topological space and \(B\) a collection of closed sets with f.i.p. Now suppose \(B\) has an empty intersection. Then \(C,\) the collection
of complements of elements in B, is an open cover of S. Therefore there is a finite subcover $C_1$ of C. Let $B_1 = \{b \in B : S \smallsetminus b \in C_1 \}$ and note that $B_1$ is a finite subset of B. Since $C_1$ covers S, the intersection of the elements of $B_1$ must be empty.

**Sufficiency:** Let $(S, U)$ be a topological space in which each collection of closed sets with f.i.p. has a nonvoid intersection. To show that $(S, U)$ is compact let B be an open cover of S. Now suppose no finite subset of B covers S. Let C be the collection of all complements of elements of B. Then C is a collection of closed sets with f.i.p. and therefore has a nonvoid intersection; i.e., there is in S an element x which is contained in each element of C. Therefore, by definition of C, x is not in any element of B. Therefore B does not cover S.

If we weaken our compactness condition slightly we get a property involving countable open covers (a set is countable if it can be mapped 1 to 1 into the integers).

**Definition 1.4.** A space $(S, U)$ is countably compact if each countable open cover has a finite subcover.

**Theorem 1.2.** If a topological space $(S, U)$ is compact it is countably compact. The proof consists merely of noting that a countable cover is a cover and therefore has a finite subcover.
Example 1.3. The converse of the theorem is not true, as can be seen by the example from J. L. Kelley (2, p. 163).

The condition which completely characterizes those countably compact spaces which are also compact is the following:

Definition 1.5. A space $(S, U)$ is Lindelof if each open cover has a countable subcover. Clearly any compact space is Lindelof, and also we have the following theorem:

Theorem 1.3. A space $(S, U)$ is compact iff it is countably compact and Lindelof.

Proof. The necessity of the two properties for compactness has already been noted. For the sufficiency of the two conditions let $C$ be an open cover of a countably compact Lindelof space $(S, U)$. Since $(S, U)$ is Lindelof there is a countable subcover $B$ of $C$ and since $(S, U)$ is countably compact there is a finite subcover $A$ of $B$. $A$ then is a finite subcover of $C$ and therefore $(S, U)$ is compact.

That neither of the properties is sufficient in itself to guarantee compactness is seen by Examples 1.3 and 1.4.

Example 1.4. Let $S$ be the set of rational numbers and $U$ be the relative usual topology. Let $C$ be any open cover and for each $x$ in $S$ let $V_x$ be some element of $C$ which contains $x$. Then $\{V_x : x \in S\}$ is a countable subcover of $C$. That the collection of all subsets of $S$ of the form $(a, a+1) \cap S$,
where a is a real number, has no finite subcover shows that 
(S,U) is noncompact.

Corresponding to Theorem 1.1 we have:

**Theorem 1.4.** (S,U) is countably compact iff each countable collection of closed sets with f.i.p. has a nonvoid intersection.

**Proof.** Only if: Let (S,U) be countably compact and B be a countable collection of closed sets with f.i.p. Let C be the collection of complements of the elements of B. Now suppose the intersection of B is empty. Then C is a countable open cover and therefore has a finite subcover C₁. Denote by B₁ the collection of the complements of the elements of C and note that B₁ is a subset of B. But ∩ B₁ = S - ∪ C₁ = ∅.

If: Now suppose each countable collection of closed sets with f.i.p. has a nonvoid intersection and let C be a countable open cover of S. Let B be the set of complements of C. Then the intersection of B is empty and therefore there exists a finite subcollection B₁ of B whose intersection is empty. The complements of the elements of B₁ form a finite subcover of C and therefore (S,U) is countably compact.

To arrive at several conditions which are equivalent to countable compactness we need to introduce some more terminology.
Definition 1.6. If \((S, U)\) is a topological space and \(A\) is a subset of \(S\), then a point \(p\) of \(S\) is an accumulation point of \(A\) if for each \(V\) in \(U\), such that \(p\) is in \(V\), the intersection of \(A\) and \(V\) is infinite.

Definition 1.7. Let \((x_n)\) be a sequence in \(S\) where \((S, U)\) is a topological space. A point \(p\) in \(S\) is a cluster point of \((x_n)\) if whenever \(V\) is in \(U\), \(p\) is in \(V\), and \(N\) is a positive integer; then there is an integer \(n > N\) such that \(x_n \in V\).

That the two previous ideas are equivalent is shown by combining the three following lemmas into theorem 1.5.

Lemma 1.1. Countable compactness implies each sequence has a cluster point.

Proof. Suppose \((S, U)\) is countably compact and \((x_n)\) is a sequence in \(S\). It may be assumed that the points \(x_n\) are all distinct. Let \(A\) be the range of \((x_n)\). If \((x_n)\) has no cluster point, then for each \(x\) in \(S\) there is a \(V_x\) such that \(V_x \cap A\) is finite. Now for each finite \(F \subseteq A\) let \(V_F = \bigcup \{V_x : V_x \cap A = F\}\). The collection of all \(V_F\) such that \(F\) is a finite subset of \(A\) is a countable cover for \(S\), which has no finite subcover.

Lemma 1.2. "Each sequence has a cluster point" implies each infinite set has an accumulation point.
Proof. Let \((S, U)\) be a topological space and \(A\) be an infinite subset of \(S\). Let \((x_n)\) be a sequence of distinct points of \(A\). Then \((x_n)\) has a cluster point \(p\). This cluster point is also an accumulation point of \(A\) because each open set containing \(p\) contains infinitely many terms of \((x_n)\) and therefore infinitely many points of \(A\).

Lemma 1.3. If a topological space \((S, U)\) is such that each infinite set has an accumulation point, then \((S, U)\) is countably compact.

Proof. Let \(H = \{A_n : n \in \mathbb{N}^+\}\) be a countable open cover for \(S\). Suppose \(H\) has no finite subcover. Let \(B_n\) be the union of all elements \(A_i\) of \(H\) such that \(i \leq n\) for each positive integer \(n\). Note that the collection of \(B_n\) covers \(S\), has no finite subcover, and that \(S - B_n\) is infinite for each \(n\). Now choose \(a_n \in S - B_n\) such that \(a_n = a_m\) implies \(n = m\). This can be done since \(S - B_n\) is infinite for each \(n\). Let \(A = \{a_n : n \text{ is a positive integer}\}\). Now each \(B_n\) contains only those elements \(a_i\) of \(A\) for which \(i \leq n\); i.e., \(B_n \cap A\) is finite for each \(n\). But \(A\) is infinite and therefore has an accumulation point \(p\) and \(\{B_n : n \in \mathbb{N}^+\}\) covers \(S\); so \(p \in B_n\) for some \(n\) and therefore \(B_n \cap A\) is infinite.
Theorem I.5. T.A.E. (These are equivalent) in any topological space \((S, U)\):

1. The space is countably compact.
2. Each sequence in \(S\) has a cluster point in \(S\).
3. Each infinite subset of \(S\) has an accumulation point in \(S\).

Proof. Lemmas I.1, I.2, and I.3.

An interesting result of countable compactness is that each continuous function into the reals is bounded. Compact spaces, of course, have this property, but it is convenient that it also results from either of the conditions of Theorem I.5.

Lemma I.4. Each continuous function from a countably compact space to the real numbers is bounded.

Proof. Suppose \((S, U)\) is countably compact and let \(f\) be a continuous function from \(S\) into \(R\). Using the usual topology for \(R\), consider the following countable cover of \(R\) by open sets: \(H = \{b_n : n \in \mathbb{I}\}\) where \(b_n = \left(\frac{n-1}{2}, n+1\right]\), for each \(n\). Since \(f\) is continuous, \(f^{-1}(b_i)\) is an open set for each integer \(i\). Denote by \(C\) the collection of all such subsets of \(S\). \(C\) then is an open countable cover and therefore has a finite subcover. Therefore the image of \(f\) is contained in the union of a finite subset of \(H\) and consequently is bounded.
The term accumulation point is sometimes used to describe a weaker situation. In some works an accumulation point of a set $A$ is a point $p$ such that each open set containing $p$ contains at least one other point in common with $A$, rather than infinitely many points. The infinite situation may then be referred to as a $1$-accumulation point (1) or an $\omega$-accumulation point (2); there are also other notations. The weaker condition will herein be called a limit point.

