A DETAILED PROOF OF THE PRIME NUMBER THEOREM FOR ARITHMETIC PROGRESSIONS

Andrew Vlasic, B.S.

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APPROVED:

Olav Richter, Major Professor
William Cherry, Committee Member
Joseph Iaia, Committee Member
Neal Brand, Chair of the Department of Mathematics
Sandra L. Terrell, Interim Dean of the Robert B.
Toulouse School of Graduate Studies

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We follow a research paper that J. Elstrodt published in 1998 to prove the Prime Number Theorem for arithmetic progressions. We will review basic results from Dirichlet characters and L-functions. Furthermore, we establish a weak version of the Wiener-Ikehara Tauberian Theorem, which is an essential tool for the proof of our main result.

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I. INTRODUCTION

Number theorists have investigated the distribution of prime numbers for centuries. Historically, $\pi(x)$ was defined to count the number of primes less than or equal to x. In 1793, Gauss conjectured that $\pi(x) \sim \frac{x}{\log(x)}$ as $x \to \infty$, which is the Prime Number Theorem, but this was not proved until 1896 by J. Hadamard [5] and de la Vallée Poussin [9] independently of each other. It was not until 1949 that Atle Selberg [7] and Paul Erdös [4], also independently of each other, discovered an elementary proof of the Prime Number Theorem. In 1837, Dirichlet [2] proved that for two positive integers, k and l, with no common prime factors, the sequence $\{kn + l\}_{n=0}^{\infty}$ contains infinitely many primes. Our goal is to find a way to count the number of primes in this sequence. To do this, we define the function $\pi_{k,l}(x)$ to count the number of primes less than or equal to x in the sequence. In 1896, de la Vallée Poussin [9] proved that for the sequence $\{kn + l\}_{n=0}^{\infty}$, $\pi_{k,l}(x) \sim \frac{1}{\Phi(k)} \frac{x}{\log(x)}$ *as* $x \to \infty$,(where Φ is the Euler phi-function), which is the Prime Number Theorem for arithmetic progressions. In particular, de la Vallée Poussin's result implies the Prime Number Theorem since $\pi_{1,1}(x) = \pi(x)$ and $\Phi(1) = 1$. Moreover, it implies that sequence $\{kn + l\}_{n=0}^{\infty}$ contains infinitely many primes. In 1980, D. Newman [6] gave a clever proof of the Prime Number Theorem. His proof requires complex analysis, properties of the Riemann ζ-function and a weaker version of the Wiener-Ikehara Tauberian Theorem. In 1998, J. Elstrodt followed Newman's approach to prove the Prime Number Theorem for arithmetic progressions. This thesis is devoted to giving the details of Elstrodt's proof.

The first section gives a brief review of Dirichlet characters.

The second section focuses on L-functions and their properties. The Riemann ζ-function is the easiest example of an L-function. L-functions play a significant role in analytic number theory, hence we will direct much of our attention to this section.

Section three is the basis for the proof of the Prime Number Theorem for arithmetic progressions. We will concentrate on Newman's proof of the Tauberian Theorem. The Tauberian Theorem will be an essential tool in the proof of the Prime Number Theorem for arithmetic progressions.

In the fourth section we will prove the Prime Number Theorem for arithmetic progressions. The proof requires the combined results of sections 1, 2, and 3.

1. CHARACTERS

The purpose of this section is to introduce Dirichlet characters. In particular, we need the following definitions in order to define L-functions in the next section.

Definition 1. *A character on a finite group G is a homomorphism, $\chi : G \to \mathbb{C}^*$, from G to the multipicative group $\mathbb{C}^*$*

Definition 2. *Let χ and χ' be characters on G. Then $\chi\chi'(g) := \chi(g)\chi'(g)$ and $\chi^{-1}(g) := (\chi(g))^{-1}$.*

Definition 3. *A Dirichlet character mod N is a character on the group $(\mathbb{Z}/N\mathbb{Z})^* = \{n \ (mod\ N) \mid (n, N) = 1\}$. Also, if χ is a Dirichlet character then we extend $\chi : \mathbb{Z} \to \mathbb{C}$ by*

$$\chi(n) = \begin{cases} \chi(n \ (mod\ N)) & if\ (n, N) = 1 \\ 0 & if\ (n, N) > 1\,. \end{cases}$$

The principal character $\chi_0(mod\ N)$ is given by

$$\chi_0(n) = \begin{cases} 1 & if\ \ (n, N) = 1 \\ 0 & if\ \ (n, N) > 1\,. \end{cases}$$

Note that $\chi^2 = \chi_0$ if and only if χ is a real character, *i.e.*, χ is real valued.

The following theorem is needed to prove the next two theorems.

Theorem 1. *There are $\Phi(N)$ distinct Dirichlet characters mod N.*

Proof. Recall from group theory that for every finite abelian group G, $G \cong \mathbb{Z}/d_1\mathbb{Z} \times \mathbb{Z}/d_2\mathbb{Z} \times \cdots \times \mathbb{Z}/d_j\mathbb{Z}$, where $|G| = d_1 d_2 \cdots d_j$. Hence, the number of distinct homomorphisms from $G \to \mathbb{C}^*$ is $|G|$. Therefore, there are $\Phi(N)$ distinct Dirichlet characters. $\square$

The two following orthogonality relations are important in the study of characters. The second orthogonality relation will be used in section four.