**Definition 1.8.** Let $A$ be a subset of a topological space $(S,U)$ and $p$ a point of $S$. Then $p$ is a limit point of $A$ if for each $V$ in $U$ such that $p$ is in $V$, the intersection of $V$ and $A$ contains at least one point other than $p$. The point $p$ may or may not be in $A$.

**Definition 1.9.** Let $(S,U)$ be a topological space. Then $(S,U)$ is $L$-compact if each infinite subset of $S$ has a limit point in $S$.

Countable compactness is obviously at least as strong a condition as $L$-compactness because any accumulation point is necessarily a limit point. That the limit point condition is strictly weaker can be seen from Example 1.5. The obvious question is what conditions will guarantee equivalence of the two properties. However it is convenient to insert another property here which is more restrictive than $L$-compactness and yet not as strong as countable compactness.
Theorem 1.6. \( S \) is countably compact \( \Rightarrow \) each infinite open cover of \( S \) has a proper subcover \( \Rightarrow \) \( S \) is \( L \)-compact.

**Proof.**

A) Suppose \((S,U)\) is countably compact. Let \( C \) be an infinite open cover of \( S \). Let \( B \) be a countable subset of \( C \). Let \( D \) be the union of \( C - B \). Then \( B \cup \{D\} \) covers \( S \) and is countable. Therefore there exists a finite subcover which will yield a proper subcover of \( C \) for \( S \).

B) Suppose each countable open cover of \( S \) contains a proper subcover. Let \( A \) be an infinite set and suppose \( A \) has no limit point. Then \( A \) is closed and each element \( x \) of \( A \) is contained in an open set \( V_x \) such that \( V_x \cap A = \{x\} \). Now the complement of \( A \) together with \( \{V_x : x \in A\} \) is an infinite open cover with no proper subcover.

That these conditions are not equivalent is seen by the next example.

**Example 1.5.** Let \( S \) be the positive integers. \( B_i = \{2i-1, 2i\} \) for each positive integer \( i \). Let \( B \) be the collection of all such sets and define the topology on \( S \) by letting \( B \) be a base for \( U \) and include \( \emptyset \) as an open set. Then each infinite set has a limit point in \( S \) (in fact, each set with more than one point has a limit point), but \( B \) itself is an infinite open cover with no proper subcover. Therefore \( L \)-compact does not imply each infinite open cover has a proper subcover.
Now let $S = R$ and $U$ consist of $\phi$, $S$, and sets of the form $V = \{x \in R : x < r\}$ for some $r \in R$. Clearly every open cover consisting of more than two sets has a proper subcover though the only open covers with finite subcovers are those containing $S$ itself.

In Theorem 1.6 all the properties become equivalent if the spaces considered are restricted to satisfy the separation axiom $t_1$. (A space is $t_1$ if for each pair of distinct points $x$ and $y$ there exist open sets $V$ and $W$ such that $x \in V$, $y \in W$, $x \notin W$ and $y \notin V$.) This will be shown by linking the weakest, $L$-compactness, to the strongest property, countable compactness, by means of Theorem I.5.

**Theorem I.7.** A $t_1$ $L$-compact space is countably compact.

**Proof.** It is sufficient to show that each infinite set has an accumulation point (Theorem I.5). Let $A$ be an infinite subset of $S$. Then $A$ has a limit point $p$. If $p$ is not an accumulation point then there is an open set $V$ containing $p$ and only a finite subset of $A$. For each point of this finite subset of $A$ (excepting $p$ if $p \in A$), there is an open set about $p$ which does not contain that point because $S$ is $t_1$. The intersection of that finite collection of subsets is an open set about $p$ which contains no other elements of $A$. This contradicts the hypothesis that $A$ has a limit point and therefore completes the proof.
The natural question is: Does Theorem 1.7 require $t_1$ or is it sufficient to assume that $(S,U)$ is a $t_0$ space? (A space $S$ is $t_0$ if given any two points in $S$ there is an open set containing one of those points but not the other.)

The space in Example 1.5 has the property that each infinite open cover has a proper subcover and it is also $t_0$, but not countably compact; so Theorem 1.7 does require $t_1$. However, it might be possible for $t_0$ and $L$-compactness to guarantee that each infinite open cover has a proper subcover. The following example resolves that question in a similar manner.

**Example 1.6.** Let $S$ be the open unit disc excluding $(0,0)$ in $\mathbb{R}^2$. Let the collection of open line segments from $(0,0)$ to any point in $S$ together with $\emptyset$ be a base for the topology $U$ on $S$. Then $(S,U)$ is $t_0$ and $L$-compact. (Take any point $x$ and select for a limit point of any set containing $x$ a point $y$ which is co-linear with $x$ and further from $(0,0)$.) However, the set of all base elements which are one unit in length is an infinite open cover with no proper subcover.

Sequential compactness is the first idea to be discussed which is "independent" of compactness in the sense that in both cases there exist spaces in which one property is satisfied but the other is not. Sequential compactness is obviously defined in terms of sequences and requires a definition of subsequence. A subsequence of a sequence $(x_n)$
must be more than just a sequence whose range is a subset of the range of \((x_n)\) for if not, it becomes insignificant. Recalling then that a sequence is a function whose domain is the positive integers we have:

**Definition 1.10.** Let \((x_n)\) be a sequence in \(S\). A subsequence of \((x_n)\) is a sequence \((x_{u(i)})\) where \(u\) is a function from \(I^+\) (the positive integers) into \(I^+\) which satisfies the following condition: If \(k \in I^+\) then there is an \(n \in I\) such that \(u(i) > k\) for \(i > n\). Another convenient notation replaces \(u(i)\) with \(n_i\); then \((x_{u(i)}) = (x_{n_i})\).

**Definition 1.11.** A space is sequentially compact if each sequence in the space has a convergent subsequence. This assumes the usual definition of convergence in a topological space; i.e., a sequence \((x_n)\) converges to \(p\) \((x_n \to p)\) if for each open set \(V\) which contains \(p\) there is an integer \(N\) such that if \(n > N\) then \(x_n \in V\). It should be mentioned that unless the space is \(T_2\) a sequence may converge to more than one point.

**Theorem 1.8.** If a space \((S,U)\) is sequentially compact it is countably compact.

**Proof.** Because of Theorem 1.5 it is sufficient to show that each sequence has a cluster point. Let \((x_n)\) be a
sequence in S and let \( (x_u(i)) \) be a convergent subsequence and \( x_{u(i)} \rightarrow p \). Let \( V \in U \) be such that \( p \in V \) and let \( N \) be a positive integer. Now since \( x_{u(i)} \rightarrow p \) there is an integer \( k_1 \) such that if \( i \geq k_1 \) then \( x_{u(i)} \in V \). Since \( (x_{u(i)}) \) is a subsequence of \( (x_n) \) there exists a \( k_2 \) such that if \( i \geq k_2 \) \( u(i) > N \). Let \( k > \max (k_1, k_2) \). Then \( u(k) > N \) and \( x_{u(k)} \in V \). Therefore \( p \) is a cluster point of \( (x_n) \).

The above proof demonstrates that if a subsequence of \( (x_n) \) converges to \( p \), then \( p \) is a cluster point of \( (x_n) \). The converse need not be true.

Example 1.7. Kelley (2, p. 77) gives an example of a space in which there exists a sequence with a cluster point to which no subsequence converges.