Theorem 2 **(1^{st} Orthogonality Relation).** *Let χ_1 and χ_2 be Dirichlet characters mod N. Then*

$$\frac{1}{\Phi(N)} \sum_{n \ (mod \ N)} \chi_1(n)\overline{\chi}_2(n) = \begin{cases} 1 & if \ \chi_1 = \chi_2 \\ 0 & if \ \chi_1 \neq \chi_2 \, . \end{cases}$$

Proof. Note that if $\chi_1 = \chi_2$ then $\chi_1(n)\overline{\chi}_2(n) = 1$ for every $n \ mod \ N$. Thus by the previous theorem we have that $\frac{1}{\Phi(N)} \sum_{n \ (mod \ N)} \chi_1(n)\overline{\chi}_2(n) = 1$. Now assume that $\chi_1 \neq \chi_2$. Let $m \in \mathbb{N}$ such that $(m, N) = 1$ and $\chi_1(m)\overline{\chi}_2(m) \neq 1$. Hence

$$\begin{aligned}
&(1 - \chi_1(m)\overline{\chi}_2(m)) \sum_{n \ (mod \ N)} \chi_1(n)\overline{\chi}_2(n) \\
&= \sum_{n \ (mod \ N)} \chi_1(n)\overline{\chi}_2(n) - \sum_{n \ (mod \ N)} \chi_1(m)\overline{\chi}_2(m)\chi_1(n)\overline{\chi}_2(n) \\
&= \sum_{n \ (mod \ N)} \chi_1(n)\overline{\chi}_2(n) - \sum_{n \ (mod \ N)} \chi_1(mn)\overline{\chi}_2(mn) \\
&= \sum_{n \ (mod \ N)} \chi_1(n)\overline{\chi}_2(n) - \sum_{n \ (mod \ N)} \chi_1(n)\overline{\chi}_2(n) = 0,
\end{aligned}$$

where the second to last equality follows from the fact that since we have that n runs through a complete system of residues of $(\mathbb{Z}/N\mathbb{Z})^*$ then so does mn. Therefore, since

$$1 - \chi_1(m)\overline{\chi}_2(m) \neq 0, \quad \sum_{n \ (mod \ N)} \chi_1(n)\overline{\chi}_2(n) = 0.$$

□

Theorem 3 **(2^{nd} Orthogonality Relation).** *Let $N \in \mathbb{N}$. Then*

$$\frac{1}{\Phi(N)} \sum_{\chi} \chi(n) = \begin{cases} 1 & if \ n \equiv 1 \ (mod \ N) \\ 0 & otherwise, \end{cases}$$

where the sum is over all Dirichlet characters mod N.

Proof. Let $n \in \mathbb{N}$ such that $n \equiv 1(mod\ N)$. Thus by Theorem 1,

$$\frac{1}{\Phi(N)} \sum_{\chi} \chi(n) = 1.$$

Now let $n \in \mathbb{N}$ such that $(n, N) > 1$. Then $\chi(n) = 0$ for every χ and hence

$$\frac{1}{\Phi(N)} \sum_{\chi} \chi(n) = 0.$$

Finally, suppose that $N, n \in \mathbb{N}$ such that $N > 2$, $(n, N) = 1$ and $n \equiv a(mod\ N)$ for some $a \neq 1$. Let χ_1 be a Dirichlet character *mod* N where $\chi_1(n) \neq 1$. Hence

$$\begin{aligned}
&(1 - \chi_1(n)) \sum_{\chi} \chi(n) \\
=& \sum_{\chi} \chi(n) - \sum_{\chi} \chi_1(n)\chi(n) \\
=& \sum_{\chi} \chi(n) - \sum_{\chi} \chi(n) = 0,
\end{aligned}$$

where the second to last equality follows from the fact that since χ runs through all characters of $(\mathbb{Z}/N\mathbb{Z})^*$ then so does $\chi_1\chi$. Therefore, since

$$1 - \chi_1(n) \neq 0, \ \sum_{\chi} \chi(n) = 0.$$

□

2. L-FUNCTIONS

In this section we discuss L-functions, which are generalizations of the Riemann ζ-function. From this point on we will write a typical complex variable as $s = \sigma + it$ for $\sigma, t \in \mathbb{R}$.

Definition 4. *Let $s \in \mathbb{C}$ such that $\sigma > 1$. If χ is a Dirichlet character mod N, then*

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

is an L-function or a Dirichlet L-series. In particular, if $N = 1$, then

$$L(s,\chi_0) = \sum_{n=1}^{\infty} \frac{\chi_0(n)}{n^s} = \sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s).$$

The next theorem characterizes the meromorphic continuation of L-function's to the complex plane. In particular, all we need is that $L(s,\chi)$ is meromorphic when $\sigma > 0$.

Theorem 4. *a) For the principal character χ_0 (mod N), $L(s,\chi_0)$ has a meromorphic continuation to the complex plane, which is holomorphic everywhere except for a pole at s=1 with $Res(L;1) = \frac{\Phi(N)}{N}$.*
b) For $\chi \neq \chi_0$ and $N > 1$, $L(s,\chi)$ has a holomorphic continuation to the complex plane.

A proof of this theorem is given on pages 255-256 of Apostol [1].

Theorem 4.a implies the following corollary which is an interesting fact about the Riemann ζ-function.

Corollary 1. *The Riemann zeta function, $\zeta(s)$, has a meromorphic continuation to the complex plane, which is holomorphic everywhere except for a simple pole at s=1 with $Res(\zeta;1) = 1$.*

Euler products play an important role in analytic number theory.

Theorem 5 (Euler Products). *Let f(n) be a multiplicative function, i.e., f(ab)=f(a)f(b) for $a, b \in \mathbb{N}$ such that (a,b)=1, and suppose that $F = \sum_{n=1}^{\infty} f(n)$*

converges absolutely. Then $F = \prod_{p\ prime} \sum_{k=0}^{\infty} f(p^k)$ *is the Euler product for F. Moreover, if f(n) is completely multiplicative, i.e., f(ab)=f(a)f(b) for all* $a, b \in \mathbb{N}$*, then* $F = \prod_{p\ prime} \frac{1}{1-f(p)}$.

Proof. Define $P(x) = \prod_{\substack{p\ prime \\ p \leq x}} \sum_{k=0}^{\infty} f(p^k)$. Thus

$$P(x) = \prod_{\substack{p\ prime \\ p \leq x}} \sum_{k=0}^{\infty} f(p^k) = \prod_{\substack{p\ prime \\ p \leq x}} (1 + f(p) + f(p^2) + ...) = \sum_{n \in A} f(n),$$

where $A = \{n \in \mathbb{N} \mid if\ p\ is\ a\ prime\ factor\ of\ n\ then\ p \leq x\}$. Since $\sum_{k=0}^{\infty} f(p^k)$ converges absolutely, we may arrange the terms in any way. Hence $F - P(x) = \sum_{n \in B} f(n)$ where $B = \{n \in \mathbb{N} \mid there\ exists\ a\ prime\ factor\ of\ n,\ p,\ such\ that\ p > x\}$. Thus

$$\left| \sum_{n=1}^{\infty} f(n) - P(x) \right| \leq \sum_{n \in B} |f(n)| \leq \sum_{n > x} |f(n)| \underset{x \to \infty}{\longrightarrow} 0,$$

since $\sum_{n=1}^{\infty} |f(n)| < \infty$. Therefore $P(x) \underset{x \to \infty}{\longrightarrow} F$.

If $f(n)$ is completely multiplicative then $f(p^k) = f(p)^k$ for every k. Thus

$$F = \prod_{p\ prime} \sum_{k=0}^{\infty} f(p^k) = \prod_{p\ prime} \sum_{k=0}^{\infty} f(p)^k = \prod_{p\ prime} \frac{1}{1-f(p)}.$$

□

Note that if χ is a homomorphism, then $f(n) = \frac{\chi(n)}{n^s}$ is completely multiplicative. Hence we find that

$$L(s, \chi) = \prod_{p\ prime} \frac{1}{1 - \chi(p)p^{-s}}.$$

The following definitions are useful in the study of L-functions. We will point out their importance shortly.

Definition 5. *Let $n \in \mathbb{N}$. Then the Mangoldt function $\Lambda(n)$ is defined by*

$$\Lambda(n) := \begin{cases} \log(p) & \text{if } n = p^k \text{ where } p \text{ is prime and } k \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

Definition 6. *Let $x > 0$. Then the Chebyshev Ψ -function is defined by*

$$\Psi(x) = \sum_{n \leq x} \Lambda(n).$$

In combination with Newman's Tauberian Theorem, (to which the following section is devoted), the next theorem plays a key role in section four.

Theorem 6. *For $x_0 > 1$, there exists an $M \in \mathbb{R}^+$ such that $|\Psi(x)| \leq Mx$ for every $x > x_0$, i.e.,*

$$\Psi(x) = O(x) \text{ as } x \to \infty.$$

Proof. Let $\vartheta(x) := \sum_{\substack{p \ prime \\ p \leq x}} \log(p)$. Note that if p is a prime such that $n \leq p \leq 2n$, then $p \,\Big|\, \begin{pmatrix} 2n \\ n \end{pmatrix}$. Hence

$$\prod_{\substack{p \ prime \\ n \leq p \leq 2n}} p \leq \begin{pmatrix} 2n \\ n \end{pmatrix} \leq \sum_{l=1}^{2n} \begin{pmatrix} 2n \\ l \end{pmatrix} = (1+1)^{2n} = 2^{2n}.$$

Taking logarithms on both sides and setting $n = 2^{l-1}$ yields

$$\sum_{\substack{p \ prime \\ 2^{l-1} \leq p \leq 2^l}} \log(p) \leq 2^l \log(2).$$

Thus

$$\sum_{\substack{p \ prime \\ p \leq 2^l}} \log(p) \leq (2^l + 2^{l-1} + ... + 2 + 1) \log(2) \leq 2^{l+1} \log(2).$$

Let $x > 1$ and choose l such that $2^{l-1} < x \leq 2^l$. Hence

$$\begin{aligned}\vartheta(x) &= \sum_{\substack{p \ prime \\ p \leq x}} \log(p) \leq \sum_{\substack{p \ prime \\ p \leq 2^l}} \log(p) \leq 2^{l+1}\log(2) \\ &= 4(2^{l-1})\log(2) \leq 4\log(2)x = O(x).\end{aligned}$$

Thus we have that

$$\begin{aligned}\Psi(x) &= \sum_{n \leq x} \Lambda(n) = \sum_{k=1}^{\infty} \sum_{\substack{p \ prime \\ p^k \leq x}} \Lambda(p^k) \\ &= \sum_{k=1}^{\infty} \sum_{\substack{p \ prime \\ p \leq x^{\frac{1}{k}}}} \log(p) = \sum_{k \leq \log_2(x)} \sum_{\substack{p \ prime \\ p \leq x^{\frac{1}{k}}}} \log(p) \\ &= \sum_{k \leq \log_2(x)} \vartheta(x^{\frac{1}{k}}) \leq \vartheta(x) + \log_2(x)\vartheta(x^{\frac{1}{2}}) \\ &= O(x).\end{aligned}$$

□

The following two theorems provide conditions for convergence and information as to when series are holomorphic.

Theorem 7. *If the series* $\sum_{n=1}^{\infty} a_n n^{-s}$ *does not converge everywhere and does not diverge everywhere, then there exists* $\sigma_0 \in \mathbb{R}$*, called the abscissa of convergence, such that the series converges for all* s *with* $\sigma > \sigma_0$ *and diverges for all* s *with* $\sigma < \sigma_0$.

The proof of this theorem is given on pages 233 of Apostol [1].

Theorem 8. *Suppose that* $F(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ *converges for* $s = s_0 = \sigma_0 + it_0$*. Then* $F(s)$ *converges uniformly on every compact subset of* $D = \{s \in \mathbb{C} | \sigma > \sigma_0\}$*. Moreover,* $F(s)$ *is holomorphic for* $\sigma > \sigma_0$ *and*

$$F^{(k)}(s) = (-1)^k \sum_{n=1}^{\infty} (log(n))^k a_n n^{-s}$$

for every $k \in \mathbb{N}$.

The proof of this theorem is given on pages 235-236 of Apostol [1].

The following theorem is an important fact about Dirichlet series.

Theorem 9 (Landau's Theorem). *Let* $F(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ *where* $a_n \in \mathbb{R}$, $a_n \geq 0$, *and let* σ_0 *be the abscissa of convergence of* $F(s)$. *Then* $F(s)$ *has a singularity at* $s = \sigma_0$.