In order to find a condition which, together with countable compactness, will imply sequential compactness, one of the countability axioms is introduced.

Definition 1.12. The neighborhood system of a point is the collection of all neighborhoods of the point. (A neighborhood of a point is any set which contains an open set containing the point.)

Definition 1.13. A local base for a point \( x \) (base for the neighborhood system of \( x \)) is any collection of neighborhoods such that each neighborhood of \( x \) contains a member of this collection.
Definition 1.14. A space is first countable if each point has a countable local base.

Theorem 1.9. If \((S,U)\) is countably compact and first countable, then it is sequentially compact.

Proof. Let \((x_n)\) be a sequence in \(S\). Then \((x_n)\) has a cluster point \(p \in S\) by Theorem 1.5. Let \((B_n)\) be a countable local base for \(p\). Define a new local base by \(A_n = \bigcap_{i=1}^{n} B_i\). Note that \(A_n \subseteq A_{n-1}\) for each \(n\). Now for each positive integer \(i\) define \(n_i > n_{i-1}\) and \(x_{n_i} \in A_i\). Clearly this can be done because \(p\) is a cluster point of \((x_n)\). Note that \(x_{n_i}\) is a subsequence of \((x_n)\). All that is left is to show \(x_{n_i}\) converges to \(p\). Let \(V\) be an open set containing \(p\); then there is an \(A_k\) such that \(A_k \subseteq V\). Therefore for each \(i > k\), \(x_{n_i} \in A_k\) and therefore \(x_{n_i} \in V\) so \(x_{n_i} \to p\).

Sequential compactness is then stronger than countable compactness. Another condition which together with countable compactness will guarantee sequential compactness is stated in Theorem 1.10, which requires the following definition:

Definition 1.15. A sidepoint of a sequence is a cluster point to which no subsequence converges.
Theorem I.10. If a countably compact space \((S,U)\) is such that each sequence has a subsequence without sidepoints, the \((S,U)\) is sequentially compact.

Proof. First it should be noted that if \((z_p)\) is a subsequence of \((y_k)\) and \((y_k)\) is a subsequence of \((x_n)\), then \((z_p)\) is a subsequence of \((x_n)\). Now let \((x_n)\) be a sequence. Let \((y_k)\) be a subsequence without sidepoints. By Theorem I.5 \((y_k)\) has a cluster point \(p\); \(p\) is not a sidepoint of \((y_k)\), and therefore there is a subsequence \(\{y_{k_i}\}\) of \((y_k)\) which converges to \(p\). \(\{y_{k_i}\}\) is also a subsequence of \((x_n)\) and therefore \((S,U)\) is sequentially compact.

In some situations the entire space \(S\) may not be compact, but the properties of compactness may prevail in the "immediate vicinity" of any point. A precise definition of this idea is worthwhile.

Definition I.16. A space \((S,U)\) is locally compact if each point of \(S\) has a compact neighborhood; i.e., a neighborhood which together with its relative topology is compact. It is easily verified that a subset is compact if each open cover of that set has a finite subcover.

Theorem I.11. A space \((S,U)\) is compact \(\implies\) each point of \(S\) is contained in an open set whose closure is compact.
(S,U) is locally compact. The first is clear since S is a neighborhood of each of its points. For the second if each point of S is contained in an open set whose closure is compact, then the closure of that open set is a compact neighborhood. Hence (S,U) is locally compact.

Example 1.8. The Reals with the usual topology is locally compact but not compact.

That the intermediate condition in Theorem I.11 is not equivalent to local compactness even in a $t_0$ space is seen from the next example.

Example 1.9. Let $S = \{1,2,3,\ldots\}$ and the topology for S consist of $\emptyset$, S and $A_i = \{1,2,3,\ldots,i\}$ for each $i$ in S. The space is $t_0$ (if $i < j$ then $i \in A_i$ and $j \notin A_i$) and locally compact (if $i \in S$ then $A_i$ is a compact neighborhood); however, given any point $n$ in S, the closure of any set containing $n$ must include $B = \{n, n+1, n+2, \ldots\}$ and $\{A_n, A_{n+1}, A_{n+2}, \ldots\}$ is an open cover of B with no finite subcover. Therefore there can be no open set containing $n$ whose closure is compact.

In order to establish a condition guaranteeing this property whenever a space is locally compact we need the following result.

Lemma I.5. Each compact subset of a Hausdorff space is closed.
Proof. Let \((S, U)\) be a \(t_2\) space and \(H\) a compact subset of \(S\). Suppose \(p\) is a limit point of \(H\) and \(p \notin H\). Then for each \(x \in H\) there are open sets \(V_{px}\) and \(V_x\) such that \(x \in V_x\) and \(p \in V_{px}\) and \(V_x \cap V_{px} = \emptyset\). Now \(\{V_x : x \in H\}\) is an open cover of \(H\) and has a finite subcover \(\{V_{x_1}, V_{x_2}, \ldots, V_{x_n}\}\) and \(\bigcap_{i=1}^{n} V_{px_i} = V\) is disjoint from each set in this cover. So \(p \in V\) and \(V \cap H = \emptyset\); therefore \(p\) is not a limit point of \(H\) and hence \(H\) is closed.

Theorem 1.12. A locally compact \(t_2\) space has the property that each point is contained in an open set whose closure is compact.

Proof. Let \(x\) be in \(S\) and \(N\) be a compact neighborhood of \(x\). Then there is an open set \(V\) such that \(x \in V \subseteq N\) and \(N = \overline{N}\) since \(S\) is \(t_2\). Therefore \(\overline{V} \subseteq \overline{N} = N\) and \(V\) is a closed subset of the compact space \(N\) and hence compact. \(V\) is then the required open set. (Verification of the statement that a closed subset \(H\) of a compact space \(S\) is compact consists simply of noting that \(S-H\) is open. Hence any open cover of \(H\) together with \(S-H\) is an open cover of \(S\) and therefore has a finite subcover.)

Thus far the only "compactness property" discussed is that of subcollections of an open cover. The idea may also be approached by taking collections of sets, each of which
is a subset of some element in an open cover. Such a collection if required to have certain properties, yields some interesting results—for example, Theorem I.13.

**Definition I.17.** An open refinement of a cover $C$ of $S$ is a collection of open sets $B$ which covers $S$ and has the property that for each $b$ in $B$ there is a $c$ in $C$ such that $b$ is contained in $c$.

**Definition I.18.** A star-finite collection of sets $C$ is one in which each member of $C$ intersects only a finite subset of $C$.

**Definition I.19.** A space $(S,U)$ is hypocompact if each open cover has a star-finite open refinement.

Compact spaces are necessarily hypocompact since a finite subcover is a star-finite refinement, and the Reals illustrate that the converse need not be true. A slightly weaker, but very important, property is that of paracompactness.

**Definition I.20.** A collection $C$ of sets is locally finite if each $x \in S$ is contained in a neighborhood which intersects at most a finite subcollection of $C$. A space is paracompact if each open cover has a locally finite open refinement.
Lemma 1.6. A hypocompact space \((S,U)\) is paracompact.

Proof. Let \(C\) be an open cover and \(B\) a star-finite open refinement. Let \(x \in S\); then there is a \(b\) in \(B\) such that \(x \in b\). Since \(B\) is star-finite \(b\) intersects only a finite subset of \(B\), and so \(b\) is the required neighborhood of \(x\) which intersects only finitely many members of \(B\). Hence \(B\) is a locally finite refinement of \(C\) and therefore \((S,U)\) is paracompact.

Definition 1.21. A collection of subsets of \(S\) is point-finite if each point of \(S\) is contained in only a finite subcollection. A space is metacompact if each open cover has a point-finite open refinement.

Lemma 1.7. If a space is paracompact it is metacompact.