Proof. Without loss of generality let $\sigma_0 = 0$. By contradiction, suppose that $F(s)$ is holomorphic at $s = 0$. Then $F(s)$ is holomorphic for $|s| < \epsilon$ for some $\epsilon > 0$ and for $\sigma > 0$. Thus $F(s)$ has a Taylor series around $s = 1$ with a radius of convergence $R > 1$. Hence $F(s) = \sum_{k=0}^{\infty} \frac{F^{(k)}(1)}{k!}(s-1)^k$ for $|s-1| < R$. Thus there exists $\delta > 0$ such that

$$\begin{aligned}
F(-\delta) &= \sum_{k=0}^{\infty} \frac{(-\delta-1)^k}{k!} F^{(k)}(1) \\
&\underset{Theorem\ 8}{=} \sum_{k=0}^{\infty} \frac{(-\delta-1)^k}{k!} (-1)^k \sum_{n=1}^{\infty} \frac{(\log(n))^k}{n} a_n \\
&= \sum_{n=1}^{\infty} \frac{a_n}{n} \sum_{k=0}^{\infty} \frac{(1+\delta)^k}{k!} (\log(n))^k \\
&= \sum_{n=1}^{\infty} \frac{a_n}{n} e^{(1+\delta)\log(n)} \\
&= \sum_{n=1}^{\infty} a_n n^{\delta} < \infty,
\end{aligned}$$

which contradicts $\sigma_0 = 0$. □

The following theorem is an essential tool for the proof of the Prime Number Theorem for arithmetic progressions, providing a crucial property of L-functions when $\sigma = 1$.

Theorem 10. *If* $t \in \mathbb{R}$ *and* $\chi \neq \chi_0$, *then* $L(1+it, \chi) \neq 0$. *Also, if* $t \in \mathbb{R}$ *such that* $t \neq 0$ *then* $L(1+it, \chi_0) \neq 0$. *In particular,* $\zeta(1+it) \neq 0$ *when* $t \neq 0$.

Proof. By Theorem 5 and Theorem 8 we have that

$$
\begin{aligned}
-\frac{L'}{L}(s,\chi) &= \sum_{p\ prime} \frac{\log(p)\ \chi(p)\ p^{-s}}{1-\chi(p)\ p^{-s}} \\
&= \sum_{p\ prime} \log(p)\ \chi(p)\ p^{-s}(1+\chi(p)\ p^{-s}+\chi(p^2)\ p^{-2s}+...) \\
&= \sum_{p\ prime} \log(p)\ (\chi(p)\ p^{-s}+\chi(p^2)\ p^{-2s}+\cdots) \\
&= \sum_{n=1}^{\infty} \Lambda(n)\chi(n)n^{-s}.
\end{aligned}
$$

For $\sigma > 1$,

$$
\begin{aligned}
&Re\left[3\frac{L'}{L}(\sigma,\chi_0)+4\frac{L'}{L}(\sigma+it,\chi)+\frac{L'}{L}(\sigma+2it,\chi^2)\right] \\
&= -\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{\sigma}} Re\left[3\chi_0(n)+4\chi(n)n^{-it}+\chi^2(n)n^{-2it})\right].
\end{aligned}
$$

But since $3+4cos(\phi)+cos(2\phi) = 2(1+cos(\phi))^2 \geq 0,\ where\ \phi = arg(\chi(n)n^{-it})$, we find that

$$
Re\left[3\frac{L'}{L}(\sigma,\chi_0)+4\frac{L'}{L}(\sigma+it,\chi)+\frac{L'}{L}(\sigma+2it,\chi^2)\right] \leq 0.
$$

Assume that $\chi^2 \neq \chi_0$. By contradiction, suppose that L has a zero of order m at $s = 1+it$, where $m \geq 1$. Recall that if f is holomorphic in $G \subseteq \mathbb{C}$ and f has a zero of order n at s_0, then $f(s) = (s-s_0)^n g(s)$ where g is holomorphic in G and $g(s_0) \neq 0$. By Theorem 4.b we have that $L(s,\chi)$ is holomorphic for $s = 1+it$. Thus $L(s,\chi) = (s-(1+it))^m g(s)$ where $g(1+it) \neq 0$. Hence

$$
\frac{L'}{L}(\sigma+it,\chi) = \frac{m}{\sigma-1} + O(1)\ as\ \sigma \to 1^+. \tag{1}
$$

Again by Theorem 4.b we have that $L(s,\chi^2) = (s-(1+2it))^a\ h(s)$ for $a \geq 0$ and $h(1+2it) \neq 0$. Thus

$$
\frac{L'}{L}(\sigma+2it,\chi^2) = \frac{a}{\sigma-1} + O(1)\ as\ \sigma \to 1^+. \tag{2}
$$

Hence

$$Re\left[3\frac{L^{'}}{L}(\sigma,\chi_0)+4\frac{L^{'}}{L}(\sigma+it,\chi)+\frac{L^{'}}{L}(\sigma+2it,\chi^2)\right]$$
$$=\frac{1}{\sigma-1}(-3+4m+a)+O(1)\ as\ \sigma\to 1^{+}.$$

But since $m\geq 1$ we have that

$$Re\left[3\frac{L^{'}}{L}(\sigma,\chi_0)+4\frac{L^{'}}{L}(\sigma+it,\chi)+\frac{L^{'}}{L}(\sigma+2it,\chi^2)\right]\geq 0,$$

which is a contradiction.

Now assume that $\chi^2=\chi_0$. Recall that χ is a real character. Thus for $t\neq 0$, by Theorem 4, $L(s,\chi)$ and $L(s,\chi^2)$ are holomorphic and the previous proof by contradiction holds. Hence we are done for $\chi=\chi_0$. It remains to consider $L(1+it,\chi)$ when $\chi\neq\chi_0$ and $t=0$.