Proof. It suffices to show that a locally finite collection of sets is a point-finite collection. Let \(x \in S\) and \(C\) be a locally finite collection of sets. Let \(N\) be a neighborhood of \(x\) which intersects only a finite number of elements of \(C\). Then clearly no more than that finite number of sets of \(C\) can contain \(x\).

Local compactness is not "strong enough" to guarantee that a countably compact space is compact. The Lindelof property does this trivially, and there is another idea which is quite similar. A space is said to be \(\sigma\)-compact if it is the union of countably many compact subsets. Since in a
\(\sigma\)-compact space any open cover contains countably many subcollections which cover those compact sets, one can easily choose a countable subcover. Hence any countably compact \(\sigma\)-compact space is compact. Moreover, if a countably compact space is second countable (i.e., has a countable base) it is first countable, hence sequentially compact; it is also Lindelof, therefore compact. All of these properties appear almost to have been designed specifically to assure that a countably compact space will be compact. One interesting connection between the two properties is the idea of a particular kind of open refinement existing for any open cover. The following theorem is, of course, true if either hypocompact or paracompact is substituted for metacompact. On page 171 of General Topology Kelley (2) states a similar proposition but assumes also that the space be \(t_1\); this is, however, apparently unnecessary.

**Theorem I.13.** If \((S, U)\) is countably compact and metacompact, then it is compact.

The proof will consist of selecting a point-finite open refinement of any open cover and choosing a refinement of that cover which has a minimal subcover. By Theorem I.6 this minimal subcover must be finite, since \((S, U)\) is countably compact.

**Proof.** Let \(B\) be an open cover and \(C\) be a point-finite open refinement of \(B\). For each \(x \in S\) let \((C_x)\) be the finite
subcollection of $C$ each of which contains $x$ and $B_x$ the intersection of that finite family. Clearly then $D = \{B_x : x \in S\}$ is an open cover for $S$. Note that if $B_x$ and $B_y$ are in $D$ and $x \in B_y$ then $B_x \subseteq B_y$, for if $x \in B_y$, then $x$ is in each member of $(C_y)$ and hence $(C_y) \subseteq (C_x)$. Therefore $B_x = \bigcap (C_x) \subseteq \bigcap (C_y) = B_y$. Also, if $B_x \subseteq B_y$ then $(C_y) \subseteq (C_x)$.

Now, since $(C_x)$ is finite, there are only finitely many subfamilies of $(C_x)$. Therefore, there must be only finitely many elements of $D$ which contain $B_x$ since each $B_y$ in $D$ which contains $B_x$ is the intersection of a subfamily of $C_x$.

Consequently there must be a maximal $B_x'$, which contains $B_x$ where $x' \in S$ and $B_x', \in D$. Let $H$ be the set of all such maximal elements. To show that $H$ is a minimal subcover of the cover $D$ of $S$ it is necessary to show that each $B_x'$ of $H$ contains a point of $S$ not contained in any other element of $H$. The required point for the set $B_x'$ is $x'$. For if $x' \in B_y'$, then by the above argument $B_x', \subseteq B_y'$, and since $B_x'$ is maximal $B_x' = B_y'$. Therefore $H$ is a minimal cover of $S$ and hence by Theorem 1.6 it is finite. Each member of $H$ is the intersection of members of $C$, so there is for each $B_x'$ in $H$ a $C_x'$ in $C$ such that $C_x' \supseteq B_x'$. Since $C$ is a refinement of $B$ there is a $V_x'$ in $B$ such that $V_x', \supseteq C_x', \supseteq B_x'$. 
so that for each member of $H$ there is a corresponding member of $B$ containing it. \( \{v_x', \mathcal{B}_x', x \in H\} \) is a finite subcover of $B$.

Consequently, $(S, U)$ is compact.

There are two approaches to the theory of convergence which are currently in use. The idea of a net is clearly a direct generalization of sequences, while a filter is purely a set-theoretic notion. In order to show the close correlation of filters and nets and illustrate their bearing on compactness, some preliminary ground work must be done.

**Definition II.1.** A relation on a set $S$ is any subset of $S \times S$.

**Definition II.2.** A partial ordering of a set $S$ is a relation $R$ on $S$ such that if $a \in S$ then $(a,a) \in R$, and if $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R$ (i.e., $R$ is respectively reflexive and transitive). If $R$ is a partial ordering of $S$, then $S$ is said to be partially ordered by $R$, and $S$ is called a partially ordered set or poset. If $(a,b) \in R$, then $a \geq b$ is the usual notation.

**Definition II.3.** A directed set is an ordered pair $(D, \geq)$ such that $D$ is a set and $\geq$ is a partial ordering of $D$; if $a,b \in D$, then there exists an element $c$ of $D$ such that $c \geq a$ and $c \geq b$. 

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Example II.1. The reals with the usual ordering is a directed set. It is also linearly ordered; i.e., if \( a, b \in \mathbb{R} \) then either \( a \geq b \) or \( b \geq a \).

Example II.2. Let \( S \) be a set. Then any collection of subsets of \( S \) which contains \( \phi \) is a directed set with the order \( A \geq B \) if \( A \subseteq B \). The collection has a maximal element \( \phi \); the ordering is called inclusion and is very useful.

Example II.3. An example of a collection \( D \) which is partially ordered can be constructed by considering all the nonempty subsets of a set \( S \) with the inclusion order. If \( S \) contains more than one element then \((D, \geq)\) is not directed, because \( \{a\} \in D \) and \( \{b\} \in D \) and their intersection is \( \phi \) which is not in \( D \).

Definition II.4. Let \( S \) be a set. A net in \( S \) is a function which has a directed set \( D \) for its domain, and which takes its values in \( S \). The notation will be \((x_\delta : \delta \in D)\) or \((x_\delta : D)\) or if \( D \) is clear, just \((x_\delta)\), where \( x_\delta \in S \) for each \( \delta \in D \).

Example II.4. Any sequence is a net. Also, \((x_\delta : \delta \in \mathbb{R})\) with the usual ordering of \( \mathbb{R} \) and \( x_\delta = \delta \) is a net in \( \mathbb{R} \).

Example II.5. Nets, of course, are not necessarily sequences; in fact, even a countable net (i.e., one in which the domain is countable) need not be a sequence. This is still true even if the most general definition of sequence is used (i.e., a sequence is a function whose domain can be
mapped one-to-one by an order-preserving map into the positive integers). The net \((x_\delta : I)\) where \(I\) is the integers and \(x_\delta = \delta\) is countable, but not a sequence.

**Definition II.5.** A net \((x_\delta : D)\) in \(S\) is eventually in a subset \(A\) of \(S\) if there exists a \(\delta_0 \in D\) such that if \(\delta \geq \delta_0\), then \(x_\delta \in A\).

**Definition II.6.** A net \((x_\delta : D)\) in \(S\) is frequently in a subset \(A\) of \(S\) if for each \(\delta_0 \in D\) there exists a \(\delta \geq \delta_0\) such that \(x_\delta \in A\).

**Definition II.7.** A net \((x_\delta)\) in \(S\) converges to a point \(p \in S\), written \(x_\delta \to p\), if \((x_\delta)\) is eventually in each neighborhood of \(p\).

**Definition II.8.** A point \(p\) is a cluster point of a net \((x_\delta)\) if \((x_\delta)\) is frequently in each neighborhood of \(p\).

Nets can be constructed in any set; however, the ideas of convergence and cluster points require a topology on the set. There is nothing in Definition II.7 which requires that \(p\) be unique. As will later be seen the uniqueness of convergence points guarantees and requires that the topology in question be Hausdorff.

**Example II.6.** Let \(x_n\) be 1 if \(n\) is even, 2 if \(n\) is odd. Then \((x_n)\) is frequently in \((0,2)\) and \((1,3)\) and eventually in \([1,2]\) and \((1,2)\). One and two are cluster points of \((x_n)\) whatever the topology, although the sequence does not converge.
to either point unless the topology allows each open set containing one point to also contain the other.