By contradiction, suppose that $L(1,\chi)=0$. Thus $f(s)=\zeta(s)L(s,\chi)$ is holomorphic at $s=1$. For $s\in\mathbb{C}$ where $\sigma>1$ we have that $f(s)=\sum_{n=1}^{\infty}a_n n^{-s}$, and by Theorem 5 we have that $f(s)=\prod_{p\ prime}\frac{1}{1-p^{-s}}\prod_{p\ prime}\frac{1}{1-\chi(p)p^{-s}}$.

Note that if:

i) $\chi(p)=1$, then

$$\prod_{\substack{p\ prime\\ \chi(p)=1}}\left(\frac{1}{1-p^{-s}}\right)\prod_{\substack{p\ prime\\ \chi(p)=1}}\left(\frac{1}{1-p^{-s}}\right)=\prod_{\substack{p\ prime\\ \chi(p)=1}}(1+\frac{2}{p^s}+\frac{3}{p^{2s}}+\cdots)$$

ii) $\chi(p)=-1$, then

$$\prod_{\substack{p\ prime\\ \chi(p)=-1}}\left(\frac{1}{1-p^{-s}}\right)\prod_{\substack{p\ prime\\ \chi(p)=-1}}\left(\frac{1}{1+p^{-s}}\right)=\prod_{\substack{p\ prime\\ \chi(p)=-1}}(1+p^{-2s}+p^{-4s}+\cdots)$$

iii) $\chi(p)=0$, then

$$\prod_{\substack{p\ prime\\ \chi(p)=0}}\left(\frac{1}{1-p^{-s}}\right)\prod_{\substack{p\ prime\\ \chi(p)=0}}\left(\frac{1}{1+(0)p^{-s}}\right)=\prod_{\substack{p\ prime\\ \chi(p)=0}}(1+p^{-s}+p^{-2s}+\cdots).$$

Therefore, since

$$\prod_{\substack{p\ prime\\ \chi(p)=1}} \left(1+\frac{2}{p^s}+\cdots\right) \prod_{\substack{p\ prime\\ \chi(p)=-1}} \left(1+p^{-2s}+\cdots\right) \prod_{\substack{p\ prime\\ \chi(p)=0}} \left(1+p^{-s}+\cdots\right)$$
$$= \sum_{n=1}^{\infty} a_n n^{-s},$$

it follows that $a_n \in \mathbb{R},\ a_n \geq 0$ and $a_{n^2} \geq 1$. Note that

$$f\left(\frac{1}{2}\right) = \sum_{n=1}^{\infty} a_n n^{-\frac{1}{2}} \geq \sum_{n=1}^{\infty} a_{n^2} n^{-1} \geq \sum_{n=1}^{\infty} n^{-1} = \infty.$$

Let σ_0 be the abscissa of convergence for $f(s)$. Since f has a singularity at $s = \frac{1}{2}$, we have that $\sigma_0 \geq \frac{1}{2}$. Also, since $L(1,\chi) = 0$, f does not have a singularity at $s = 1$. Consequently, by Landau's Theorem we have that $\frac{1}{2} \leq \sigma_0 < 1$. Recall from Landau's Theorem that f has a singularity at $s = \sigma_0$, but by Theorem 4 we have that $\zeta(s)$ and $L(s,\chi)$ are holomorphic for $\frac{1}{2} \leq \sigma < 1$, which is a contradiction. $\square$

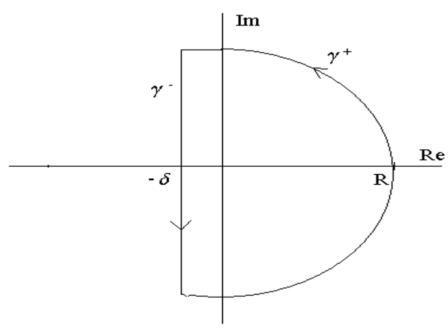
3. TAUBERIAN THEOREM

This section deals with the fundamental ingredient in the proof of the Prime Number Theorem for arithmetic progressions, Newman's Tauberian Theorem. The first theorem provides the necessary means to prove it.

Theorem 11. *Let $F : (0,\infty) \longrightarrow \mathbb{C}$ be a bounded and locally integrable function. For $s \in \mathbb{C}$, set $g(s) = \int_0^\infty F(x)e^{-sx}dx$. Then g is holomorphic for $\sigma > 0$. Moreover, suppose that g extends holomorphically to $\sigma \geq 0$. Then $\int_0^\infty F(x)dx$ exists and is equal to g(0).*

Proof. We will proceed as in Newman [6]. Let $\lambda > 0$ and define $g_\lambda(s) = \int_0^\lambda F(x)e^{-sx}dx$. Thus $g_\lambda(s)$ is an entire function. We want to show that $g_\lambda(0) \to g(0)$ *as* $\lambda \to \infty$. Let γ be the boundary of $D = \{s \in \mathbb{C} : |s| \leq R,\ \sigma \geq -\delta\}$ where $\delta > 0$ and, depending on R, δ is small enough such that

$g(s)$ is holomorphic on D and on γ.



Define $\gamma = \gamma^+ \perp \gamma^-$ where $\gamma^+ = \gamma|_{0\leq\sigma\leq R}$ and $\gamma^- = \gamma|_{-\delta\leq\sigma<0}$.
Newman's Trick: By the Cauchy integral formula, we have that

$$(g(0) - g_\lambda(0))e^{\lambda 0} = \frac{1}{2\pi i}\int_\gamma (g(s) - g_\lambda(s))e^{\lambda s}\left(\frac{1}{s} + \frac{s}{R^2}\right) ds.$$

First consider γ^+. We have

$$\left|(g(s) - g_\lambda(s))e^{\lambda s}\left(\frac{1}{s} + \frac{s}{R^2}\right)\right| = \left|\int_\lambda^\infty F(x)e^{-sx}dx[e^{\lambda s}]\left(\frac{1}{s} + \frac{s}{R^2}\right)\right|.$$

By assumption there exists $B \in \mathbb{R}^+$ such that $|F(s)| \leq B$. Thus

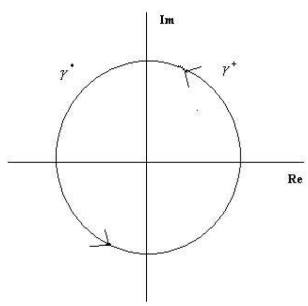
$$
\begin{aligned}
&\left| \int_\lambda^\infty F(x) e^{-sx} dx [e^{\lambda s}] \left(\frac{1}{s} + \frac{s}{R^2} \right) \right| \\
&\leq \int_\lambda^\infty B e^{-\sigma x} dx \ e^{\lambda \sigma} \left| \frac{1}{s} + \frac{s}{R^2} \right| = \frac{B}{\sigma} \left| \frac{1}{s} + \frac{s}{R^2} \right| \\
&\underset{R=|s|}{=} \frac{B}{\sigma} \left| \frac{s + \overline{s}}{R^2} \right| = \frac{2B}{R^2}.
\end{aligned}
$$

Hence

$$
\left| \frac{1}{2\pi i} \int_{\gamma^+} (g(s) - g_\lambda(s)) e^{\lambda s} \left(\frac{1}{s} + \frac{s}{R^2} \right) ds \right| \leq \frac{1}{2\pi} \int_{\gamma^+} \frac{2B}{R^2} |ds| = \frac{B}{R}. \tag{3}
$$

Now consider γ^-. Note that

$$
|g_\lambda(s)| = \left| \int_0^\lambda F(x) e^{-sx} dx \right| \leq B \int_0^\lambda e^{-\sigma t} dt < \frac{B}{|\sigma|} e^{-\lambda \sigma}.
$$



Set $\gamma^*(\phi) = Re^{i\phi}$ for $\frac{\pi}{2} \leq \phi \leq \frac{3\pi}{2}$. Since $g_\lambda(s)$ is entire,

$$
\begin{aligned}
&\left| \frac{1}{2\pi i} \int_{\gamma^-} g_\lambda(s) e^{\lambda s} \left(\frac{1}{s} + \frac{s}{R^2} \right) ds \right| = \left| \frac{1}{2\pi i} \int_{\gamma^*} g_\lambda(s) e^{\lambda s} \left(\frac{1}{s} + \frac{s}{R^2} \right) ds \right| \\
&\leq \frac{1}{2\pi} \int_{\gamma^*} |g_\lambda(s)| (e^{\lambda \sigma}) \left| \frac{1}{s} + \frac{s}{R^2} \right| |ds| \leq \frac{1}{2\pi} \int_{\gamma^*} \frac{B}{|\sigma|} e^{\lambda \sigma} e^{-\lambda \sigma} \frac{2|\sigma|}{R^2} |ds| = \frac{B}{R}.
\end{aligned} \tag{4}
$$

Finally we will estimate $\int_{\gamma^-} g(s)e^{\lambda s}\left(\frac{1}{s}+\frac{s}{R^2}\right)ds$. Let $\epsilon > 0$, set $R = \frac{B}{\epsilon}$ and choose $\delta > 0$ small enough such that $g(s)$ is holomorphic on D and γ. Also define $C = \max_{s\in[\gamma^-]}\left|g(s)\left(\frac{1}{s}+\frac{s}{R^2}\right)\right|$. Now pick $\mu > 0$ such that

$$\left|\frac{1}{2\pi i}\int_{\substack{\gamma^- \\ -\mu\leq\sigma<0}} g(s)e^{\lambda s}\left(\frac{1}{s}+\frac{s}{R^2}\right)ds\right| \leq \frac{C}{2\pi}\int_{\substack{\gamma^- \\ -\mu\leq\sigma<0}} |ds| < \epsilon. \tag{5}$$

Furthermore, for $\sigma \leq -\mu$, we have that

$$\left|\frac{1}{2\pi i}\int_{\substack{\gamma^- \\ \sigma\leq-\mu}} g(s)e^{\lambda s}\left(\frac{1}{s}+\frac{s}{R^2}\right)ds\right| \leq \frac{C}{2\pi}e^{-\lambda\mu}\int_{\substack{\gamma^- \\ \sigma\leq-\mu}} |ds| \leq \frac{C}{2}Re^{-\lambda\mu}.$$

Since for $\epsilon > 0$, C and R are very large and μ is very small, we may vary λ. Thus for every $\epsilon > 0$ there exists $\lambda_0 > 0$ such that for $\sigma \leq -\mu$,

$$\left|\frac{1}{2\pi i}\int_{\substack{\gamma^- \\ \sigma\leq-\mu}} g(s)e^{\lambda s}\left(\frac{1}{s}+\frac{s}{R^2}\right)ds\right| < \epsilon \tag{6}$$

for all $\lambda \geq \lambda_0$. Thus by equations (3), (4), (5) and (6) we get that $|g(0) - g_\lambda(0)| < 4\epsilon$. Therefore $g_\lambda(0) \longrightarrow g(0)$. □

The next theorem is the last (and most important) before the proof of the Prime Number Theorem for arithmetic progressions.

Theorem 12 (Newman's Tauberian Theorem). *Let $f : [1,\infty) \to [0,\infty)$ be a nonnegative, nondecreasing function and $f(x) = O(x)$ as $x \to \infty$, so that it's Mellin Transform, $g(s) = s\int_1^\infty f(x)x^{-s-1}dx$, is a well-defined and holomorphic for $\sigma > 1$. Suppose further that for some constant $c > 0$, the function $g(s) - \frac{c}{s-1}$ can be continued holomorphically to a neighborhood of the line $\sigma = 1$. Then*

$$f(x) \sim cx \text{ as } x \to \infty, \text{ i.e., } \lim_{x\to\infty}\frac{f(x)}{x} = c.$$

Proof. Define $F(x) = e^{-x}f(e^x) - c$. Hence $F(x)$ is bounded and on $(0, \infty)$. Note that for

$$\begin{aligned}
G(s) := & \int_0^\infty F(x)e^{-sx}dx \\
= & \int_0^\infty (e^{-x}f(e^x) - c)e^{-sx}dx \\
\underset{y=e^x}{=} & \int_1^\infty f(y)y^{-s-2}dy - \frac{c}{s} \\
= & \frac{1}{s+1}g(s+1) - \frac{c}{s}.
\end{aligned}$$

By assumption, $G(s)$ can be continued holomorphically to a neighborhood of $\sigma \geq 0$. Thus by Theorem 11 we have that

$$\int_0^\infty (e^{-x}f(e^x) - c)dx = \int_1^\infty \frac{f(y) - cy}{y^2}dy$$

exists.