An unfortunate property of sequences is that it is possible for a sequence to have a cluster point \( p \) and no subsequence converging to \( p \). This is not the case with nets, as will be seen in Theorem II.1. In order to arrive at Theorem II.1 it is necessary to define a subnet of a net \((x_\delta)\).

**Definition II.9.** Let \( D \) and \( D' \) be directed sets and \( u \) a function from \( D' \) to \( D \). Then \( u \) is finalizing if for each \( \delta \) in \( D \) there exists an \( \alpha_0 \in D' \) such that if \( \alpha > \alpha_0 \) then \( u(\alpha) > \delta \).

**Example II.7.** Consider the directed sets \( \mathbb{R}^+ \) and \( \mathbb{I}^+ \) with the usual order and the functions \( f: \mathbb{I}^+ \to \mathbb{R}^+: n \to n \) and \( g: \mathbb{I}^+ \to \mathbb{R}^+: n \to \frac{1}{n} \). The function \( f \) is finalizing and \( g \) is not.

**Definition II.10.** Suppose \((x_\delta:D)\) is a net. A subnet of \((x_\delta)\) is a net \((x_{u(\alpha)}: \alpha \in D')\) where \( u:D' \to D \) is finalizing.

**Example II.7.** Let \((x_n)\) be a sequence. Then \((x_{u(\alpha)}, \mathbb{R}^+)\) where \( u(\alpha) = n \) when \( n-1 < \alpha \leq n \) is a subnet of \((x_n)\). It is worth noting, however, that \((x_{u(\alpha)})\) is not a subsequence of \((x_n)\).

**Lemma II.1.** Let \((x_\delta:D)\) be a net in \( S \) and \( H \) be a subset of \( S \). Then if \((x_\delta)\) is eventually in \( H \), each subnet of \((x_\delta)\) is eventually in \( H \).
Proof. Let \((x_{u(a)}:D')\) be a subnet of \((x_\delta:D)\). Since \((x_\delta)\) is eventually in \(D\) there exists a \(\delta_0 \in D\) such that if \(\delta \geq \delta_0\), \(x_\delta \in \overline{H}\). Since \(u\) is finalizing, there exists an \(\alpha_0 \in D'\) such that if \(\alpha \geq \alpha_0\), then \(u(\alpha) \geq \delta_0\); therefore \(x_{u(a)}(\alpha) \in \overline{H}\). Hence \((x_{u(a)}:D')\) is eventually in \(\overline{H}\).

Corollary. If \(x_\delta \to p\), then each subnet of \((x_\delta)\) converges to \(p\).

Lemma II.2. If \((x_\delta)\) is eventually in \(\overline{H} \subseteq S\), then each cluster point of \((x_\delta)\) is in \(\overline{H}\).

Proof. Let \(p \in S-\overline{H}\). Then there exists an open set \(V\) such that \(p \in V\) and \(V \cap \overline{H} = \emptyset\). Since \((x_\delta)\) is eventually in \(\overline{H}\) it is not frequently in \(V\); hence \(p\) is not a cluster point. So if \(q\) is a cluster point of \((x_\delta)\), then \(q \notin S-\overline{H}\), i.e., \(q \in \overline{H}\).

Theorem II.1. A point \(p\) is a cluster point of a net \((x_\delta)\) iff there is a subnet of \((x_\delta)\) which converges to \(p\).

Proof. If \(\delta_0\) be in \(D\) be a neighborhood of \(p\) and \(\delta_0\) be in \(D\). It is necessary to find a \(\delta \geq \delta_0\) for which \(x_\delta \in N\). Since \(u\) is finalizing, there exists an \(\alpha_0 \in D'\) such that if \(\alpha \geq \alpha_0\) then \(u(\alpha) \geq \delta_0\). Also there is an \(\alpha_1 \in D'\) such that if \(\alpha \geq \alpha_1\) then \(x_{u(a)}(\alpha) \in N\), since \(x_{u(a)}(\alpha) \to p\).

Because \(D'\) is directed, there is an \(\alpha_2 \in D'\) such that \(\alpha_2 \geq \alpha_1\)
and $\alpha_2 \geq \alpha_0$. Therefore $u(\alpha_2) = \delta_2 \in D$ and $\delta_2 \geq \delta_0$ and $x_\delta \in N$. Hence $p$ is a cluster point of $(x_\delta)$. 

Only if) Suppose $p$ is a cluster point of $(x_\delta; D)$. Let $H$ be the collection of all neighborhoods of $p$. Then $H$ is a directed set partially ordered by inclusion. At first glance this set might appear to be sufficient as the domain of a subnet converging to $p$ by defining, for each $N \in H$, $u(N) = \delta$ where $x_\delta \in N$. This function, however, fails to be finalizing and so we must have a slightly more complicated directed set. Let $D' = \{(\delta, N) \mid N \in H$ and $x_\delta \in N\}$. $D'$ is a directed set with the following definition: $(\delta_1, N_1) \geq (\delta_2, N_2)$ if $\delta_1 \geq \delta_2$ in $D$ and $N_1 \subseteq N_2$. Now define $u: D' \rightarrow D:(\delta, N) \mapsto \delta$. To show that $u$ is finalizing, let $\delta_0 \in D$ and $N$ be a neighborhood of $p$. Then there is a $\delta_1 \geq \delta_0$ such that $x_{\delta_1} \in N_1$ so $(\delta_1, N_1) \in D'$. Let $(\delta, N) \geq (\delta_1, N_1)$; then $u(\delta, N) = \delta \geq u(\delta_1, N_1) = \delta_1 \geq \delta_0$. Hence $D$ is finalizing. Now $(x_{u(\delta, N)})$ converges to $p$. Let $N_1$ be a neighborhood of $p$. Let $\delta_1 \in D$ be such that $x_{\delta_1} \in N_1$. (There exists such a $\delta_1$ because $p$ is a cluster point of $(x_\delta)$.) Then $(\delta_1, N_1) \in D'$. Let $(\delta, N) \geq (\delta_1, N_1)$. Then $\delta \geq \delta_1$ and $N \subseteq N_1$. Therefore $x_{u(\delta, N)} = x_\delta \in N \subseteq N_1$. Hence $(x_{u(\delta, N)})$ is eventually in $N_1$; i.e., $(x_{u(\delta, N)})$ converges to $p$.

The last few pages have prepared the way for the association of compactness with certain properties of nets. The
connection will be made via Theorem I.1, which is as follows: Compactness is equivalent to the statement that each collection of closed sets with f.i.p. has a nonvoid intersection. The following definition will yield the collection of closed sets and prepare for Theorem II.2.

**Definition II.10.** A tail of a net \((x_\delta)\) is any set of the form \(T_{\delta_0} = \{x_\delta : \delta \geq \delta_0\}\).

**Lemma II.3.** The collection of tails of a net has the finite intersection property.

**Proof.** Let \(T_{\delta_1}\) and \(T_{\delta_2}\) be tails of a net \((x_\delta: D)\). Then there exists a \(\delta_3 \in D\) such that \(\delta_3 \geq \delta_1\) and \(\delta_3 \geq \delta_2\). Hence \(x_{\delta_3} \in T_{\delta_1}\) and \(x_{\delta_3} \in T_{\delta_2}\); so \(x_{\delta_3} \in T_{\delta_1} \cap T_{\delta_2} \neq \emptyset\). Induction will extend this to any finite collection of tails.

**Lemma II.4.** The set of cluster points of a net \((x_\delta)\) is the intersection of the collection of tails of \((x_\delta)\).

**Proof.** A) Let \(p\) be a cluster point of \((x_\delta)\). Suppose \(p \notin \bigcap_{\delta \in D} T_\delta\); then there is a \(\delta_0 \in D\) such that \(p \notin T_{\delta_0}\). Therefore \(p \in S-T_{\delta_0}\) which is an open set; hence \((x_\delta)\) is frequently in \(S-T_{\delta_0}\). But this cannot be, for \((x_\delta)\) is obviously eventually in \(T_{\delta_0}\). Therefore \(p \in \bigcap_{\delta \in D} T_\delta\).

B) Let \(p \in \bigcap_{\delta \in D} T_\delta\). If \(p\) is not a cluster point, then there exists a neighborhood \(N\) of \(p\) such that
(x_\delta) is not frequently in N; i.e., there is a \delta_0 \in D such that if \delta \geq \delta_0, then x_\delta \notin N. Then N is a neighborhood of p which does not intersect \{x_\delta: \delta \geq \delta_0\} so p \notin \bigcap_{\delta \in D} \{x_\delta: \delta \geq \delta_0\} = T_{\delta_0} and hence p \notin \bigcap_{\delta \in D} T_\delta. This contradicts the hypothesis; consequently, p must be a cluster point of (x_\delta).

**Theorem II.2.** A topological space (S,U) is compact iff each net in S has at least one cluster point in S.

**Proof.** Only if) Suppose (S,U) is compact. Let (x_\delta) be a net in S. By definition the collection T of tails of (x_\delta) is closed and by Lemma II.3 has f.i.p. By Lemma II.4 the intersection of T is equal to the set of cluster points of (x_\delta) and by Theorem 1.1 that intersection is nonempty. Hence (x_\delta) has a cluster point.

If) Suppose each net in S has a cluster point in S. Now let H be any collection of closed sets with f.i.p. Let D be the collection of finite intersections ordered by inclusion. Clearly D is a directed set. Now for each \delta \in D let x_\delta \in \delta. Then (x_\delta) is a net in S. Because \delta \geq \delta_1 implies \delta \subseteq \delta_1 it follows that if \delta \geq \delta_1, x_\delta \in \delta_1. Therefore

\[ T_{\delta_1} \subseteq \delta_1 \text{ since both are closed sets.} \]

Therefore

\[ \bigcap_{\delta \in D} T_{\delta} \subseteq \bigcap_{\delta \in D} \delta = \bigcap_{h \in H} h, \text{and since (x_\delta) has a cluster point,} \]

\[ \bigcap_{\delta \in D} T_{\delta} \neq \emptyset. \text{So } \bigcap_{h \in H} h \neq \emptyset; \text{ i.e., (S,U) is compact.} \]
Corollary. A space \((S,U)\) is compact iff each net in \(S\) has a subnet which converges in \(S\).

Proof. This follows immediately from Theorem II.1.

The other approach to convergence, filters, involves only sets. The two ideas can be developed in complete independence, or they may be, as in this paper, very closely related.

**Definition II.12.** A filter in a set \(H\) is a nonempty collection \(F\) of nonvoid subsets of \(H\) such that:

1) If \(A\) and \(B\) are in \(F\) then \(A \cap B\) is in \(F\).
2) If \(A \in F\) and a subset \(B\) of \(H\) contains \(A\), then \(B \in F\).

**Example II.8.** The collection of all neighborhoods of a point \(p\) is a filter called the neighborhood filter.

**Definition II.13.** A filter \(F\) converges to a point \(p\) if it contains the neighborhood system (collection of all neighborhoods) of \(p\). This is written \(F \rightarrow p\).

Examples II.9 and II.10 show the correlation between nets and filters.

**Example II.9.** Corresponding to each filter one can construct a net in such a fashion that the net and filter converge together. Let \(F\) be a filter in \(H\). Let \(D = \{(a,A): a \in A \in F\}\) and order \(D\) by the following: \((a,A) \geq (b,B)\) if \(A \subseteq B\). \(D\) is a directed set and \(x_\delta = a\) where \(\delta = (a,A) \in D\) defines a net in \(H\).
Definition II.14. The net in Example II.9 is called the associated net of the filter $F$.

Lemma II.5. A filter $F$ converges to a point $p$ iff the associated net of $F$ converges to $p$.

Proof. Let $F$ be a filter and $(x_\delta : D)$ the associated net in $(S, U)$.

Only if) Suppose $F \rightarrow p$. Let $N$ be a neighborhood of $p$. Then $N \in F$ and $(p, N) \in D$. Let $\delta_0 = (p, N)$ and suppose $\delta \geq \delta_0$; then $\delta = (a, A)$ where $a \in A$ and $A \subseteq N$. Therefore $x_\delta = a \in N$; i.e., $(x_\delta)$ is eventually in $N$ and hence converges to $p$.

If) Suppose $F \not\rightarrow p$. Then there is a neighborhood $N$ of $p$ which is not in $F$. Note that $A - N \neq \emptyset$ for each $A$ in $F$ (if $A - N = \emptyset$, then $A \subseteq N$ and hence $N \in F$). Let $\delta_0 = (a, A) \in D$ and let $b \in A - N$. Then $\delta = (b, A) \in D$ and $x_\delta \notin N$. Since $\delta_0$ was arbitrary, it follows that $(x_\delta)$ is frequently "outside of $N"; i.e., frequently in $S - N$ and hence not eventually in $N$. Therefore $x_\delta \not\rightarrow p$.

It might appear natural to define a cluster point of a filter to be "any point which is in each element of the filter." The inconsistency of such a definition is seen by considering the usual topology on the reals and the following filter: Let $p$ be a real number and $H$ be the neighborhood filter of $p$. Let $P = \{x : x > p\}$ and $F = H \cup \{P \cap A : A \in H\}$. $F$ is a filter and converges to $p$ but $p$ is not a cluster point of $F$, using the suggested definition.
A cluster point of a filter could also be defined as "any point which is a cluster point of the associated net." This would then require proving Definition II.15 as a theorem. Here the first "definition" will be modified slightly, to keep the definition purely in terms of filters, and the relation to the associated net then becomes Lemma II.6.

**Definition II.15.** Let $F$ be a filter in $(S,U)$. Then $p$ is a cluster point of $F$ if $p \in \bigcap \{A : A \in F\}$.

**Lemma II.6.** A point $p$ is a cluster point of a filter $F$ iff $p$ is a cluster point of the associated net $(x_\delta : D)$ of $F$.

If) Suppose $p$ is a cluster point of $(x_\delta)$. Let $A \in F$. By Definition II.14 $(x_\delta)$ is eventually in $A$ and by Lemma II.2 $p$ is then in $\overline{A}$. Therefore $p$ is a cluster point of $F$.

Only if) Suppose $p$ is not a cluster point of $(x_\delta)$. Then there exists an open set $V$ containing $p$ such that $(x_\delta)$ is not frequently in $V$; i.e., there exists a $\delta_0 \in D$ such that if $\delta \geq \delta_0$, then $x_\delta \notin V$. Let $\delta_0 = (a,A)$. Now if $b \in A$ then $b = x_\delta$ where $\delta = (b,A)$ and $(b,A) \geq \delta_0$ for each $b \in A$; hence $A \cap V = \emptyset$. Consequently $p \notin \overline{A}$, since $p$ is in $V$, an open set. Therefore $p$ is not a cluster point of $F$.

**Example II.10.** Corresponding to Example II.9 we have the following: Let $(x_\delta)$ be a net in $S$. Let $F = \{A \subseteq S : (x_\delta) \text{ is eventually in } A\}$. Then $F$ is a filter.
Definition II.16. The filter of Example II.10 is called the related filter of \((x_0)\).

Lemma II.7. A net \((x_0)\) converges to a point \(p\) iff the related filter \(F\) converges to \(p\).

Proof. The proof consists simply of noting that a neighborhood \(N\) of \(p\) is in \(F\) iff \((x_0)\) is eventually in \(N\).