Suppose that there is some $a > 1$ such that for an arbitrarily large x_0 we have $f(x) \geq acx$ for $x \geq x_0$ This would imply that

$$\begin{aligned}
& \int_x^{ax} \frac{f(y) - cy}{y^2}dy \geq \int_x^{ax} \frac{f(x) - cy}{y^2}dy \\
\geq & \int_x^{ax} \frac{acx - cy}{y^2}dy = \int_1^a \frac{acx - c(ux)}{(ux)^2}xdu \\
= & c\int_1^a \frac{a-u}{u^2}du > 0,
\end{aligned}$$

which is a contradiction by the Cauchy criterion.

Now suppose that there is some $0 < a < 1$ such that for an arbitrarily large x_0 we have $f(x) \leq acx$ for $x \geq x_0$. This would imply that

$$\begin{aligned}
& \int_{ax}^{x} \frac{f(y) - cy}{y^2}dy \leq \int_{ax}^{x} \frac{f(x) - cy}{y^2}dy \\
\leq & \int_{ax}^{x} \frac{acx - cy}{y^2}dy = c\int_a^1 \frac{a-u}{u^2}du < 0,
\end{aligned}$$

again a contradiction. Therefore

$$f(x) \sim cx \; as \; x \to \infty.$$

$\square$

4. PROOF OF THE PRIME NUMBER THEOREM FOR ARITHMETIC PROGRESSONS

The following theorem is the main result of the thesis and the proof is due to Elstrodt [3].

Theorem 13. *Let $k, l \in \mathbb{N}$ such that $(k,l) = 1$ and define $\pi_{k,l}(x)$ to count the number of prime numbers in the sequence $\{kn+l\}_{n=0}^{\infty}$ that are less than or equal to x. Then $\pi_{k,l}(x) \sim \frac{1}{\Phi(k)} \frac{x}{\log(x)}$ as $x \to \infty$.*

Proof. Let χ be a Dirichlet Character $(mod\ k)$, and $s \in \mathbb{C}$ such that $\sigma > 1$. By Theorem 3 and the proof of Theorem 10 we have that

$$-\frac{1}{\Phi(k)} \sum_{\chi} \overline{\chi}(l) \frac{L'(s,\chi)}{L(s,\chi)} = \sum_{\nu=1}^{\infty} \sum_{\substack{p \ prime \\ p^{\nu} \equiv l (mod k)}} \log(p) p^{-\nu s} = \sum_{n=1}^{\infty} \Lambda_{k,l}(n) n^{-s}$$

where

$$\Lambda_{k,l}(n) := \begin{cases} \log(p) & if\ n = p^{\nu}\ and\ n \equiv l (mod\ k) \\ 0 & otherwise. \end{cases}$$

The definition of $\Lambda_{k,l}(n)$ allows us to define the Chebyshev function,

$$\Psi_{k,l}(x) = \sum_{n \le x} \Lambda_{k,l}(n) = \sum_{\substack{p^{\nu} \le x \\ p^{\nu} \equiv l (mod k)}} \log(p).$$

Note that $\Psi_{k,l}(x) \le \Psi(x) \underset{Theorem\ 6}{=} O(x)$, *i.e.*, $\Psi_{k,l}(x) = O(x)$ *as* $x \to \infty$. Thus

$$\begin{aligned}
&- \frac{1}{\Phi(k)} \sum_{\chi} \overline{\chi}(l) \frac{L'(s,\chi)}{L(s,\chi)} = \sum_{n=1}^{\infty} \Lambda_{k,l}(n) n^{-s} \\
&= \sum_{n=1}^{\infty} [\Psi_{k,l}(n) - \Psi_{k,l}(n-1)] n^{-s} = \lim_{\alpha \to \infty} \sum_{n=1}^{\alpha} [\Psi_{k,l}(n) - \Psi_{k,l}(n-1)] n^{-s} \\
&= \lim_{\alpha \to \infty} \{[\Psi_{k,l}(1) - \Psi_{k,l}(0)](1^{-s}) + \cdots + [\Psi_{k,l}(\alpha) - \Psi_{k,l}(\alpha-1)](\alpha^{-s})\} \\
&= \lim_{\alpha \to \infty} \sum_{n=1}^{\alpha-1} \Psi_{k,l}(n)[n^{-s} - (n+1)^{-s}] + \lim_{\alpha \to \infty} \Psi_{k,l}(\alpha)(\alpha)^{-s} \\
&\underset{\substack{\Psi_{k,l}(x) = O(x) \\ \sigma > 1}}{=} \sum_{n=1}^{\infty} \Psi_{k,l}(n)[n^{-s} - (n+1)^{-s}] = \sum_{n=1}^{\infty} \Psi_{k,l}(n) s \int_{n}^{n+1} x^{-s-1} dx \\
&= s \int_{1}^{\infty} \Psi_{k,l}(x) x^{-s-1} dx.
\end{aligned}$$

Since $\Psi_{k,l}(x) = O(x)$ *as* $x \to \infty$, $\Psi_{k,l} : [1, \infty) \to [0, \infty)$, and $\Psi_{k,l}$ is a nondecreasing function, we may apply Theorem 12 to $\Psi_{k,l}(x)$. We need that the function

$$q(s) := -\frac{1}{\Phi(k)} \sum_{\chi} \overline{\chi}(l) \frac{L^{'}(s,\chi)}{L(s,\chi)} = s \int_{1}^{\infty} \Psi_{k,l}(x) x^{-s-1} dx$$

satisfies the hypothesis of $g(s)$ in Theorem 12.