Just as a net may have a cluster point to which it does not converge so may a filter. It is possible to "reduce the size" in the case of nets and get subnets which will converge to those cluster points. This is inapplicable to filters, because if a filter \(F\) does not converge to a point \(p\) then certainly no "subfilter" of \(F\) would contain all the neighborhoods of \(p\). A look at Example II.10 might suggest an alternative, for if a net is eventually in a set \(A\), then any subnet would be eventually in \(A\). Hence the related filter of a subnet would be "larger" than the related filter of the net. Consequently we have the following idea.

Definition II.17. Let \(F\) and \(G\) be filters. Then \(F\) is finer than \(G\) if \(F \supset G\). In that case \(F\) is said to be a refinement of \(G\).

Theorem II.3. A point \(p\) is a cluster point of a filter \(F\) iff there is a refinement of \(F\) which converges to \(p\).

Proof. Let \(F\) be a filter and \(p\) a point of \(S\).

Only if) Suppose \(p\) is a cluster point of \(F\). Then \(p\) is a cluster point of the associated net \((x_0:D)\) of \(F\). Let
(x_\alpha D') be a subnet of (x_\delta) which converges to p. By Lemma II.7 the related filter F' of (x_\alpha D') converges to p. Let A \in F. Then (x_\delta) is eventually in A and hence (x_\alpha D') is eventually in A. Therefore by Definition II.16 A \in F', so F \subseteq F' and F' is the convergent refinement.

If) Suppose there exists a refinement F' of F converging to p. Let A \in F and V be an open set about p. Then A \in F' because F \subseteq F' and V \in F' since F' \to p. Hence V \cap A \neq \emptyset. Therefore p \in \overline{A}, and consequently p is a cluster point of F.

Lemma II.8. A point p is a cluster point of a net (x_\delta D) iff p is a cluster point of the related filter F.

Proof. Suppose p is a cluster point of (x_\delta). Let (x_\alpha D') be a subnet of (x_\delta) converging to p. Let F and F' be the related filters of (x_\delta) and (x_\alpha) respectively. Now F' is finer than F because if (x_\delta) is eventually in A, then (x_\alpha) is eventually in A. Also F' converges to p by Lemma II.7. Hence, by Theorem II.3 p is a cluster point of F.

Suppose p is a cluster point of F. Note that (x_\delta) is eventually in each of its tails. Hence T_\delta \in F for each \delta \in D. Now p \in \bigcap \{\overline{A} : A \in F\} \subseteq \bigcap \{T_\delta : \delta \in D\}, but T_\delta is closed for each \delta. Therefore p \in \bigcap \{T_\delta : \delta \in D\}, which is precisely
the set of cluster points of \((x^\delta:D)\). Hence \(p\) is a cluster point of \((x^\delta)\).

The next theorem for filters is analogous to Theorem II.2 for nets.

**Theorem II.4.** In order that a topological space \((S,U)\) be compact, it is necessary and sufficient that each filter in \(S\) have a cluster point in \(S\).

**Proof.** Let \((S,U)\) be a topological space.

Necessity) Suppose \((S,U)\) is compact. Let \(F\) be a filter. The associated net has a cluster point by Theorem II.2, and hence, by Lemma II.6, \(F\) has a cluster point.

Sufficiency) Suppose each filter in \(S\) has a cluster point in \(S\). Let \((x^\delta)\) be a net in \(S\) and \(F\) be the related filter. Then by the hypothesis and by Lemma II.8 \((x^\delta)\) has a cluster point and by Theorem II.2 \((S,U)\) is compact.

**Corollary.** A space \((S,U)\) is compact iff each filter in \(S\) has a refinement which converges to a point in \(S\).

**Proof.** Theorem II.3.

The next idea has the usual analogy in nets, but for sequences becomes completely trivial.

**Definition II.8.** An ultrafilter is a maximal filter; i.e., one which is a proper subset of no other filter.

**Example II.11.** Let \(S\) be a set and \(p \in S\). Let \(F = \{A \subseteq S : p \in A\}\). Then \(F\) is a filter. If \(B \notin F\), then \(p \notin B\) and
since \{p\} ∈ F and \{p\} ∩ B = ∅, B cannot be in any filter containing F. Hence F is an ultrafilter.

**Lemma II.9.** If p is a cluster point of an ultrafilter F, then F converges to p.

**Proof.** By Theorem II.3 F has a convergent refinement which must be F.

**Lemma II.10.** If F is a filter, then there exists an ultrafilter which is finer than F.

**Proof.** Let F be a filter. Let \( K = \{ F_\alpha : F_\alpha \text{ is a refinement of } F \} \). Define \( F_1 \supseteq F_2 \) if \( F_1 \supseteq F_2 \). Then K is a partially ordered set and hence has a maximal chain \( K' \). Then \( F' = \bigcup \{ F_\alpha : F_\alpha \in K' \} \) is an ultra filter because:

1) Clearly \( \emptyset \) is not in \( F' \).

2) Let \( A, B \in F' \). Then \( A \in F_1 \in K' \) and \( B \in F_2 \in K' \) and \( F_1 \subseteq F_2 \) or \( F_2 \supseteq F_1 \). So \( A \in F_2 \) or \( B \in F_1 \); i.e., \( A \cap B \in F_1 \) or \( A \cap B \in F_2 \). In either case \( A \cap B \in F' \).

3) If \( A \in F' \) and \( A \subseteq B \), then \( A \in F_1 \in K' \). Hence \( B \in F_1 \), and therefore \( B \in F' \).

4) Also if \( F' \subseteq H \), then \( H \in K' \). Hence \( H \subseteq F' \) and \( F' \) is maximal.

**Theorem II.5.** Compactness is equivalent to the convergence of each ultrafilter.

**Proof.** Theorem II.4 and Lemma II.9.
Definition II.19. A universal net in $S$ is a net $(x_\delta:D)$ such that if $A \subseteq S$, then $(x_\delta:D)$ is eventually in either $A$ or $S-A$.

The following is an extremely useful criterion for determining whether or not a filter is an ultrafilter. Together with Definition II.19 it makes the relationship between universal nets and ultrafilters obvious.

Lemma II.11. Let $F$ be a filter in $S$. Then $F$ is an ultrafilter iff for each subset $A$ of $S$, either $A \in F$ or $S-A \in F$.

**Proof.** Let $F$ be a filter.

Suppose $F$ is an ultrafilter. Let $A \subseteq S$ and suppose $S-A \notin F$. Then for each $B \in F$, $B \subseteq S-A$. Therefore $A \cap B \neq \emptyset$ for each $B \in F$. Let $K = F \cup \{A \cap B : B \in F\}$. Note that $\emptyset \notin K$, and if $C \in K$ and $D \in K$, then there is an $E \in K$ such that $E \subseteq C \cap D$. Therefore $K' = \{H : \text{there is a } C \in K \text{ such that } C \subseteq H\}$ is a filter finer than $F$. But since $F$ is an ultrafilter $K' = F$, and since $A \in K'$ it follows that $A \in F$.

Suppose for each $A \subseteq S$ either $A \in F$ or $S-A \in F$. Let $F'$ be an ultrafilter refinement of $F$ and let $B \in F'$. Then $S-B \notin F'$ since $B \cap (S-B) = \emptyset$; moreover, $S-B \notin F$. Hence $B \in F$ and $F' \subseteq F$, and consequently $F' = F$; i.e., $F$ is an ultrafilter.

Lemma II.12. The related filter of a universal net is an ultrafilter.
**Proof.** Let \((x_\delta)\) be a universal net and \(F\) be the related filter. Let \(A \subseteq S\). Then \((x_\delta)\) is eventually in either \(A\) or \(S-A\) and therefore, by the definition of \(F\), either \(A \in F\) or \(S-A \in F\).