It is sufficient to show that $\frac{L^{'}(s,\chi)}{L(s,\chi)}$ is holomorphic in a neighborhood of $\sigma = 1$. Recall from Theorem 10 that for $\chi \neq \chi_0$, $L(1+it,\chi) \neq 0$ for every $t \in \mathbb{R}$. Thus for χ $(mod\ k)$, $\frac{L^{'}(s,\chi)}{L(s,\chi)}$ is holomorphic in a neighborhood of $\sigma = 1$. Hence we must consider χ_0 $(mod\ k)$, since $L(1+it,\chi_0) \neq 0$ for $t \neq 0$. Note that

$$\begin{aligned} L(s,\chi_0) &= \prod_{\substack{p\ prime \\ (p,k)=1}} \frac{1}{1-p^{-s}} \\ &= \prod_{\substack{p\ prime \\ p|k}} (1-p^{-s}) \prod_{p\ prime} \frac{1}{1-p^{-s}} \\ &= \prod_{\substack{p\ prime \\ p|k}} (1-p^{-s}) \zeta(s). \end{aligned}$$

Thus

$$L^{'}(s,\chi_0) = \prod_{\substack{p\ prime \\ p|k}} (1-p^{-s}) \zeta^{'}(s) + \zeta(s) \sum_{\substack{p\ prime \\ p|k}} \log(p) p^{-s}.$$

Hence

$$-\frac{L^{'}(s,\chi_0)}{L(s,\chi_0)} = -\frac{\zeta^{'}(s)}{\zeta(s)} - \sum_{\substack{p\ prime \\ p|k}} \frac{\log(p)}{p^s - 1} = \frac{1}{s-1} + h(s)$$

where $h(s)$ is holomorphic in a neighborhood of $\sigma = 1$ by Corollary 1 of Theorem 4. Thus $q(s) - \frac{1}{\Phi(k)} \frac{1}{s-1}$ is holomorphic in a neighborhood of $\sigma = 1$. Therefore by Theorem 12 we have that

$$\Psi_{k,l}(x) \sim \frac{1}{\Phi(k)} x \ as\ x \to \infty.$$

Note that

$$
\begin{aligned}
\Psi_{k,l}(x) &= \sum_{\substack{p^\nu \le x \\ p^\nu \equiv l (mod k)}} \log(p) \\
&= \sum_{\substack{p \le x \\ p \equiv l (mod k)}} \log(p) + \sum_{\substack{p^2 \le x \\ p^2 \equiv l (mod k)}} \log(p) + \sum_{\substack{p^3 \le x \\ p^3 \equiv l (mod k)}} \log(p) + \cdots \\
&= \sum_{\substack{p \le x \\ p \equiv l (mod k)}} \log(p) + \sum_{\substack{p \le x^{\frac{1}{2}} \\ p^2 \equiv l (mod k)}} \log(p) + \sum_{\substack{p \le x^{\frac{1}{3}} \\ p^3 \equiv l (mod k)}} \log(p) + \cdots \\
&\le \sum_{\substack{p \le x \\ p \equiv l (mod k)}} \log(p) + \sum_{p \le x^{\frac{1}{2}}} \log(p) + \sum_{p \le x^{\frac{1}{3}}} \log(p) + \cdots \\
&\le \sum_{\substack{p \le x \\ p \equiv l (mod k)}} \log(p) + \Psi(x^{\frac{1}{2}}) + \Psi(x^{\frac{1}{3}}) + \cdots .
\end{aligned}
$$

Since $\Psi(x^{\frac{1}{2}}) = O(x^{\frac{1}{2}})$, $\Psi(x^{\frac{1}{2}}) \ge \Psi(x^{\frac{1}{n}})$ for $n \ge 2$, and there are at most $O(\log(x))$ terms, we have that $\Psi(x^{\frac{1}{2}}) + \Psi(x^{\frac{1}{3}}) + \cdots \le O((x^{\frac{1}{2}}) \log(x))$. Also since $\log(x) \ge \log(p)$ for every $p \le x$, p prime, and

$$
\sum_{\substack{p \le x \\ p \equiv l (mod k)}} \log(p) \le \log(x) \sum_{\substack{p \le x \\ p \equiv l (mod k)}} 1 = \log(x)\pi_{k,l}(x),
$$

we have that

$$
\Psi_{k,l}(x) \le \log(x)\pi_{k,l}(x) + O((x^{\frac{1}{2}}) \log(x)).
$$

Thus

$$
\liminf_{x \to \infty} \pi_{k,l}(x) \frac{\log(x)}{x} \ge \frac{1}{\Phi(k)}, \tag{7}
$$

since $\Psi_{k,l}(x) \sim \frac{1}{\Phi(k)} x$ *as* $x \to \infty$.

Now let $0 < y < x$. Hence

$$
\begin{aligned}
\pi_{k,l}(x) &= \pi_{k,l}(y) + \sum_{\substack{y < p \le x \\ p \equiv l (mod k)}} 1 \\
&\le \pi_{k,l}(y) + \sum_{\substack{y < p \le x \\ p \equiv l (mod k)}} \frac{\log(p)}{\log(y)} \\
&\le y + \frac{1}{\log(y)} \Psi_{k,l}(x).
\end{aligned}
$$

Letting $y = \frac{x}{\log^2(x)}$ gives us that

$$\pi_{k,l}(x)\frac{\log(x)}{x} \leq \frac{1}{\log(x)} + \frac{\Psi_{k,l}(x)}{x}\frac{\log(x)}{\log(x) - 2\log\log(x)}.$$

Thus

$$\limsup_{x\to\infty} \pi_{k,l}(x)\frac{\log(x)}{x} \leq \frac{1}{\Phi(k)}. \tag{8}$$

Equations (7) and (8) yield

$$\pi_{k,l}(x) \sim \frac{1}{\Phi(k)}\frac{x}{\log(x)}.$$

$\square$

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