**Lemma II.13.** The associated net of an ultrafilter is a universal net.

**Proof.** Let \(F\) be an ultrafilter and \((x_\delta)\) the associated net. If \(A \subseteq S\), then \(A \in F\) or \(S-A \in F\). So \((x_\delta)\) is eventually in either \(A\) or \(S-A\) because the associated net is eventually in each element of the filter.

While it is true that each net has a universal subnet, it is not necessary to the proof of Theorem II.6 and hence is omitted. However, one more result is required.

**Lemma II.14.** If a universal net has a cluster point \(p\), then it converges to \(p\).

**Proof.** Let \((x_\delta)\) be a universal net with a cluster point \(p\). By Lemma II.12 the related filter \(F\) of \((x_\delta)\) is an ultrafilter. By Lemma II.8 \(p\) is a cluster point of \(F\), and by Lemma II.9 \(F \to p\). Hence, by Lemma II.7, \(x_\delta \to p\). As the above proof involves a good deal of tracking, the following direct proof is given: Let \(N\) be a neighborhood of \(p\); then \((x_\delta)\) is frequently in \(N\) and hence not eventually in \(S-N\). Therefore since \((x_\delta)\) is universal, it is eventually in \(N\); i.e., \(x_\delta \to p\).
Theorem II.6. In order that a space be compact it is necessary and sufficient that each universal net in $S$ converge to a point in $S$.

Proof.

Necessity) Let $(S,U)$ be compact. Then by Theorem II.2 each net has a cluster point; therefore, by Lemma II.14, each universal net converges.

Sufficiency) Suppose each universal net converges. Let $F$ be an ultrafilter. Then the associated net is universal by Lemma II.13 and converges by hypothesis. Therefore, by Lemma II.5, $F$ converges, and hence, by Theorem II.5, $(S,U)$ is compact.

The equivalence of certain convergence conditions to compactness can be extended to countable compactness. To do this, it is necessary to define a countable filter and a countable net. The former requires the concept of a base for a filter.

Definition II.20. A filter base or base for a filter is a collection $\beta$ of sets which satisfy the following conditions:

1) $\phi$ is not an element of $\beta$.

2) If $B_1 \in \beta$ and $B_2 \in \beta$, then $B_1 \cap B_2 \in \beta$.

The filter generated by $\beta$ is $\{A \subseteq S: \text{there is a } B_1 \in \beta \text{ such that } B_1 \subseteq A\}$. 
Definition II.21. A subbase in $S$ is any collection $L$ of subsets of $S$ having f.i.p. The base $\beta$ generated by $L$ is the collection of all finite intersections of $L$. The filter generated by $L$ is the filter generated by $\beta$.

Lemma II.15. If $\beta$ is a base for a filter $F$ then $\bigcap \{\overline{B}:B \in \beta\} = \bigcap \{\overline{A}:A \in F\}$.

Proof. Since $\beta \subseteq F$, it follows that $\bigcap \{\overline{A}:A \in F\} \subseteq \bigcap \{\overline{B}:B \in \beta\}$. Also if $A \in F$, then there is a $B \in \beta$ such that $B \subseteq A$. Hence $\bigcap \{\overline{B}:B \in \beta\} \subseteq \bigcap \{\overline{A}:A \in F\}$. Therefore $\bigcap \{\overline{B}:B \in \beta\} = \bigcap \{\overline{A}:A \in F\}$.

Definition II.22. A countable filter is a filter which has a countable base.

Theorem II.7. In order that a space be countably compact, it is necessary and sufficient that each countable filter have a cluster point.

Proof.

Necessity) Suppose $(S, U)$ is countably compact, and let $F$ be a countable filter. If $\beta$ is a base for $F$, then $\beta$ has f.i.p. Therefore $\{\overline{B}:B \in \beta\}$ is a countable collection of closed sets with f.i.p., which by Theorem I.4 has a nonvoid intersection. Consequently, by Lemma II.15, $\bigcap \{\overline{A}:A \in F\} = \bigcap \{\overline{B}:B \in \beta\} \neq \emptyset$, which means $F$ has a cluster point.

Sufficiency) Suppose each countable filter has a cluster point. Let $L$ be a countable collection of closed sets with f.i.p. Then the filter $F$ generated by $L$ as a subbase has a cluster point. Now $\bigcap \{J:J \in L\} \supseteq \bigcap \{\overline{A}:A \in F\} \neq \emptyset$, 
because \( L \subseteq F \). Therefore \( L \) has a nonempty intersection, and by Theorem 1.4 \((S,U)\) is countably compact.

**Theorem II.8.** A topological space \((S,U)\) is countably compact iff for each countable filter, there is a finer filter which converges.

**Proof.** Theorem II.3 and Theorem II.7.

As should be expected, countable filters have their counterpart in nets.

**Definition II.23.** A net \((x_\delta:D)\) is countable if \(D\) is countable.

**Theorem II.9.** A topological space is countably compact iff each countable net has a cluster point.

**Proof.**

Only if) Let \((S,U)\) be countably compact and \((x_\delta:D)\) be a countable net in \(S\). Then the collection of tails of \((x_\delta)\) is a countable collection of closed sets with f.i.p. Hence their intersection is nonempty, by Theorem I.4, and by Lemma II.4 \((x_\delta)\) has a cluster point.

If) Suppose each countable net has a cluster point, and let \(H\) be a countable collection of closed sets with f.i.p. Then the collection \(D\) of all finite intersections of \(H\) is directed by inclusion and countable. For each \(\delta \in D\), let \(x_\delta \in \delta\) and note that \((x_\delta:D)\) is a countable net. Then by the hypothesis \((x_\delta)\) has a cluster point. Therefore, by Lemma II.4,
∩ {T_δ:δ∈D}≠∅. Now each closed set in H is a tail of (x_δ).
(If h ∈ H, then h ∈ D; so if δ ≥ h, then δ ⊆ h and x_δ ∈ δ ⊆ h.)
Therefore H ⊆ {T_δ:δ∈D}, and ∩{h:heH} ⊆ ∩{T_δ:δ∈D} ≠ ∅;
i.e., H has a nonvoid intersection. By Theorem I.4, then,
(S,U) is countably compact.

Theorem II.10. Countable compactness is equivalent to
the statement that each countable net has a convergent subnet.

Proof. This follows immediately from Theorem II.1 and
Theorem II.9.

Theorem II.10 would seem to indicate that countable
compactness guarantees sequential compactness. This, how-
ever, is not the case since Theorem II.10 only guarantees
that a sequence (which is a countable net) has a convergent
subnet. There is no assurance that this subnet will be a
subsequence.

Example II.13. Let x_n = n where n is a positive integer.
Then (x_n) is a sequence and (x_{u(α)}:R^+) is a net, where
u(α) = n if n-1 ≤ δ < n and R^+ is the set of positive reals.
(x_{u(α)}) is a subnet of the net (x_n), but it is not a sub-
sequence, nor even a countable net.
Fig. 1--Selected relationships among various types of compactness.
THE EQUIVALENCE OF COMPACTNESS AND CONVERGENCE PROPERTIES

In any topological space \((S,U)\) the following are equivalent:

1. \((S,U)\) is compact.
2. Each net in \(S\) has a cluster point in \(S\).
3. Each net in \(S\) has a convergent subnet.
4. Each universal net in \(S\) converges to a point in \(S\).
5. Each filter in \(S\) has a cluster point in \(S\).
6. Each filter in \(S\) has a convergent refinement.
7. Each ultrafilter in \(S\) converges to a point in \(S\).

In any topological space \((S,U)\) the following are equivalent:

1. \((S,U)\) is countably compact.
2. Each countable net in \(S\) has a cluster point in \(S\).
3. Each countable net in \(S\) has a convergent refinement.
4. Each countable filter in \(S\) has a cluster point in \(S\).
5. Each countable filter in \(S\) has a convergent refinement.
